§13. Manifolds

§13.1. Basics

Defn. A subset $M \subset \mathbb{R}^N$ is called an n-dimensional (embedded) manifold, if it locally looks like a graph: $\forall x_0 \in M$ $\exists$ open set $U_0 \subset \mathbb{R}^N$ s.t. $x_0 \in U$

such that $M \cap U_0 = \{ x'' = F(x') \mid x' \in V_0 \}$

where we took some splitting

$$\mathbb{R}^N_x \simeq \mathbb{R}^n_{x'} \times \mathbb{R}^{N-n}_{x''}$$

(might need to permute vector entries

e.g. $x = (x_1, x_2, x_3, x_4)$
$x' = (x_2, x_3)$, $x'' = (x_1, x_4)$),

$V_0 \subset \mathbb{R}^n$ is some open set,

and $F : V_0 \to \mathbb{R}^{N-n}$ is a C^∞ map.

Fundamental example:

$$M = \{ x \in U \mid G(x) = 0 \}$$

where $U \subset \mathbb{R}^N$ is some open set and $G : U \to \mathbb{R}^{N-n}$ is a C^∞ map.

If dG is onto at each point of M, then M is an n-dimensional manifold.

The proof uses Inverse Mapping Thm: fix $x_0 \in M$ & choose a splitting $x = (x', x'')$ such that $\partial_{x''} G(x_0)$ is invertible (possible since $dG(x_0)$ is onto)

Then the map

$$\Phi : x \mapsto (x', G(x', x''))$$

has $d\Phi(x_0) = \begin{pmatrix} I & \partial_{x'} G(x_0) \\ 0 & \partial_{x''} G(x_0) \end{pmatrix}$ invertible

So by the Inverse Mapping Thm Φ is a diffeomorphism when restricted to some neighborhood U_0 of x_0.

If its inverse is

$$\Phi^{-1}(x) = (x', A(x))$$

where $A: \mathbb{R}^N \to \mathbb{R}^{N-n}$ is a C^∞ map,

$W_0 := \Phi(U_0)$ open, then

$$\begin{aligned} M \cap U_0 &= \Phi^{-1}(W_0 \cap \{x'' = 0\}) \\ &= \{(x', A(x', 0))\} \\ &= \{x'' = F(x')\} \end{aligned}$$

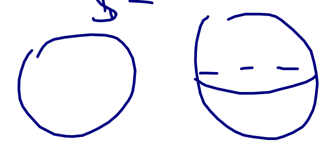
where $F(x') := A(x', 0)$.

(In effect we are reproving the Implicit Function Theorem...) $\square$

Some examples:

- Any open $U \subset \mathbb{R}^n$ is an n-dim mfld
- $S^n \subset \mathbb{R}^{n+1}$ is an n-dim compact mfld

where $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$

(Here $G(x) = |x|^2 - 1$ say 

and $dG(x) = 2x \neq 0$ on S^n)

Coordinates & Parametrizations:

if $M \cap U_0 = \{x'' = F(x') : x' \in V_0\}$ then

$x \in M \cap U_0 \overset{\varkappa}{\longmapsto} x'$ is a (local) coordinate system on M

$x' \in V_0 \longmapsto (x', F(x')) \in M$ is a parametrization

Transitions: if $x : M \cap U_0 \to V_0$, $\tilde{x} : M \cap U_0 \to \tilde{V}_0$ are 2 coordinates then

$\exists$ a C^∞ diffeomorphism $\Psi : V_0 \to \tilde{V}_0$

s.t. the diagram

$$\begin{tikzcd} & M \cap U_0 \arrow[dl, "x"'] \arrow[dr, "\tilde{x}"] & \\ V_0 \arrow[rr, "\Psi"] & & \tilde{V}_0 \end{tikzcd}$$

is commutative.

We are generally interested in <u>intrinsic</u> objects on M (those depending only on M & the "C^∞ structure" given by local coordinates) rather than <u>extrinsic</u> ones (those depending on the way M is embedded into $\mathbb{R}^N$)

In fact, it would be better to use <u>abstract manifolds</u>:

<u>"Defn."</u> An n-dimensional (abstract) manifold is a metrizable topological space M together with a system of coordinate charts:

$$(\text{open subset of } M) \xrightarrow[\text{morphism}]{\text{homeo}} (\text{open subset of } \mathbb{R}^n)$$

such that the transition maps are C^∞ diffeomorphisms

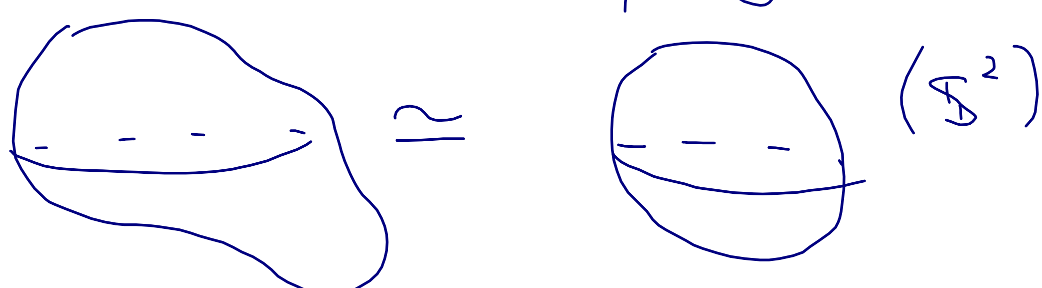
For more details, see $\boxed{18.101}$.

§13.2. Basic objects on a manifold

Assume $M \subset \mathbb{R}^N$ is an n-dim. manifold.

- $C^\infty(M)$: consists of functions $f: M \to \mathbb{C}$ s.t. in each coordinate system $\varkappa: M \cap U_0 \to V_0 (\subset \mathbb{R}^n$ open$)$ the map $f \circ \varkappa^{-1}: V_0 \to \mathbb{C}$ is C^∞
- Can define C^∞ maps & diffeomorphisms between manifolds. Intrinsic objects should transform naturally by diffeos.

Picture:



- Tangent Space: if $x_0 \in M$ & $\varkappa$ is a coordinate system then $T_{x_0}M$ (tangent space to M at x_0) is the range of $d\varkappa^{-1}(\varkappa(x_0))$.

($\varkappa^{-1}$ = parametrization map)

It's an n-dimensional subspace of $\mathbb{R}^N$, since $\varkappa^{-1}: V_0 \to \mathbb{R}^N$.

- If $M = \{x \in U : G(x) = 0\}$ then $T_{x_0}M = \{v \in \mathbb{R}^N : dG(x_0)v = 0\}$

  i.e. the kernel of $dG(x_0)$.

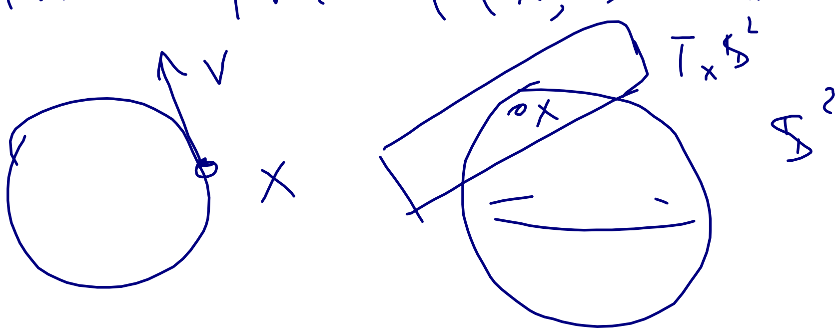
- <u>Tangent bundle</u>:

$$TM := \{(x,v) \in \mathbb{R}^N \times \mathbb{R}^N \mid x \in M,\ v \in T_x M\}$$

is a $2n$-dimensional manifold.

<u>Example</u>: $M = S^1 = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$.

Then $TM = \{(x,v) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1,\ x_1 v_1 + x_2 v_2 = 0\}$



- <u>Smooth vector field</u>: a "C^∞ section" of the tangent bundle, i.e. X is a C^∞ vector field on M (write $X \in C^\infty(M; TM)$) if it is a C^∞ map $X : M \to \mathbb{R}^N$ such that $\forall x \in M$, $X(x) \in T_x M$.

• Cotangent bundle:

$T^*M := \{(x,\xi) \mid x \in M, \xi \in T_x^*M\}$

where T_x^*M is the dual space to T_xM,

i.e. $T_x^*M = \{\xi : T_xM \to \mathbb{R} \text{ linear}\}$

T^*M is a $2n$-dimensional manifold

Can identify $T^*M \simeq TM$ extrinsically

(depending on the embedding $M \subset \mathbb{R}^N$)

by mapping $\xi \in T_x^*M$ to $v \in T_xM$

such that $\xi(w) = v \cdot w \quad \forall w \in T_xM$

Euclidean inner product

• 1-forms: C^∞ sections of T^*M

i.e. $\omega : x \in M \mapsto \omega(x) \in T_x^*M$

s.t. the map $\underset{M}{\underset{\cap}{x}} \mapsto (x, \underset{T_x^*M}{\underset{\cap}{\omega(x)}})$ is C^∞.

- **Differential**: if $f \in C^\infty(M)$

then $df \in C^\infty(M; T^*M)$ is a 1-form.

Defined as follows: if $x \in M$, $v \in T_x M$

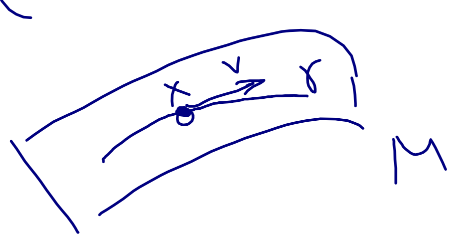
then $df(x)v$ is the derivative of f along v:

take any C^∞ curve $\gamma : (-\varepsilon, \varepsilon) \to M$

$\gamma(0) = x, \quad \gamma'(0) = v$

then $df(x)v = \frac{d}{dt}\Big|_{t=0} f(\gamma(t))$

(Such γ exist & $df(x)v$ is independent of γ ...)



df is intrinsic: if $\Phi : M \to \tilde{M}$ is a diffeomorphism and $f \in C^\infty(\tilde{M})$

then $d(\Phi^* f) = \Phi^* df$ in the sense that $\forall\, x \in M$, $v \in T_x M$ we have

$d(\Phi^* f)(x)v = df(\Phi(x))\, d\Phi(x) v$

Here $d\Phi(x) : T_x M \to T_{\Phi(x)} \tilde{M}$

- <u>Riemannian metric</u> on M:

g is a Riem. metric on M if

$\forall x \in M$, $g(x)$ is an inner product on $T_x M$.

And $g \in C^\infty$ in the sense that

$\forall$ vector fields $X, Y \in C^\infty(M; TM)$

the function

$$x \in M \mapsto g(x)(X(x), Y(x)) \text{ is } C^\infty.$$

<u>Example</u>: (extrinsic!!) can put

$$g(x)(v, w) = v \cdot w \quad \text{Euclidean inner product}$$

$\forall x \in M$, $v, w \in T_x M \subset \mathbb{R}^N$

- If $M = S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$

then this example gives the <u>round metric</u> on S^n

§13.3. Distributions on manifolds

We will fix a Riemannian metric g on a manifold M.

(Major cheating... we did not need to do this at all. A better way would be to use the bundle of densities, see e.g. [Hörmander, §6.3])

Integration: for $f: M \to \mathbb{C}$ define $\int_M f(x)\, d\mathrm{Vol}_g(x)$ as follows: (as usual, the result might be $=\infty$ or not exist)

- if we have a coordinate system $\varkappa: U_0 \to V_0 \subset \mathbb{R}^n$ and $\operatorname{supp} f \subset U_0$

then $$\int_M f(x)\, d\mathrm{Vol}_g(x) := \int_{V_0} f(\varkappa^{-1}(y))\, J(y)\, dy$$

where $J(y) = \sqrt{\det\, g_{jk}(y)_{j,k=1}^n}$ $\qquad e_j = (0,\ldots,\overset{j}{1},0\ldots0)$

and $g_{jk}(y) = g(\varkappa^{-1}(y))(d\varkappa^{-1}(y)\, e_j,\, d\varkappa^{-1}(y) e_k)$ are the coefficients of g w.r.t. $\varkappa$

- Changing coordinates will not change the $\int_M f \, dVol_g$ defined above: can be shown using Jacobi's formula (see Pset 9)
- In general split f into functions supported in a single U_0 (partition of unity)
- This gives the Lebesgue $\int$ w.r.t. the Riemannian volume measure on (M, g) (which we kind of defined above: $vol_g(U) := \int_M 1_U \, dVol_g$)
- Can define spaces $L^p(M, g)$.

The spaces $L^p_{loc}(M)$ (locally L^p), $L^p_c(M)$ (compactly supported) are actually independent of the choice of g.

- Define $C_c^\infty(M)$ similarly to $C_c^\infty(U)$, $U \subset \mathbb{R}^n$, can define convergence using coordinates

More precisely, a sequence $\varphi_k \in C_c^\infty(M)$ converges to 0 in $C_c^\infty(M)$ if:

1. $\exists$ compact $K \subset M$: $\forall k$, $\operatorname{supp} \varphi_k \subset K$
2. $\forall$ coordinate system $x: \underset{M}{U_0} \to \underset{\mathbb{R}^n}{V_0}$ ($U_0 \subset M$, $V_0 \subset \mathbb{R}^n$), the functions $\varphi_k \circ x^{-1} \in C^\infty(V_0)$ converge to 0 in $C^\infty(V_0)$.

<u>Defn</u> A <u>distribution</u> on M is a linear map $u: C_c^\infty(M) \to \mathbb{C}$ such that $\forall \varphi_k \to 0$ in $C_c^\infty(M)$, we have $(u, \varphi_k) \to 0$.

To embed functions into distributions, use the pairing

$$(f, \varphi) := \int_M f(x)\varphi(x)\, d\mathrm{Vol}_g(x)$$

$\forall f \in L^1_{loc}(M)$, $\varphi \in C_c^\infty(M)$.

(depends on $d\mathrm{Vol}_g$, not ideal... better to define distr. as dual to C_c^∞ densities...)

- Denote by $\mathcal{D}'(M)$ the space of distributions on M
- $C_c^\infty(M)$ is still dense in $\mathcal{D}'(M)$...
- Can define $u|_W$ for $u \in \mathcal{D}'(M)$, $W \subset M$ open

& can define $\operatorname{supp} u \subset W$ closed

- $\mathcal{E}'(M)$ distributions with compact support on M (dual to $C^\infty(M)$)
- The sheaf property still holds on $\mathcal{D}'(M)$
- If $x : \underset{\underset{M}{\cap}}{U_0} \to \underset{\underset{\mathbb{R}^n}{\cap}}{V_0}$ is a coordinat system

then can define $x^* : \mathcal{E}'(V_0) \to \mathcal{E}'(U_0) \subset \mathcal{E}'(M)$

$x^{-*} = (x^{-1})^* : \mathcal{D}'(M) \to \mathcal{D}'(U_0) \to \mathcal{D}'(V_0)$

which lets us think of distributions on M in local coordinates

- If $M, \tilde{M}$ are manifolds and

$\Phi : M \to \tilde{M}$ is a submersion

(i.e. $d\Phi(x)$ is onto $T_{\Phi(x)}\tilde{M}$ at every $x \in M$)

then can define $\Phi^* : \mathcal{D}'(\tilde{M}) \to \mathcal{D}'(M)$ (using local coordinates etc.)

- **Sobolev spaces** Let $s \in \mathbb{R}$.

Define $H^s_{loc}(M) \subset \mathcal{D}'(M)$ as follows:

$u \in \mathcal{D}'(M)$ lies in $H^s_{loc}(M)$

if and only if $\forall$ coordinate system

$\varkappa : \underset{M}{\underset{\cap}{U_0}} \to \underset{\mathbb{R}^n}{\underset{\cap}{V_0}}$, the pullback

$\varkappa^{-*} u \in \mathcal{D}'(V_0)$ lies in $H^s_{loc}(V_0)$

Define $H^s_c(M) = H^s_{loc}(M) \cap \mathcal{E}'(M)$.

Note: this is a reasonable definition

because H^s_{loc} is invariant under

pullbacks by diffeomorphisms.

(Pset 8, Exercise 7).

In particular, if $v \in H^s_c(V_0) \cap \mathcal{E}'(V_0)$

then $\varkappa^* v \in \mathcal{E}'(M)$ lies in $H^s_c(M)$.

Indeed, if $\tilde{\varkappa} : \tilde{U}_0 \to \tilde{V}_0$ is another coord. system

then $\tilde{\varkappa}^{-*}\varkappa^* v \in \mathcal{D}'(\tilde{V}_0)$ is given by

$\tilde{x}^{-*} x^* v = \Phi^* v$ where

$$\Phi = x \circ \tilde{x}^{-1} : \tilde{V}_0 \cap x(U_0) \to$$
$$\to V_0 \cap x(\tilde{U}_0)$$

is a C^∞ diffeomorphism:

$$\begin{tikzcd} & U_0 \cap \tilde{U}_0 \arrow[dl, "\tilde{x}"'] \arrow[dr, "x"] & \\ \tilde{V}_0 \cap x(U_0) \arrow[rr, "\Phi"] & & V_0 \cap x(\tilde{U}_0) \end{tikzcd}$$

is commutative.

Since $v \in H^s_c(V_0) \subset H^s_{loc}(V_0)$ we have

$$\Phi^* v \in H^s_{loc}(\tilde{V}_0 \cap x(U_0))$$

and $\mathrm{supp}(\Phi^* v)$ is the intersection of $\tilde{V}_0$ with a compact set $\Rightarrow$

$\Rightarrow$ can extend $\Phi^* v$ by 0 to $\mathcal{D}'(\tilde{V}_0)$

and this will be in $H^s_{loc}(\tilde{V}_0)$ ...

- <u>Multiplication</u>: if $a \in C^\infty(M)$ then can define $u \mapsto au$ as an operator $\mathcal{D}'(M) \to \mathcal{D}'(M)$, $H^s_{loc}(M) \to H^s_{loc}(M)$.