§9. Entropy

An entropy of a map/flow, roughly speaking, is a nonnegative number which describes how "complicated" the map/flow is, in the sense of the growth of the number of "different" trajectories of length n as $n \to \infty$.

Very roughly speaking,

$$\#\left(\begin{array}{c}\text{different trajectories}\\ \text{of length} \leq n\end{array}\right) \sim \exp(\text{entropy} \cdot n)$$

§9.1. Topological entropy

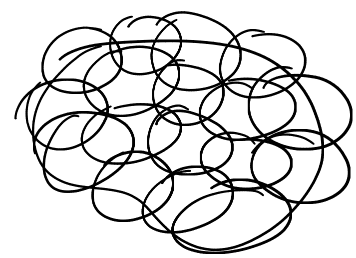
Let us start with the case of maps (not necessarily invertible, for now):

- (X,d) compact metric space
- $\varphi : X \circlearrowleft$ continuous map

Before we bring in φ,
we discuss various ways of counting how many ε-sized balls pack in X etc.:

<u>Defn</u> Let (X,d) be a compact metric space. Let $\varepsilon>0$. Define:

- $S_d(\varepsilon)$ = the <u>minimal</u> number N of points $x_1,\dots,x_N \in X$ such that



$$X \subset \bigcup_{j=1}^{N} B_d(x_j,\varepsilon)$$

where $B_d(a,\varepsilon) = \{x \in X : d(x,a) \le \varepsilon\}$ is a metric ball

- $D_d(\varepsilon)$ = the <u>minimal</u> number N of sets $B_1,\dots,B_N \subset X$ such that $X \subset \bigcup_{j=1}^{N} B_j$ and $\operatorname{diam} B_j \le \varepsilon \ \forall j$

Here $\operatorname{diam} B = \sup\{d(x,y) \mid x,y \in B\}$

- $N_d(\varepsilon)$ = the maximal number N of points $x_1, \dots, x_N \in X$ such that $d(x_j, x_k) > \varepsilon \quad \forall j \neq k$.

Each of these is a slightly different way of counting how many different points X has if we look at "resolution" ε (w.r.t. the metric d)

They are all roughly equivalent:

Lemma 1 We have

(1) $D_d(2\varepsilon) \le S_d(\varepsilon) \le D_d(\varepsilon)$ and

(2) $N_d(2\varepsilon) \le S_d(\varepsilon) \le N_d(\varepsilon)$

Proof $S_d(\varepsilon) \le D_d(\varepsilon)$:

if $X \subset \bigcup_{j=1}^{D_d(\varepsilon)} B_j$, $\operatorname{diam} B_j \le \varepsilon$, then take any $x_j \in B_j$ & we have $X \subset \bigcup_{j=1}^{D_d(\varepsilon)} B(x_j, \varepsilon)$

(as $B_j \subset B(x_j, \varepsilon)$)

$D_d(2\varepsilon) \leq S_d(\varepsilon)$:

if $X \subset \bigcup_{j=1}^{S_d(\varepsilon)} B(x_j, \varepsilon)$

then $X \subset \bigcup_{j=1}^{S_d(\varepsilon)} B_j$ where $B_j := B(x_j, \varepsilon)$

and $\operatorname{diam}(B_j) \leq 2\varepsilon$.

$S_d(\varepsilon) \leq N_d(\varepsilon)$: if $x_1, \ldots, x_{N_d(\varepsilon)}$ are ε-separated (i.e. $d(x_j, x_k) > \varepsilon \;\; \forall j \neq k$)

then $X \subset \bigcup_{j=1}^{N_d(\varepsilon)} B(x_j, \varepsilon)$.

Indeed, otherwise $\exists\, x \in X : d(x_j, x) > \varepsilon \;\; \forall j$

but then $x, x_1, \ldots, x_{N_d(\varepsilon)}$ would be ε-separated.

(contradicting the maximality of $N_d(\varepsilon)$)

$N_d(2\varepsilon) \leq S_d(\varepsilon)$: if $x_1, \ldots, x_{N_d(2\varepsilon)}$ are 2ε-separated, then $\forall j \neq k$, x_j and x_k cannot lie in the same ball of radius ε. So we need at least $N_d(2\varepsilon)$ balls of radius ε to cover the points $x_1, \ldots, x_{N_d(2\varepsilon)}$. $\square$

Remark: (won't be useful right away...)

if X is a (compact) manifold, $\dim X = r$, then as $\varepsilon \to 0$ we have

$$S_d(\varepsilon) \sim D_d(\varepsilon) \sim N_d(\varepsilon) \sim \varepsilon^{-r}$$

in the sense that $\exists\, C>0\ \forall \varepsilon<1$

$$\frac{1}{C}\varepsilon^{-r} \leq S_d(\varepsilon) \leq C\varepsilon^{-r} \ldots$$

Now we bring in the map $\varphi : X \circlearrowleft$.

Fix a metric d on X and for $n \geq 0$ define the refined metric

$$d_n(x,y) = \max_{j=0}^{n-1} d(\varphi^j(x), \varphi^j(y))$$

$$= \max(d(x,y), d(\varphi(x),\varphi(y)), \ldots, d(\varphi^{n-1}(x), \varphi^{n-1}(y)))$$

That is, $d_n(x,y) \leq \varepsilon$ if

$\forall j = 0, \ldots, n-1$, $d(\varphi^j(x), \varphi^j(y)) \leq \varepsilon$

i.e. the trajectories $\varphi^j(x)$, $\varphi^j(y)$ stay ε-close for $0 \leq j < n$.

(If we are looking at a "resolution" ε then such trajectories look the same)

Define $\boxed{D_\varphi(\varepsilon, n) = D_{d_n}(\varepsilon)}$

= minimal number N of sets $B_1, \dots, B_N$ such that

$X \subset \bigcup_{\ell=1}^{N} B_\ell$ (they cover X) and

$\forall x, y \in B_\ell,\ d_n(x,y) \le \varepsilon$

(each B_ℓ has all trajectories ε-close for time $0 \le j \le n-1$)

The quantity D_φ is useful to us because it is <u>submultiplicative</u> w.r.t. n:

<u>Lemma 2</u> We have $\forall n, m \ge 0$

$$D_\varphi(\varepsilon, n+m) \le D_\varphi(\varepsilon, n)\, D_\varphi(\varepsilon, m)$$

<u>Proof</u> Assume that

$(B_\ell)_{\ell=1}^{D_\varphi(\varepsilon,n)}$, $(C_r)_{r=1}^{D_\varphi(\varepsilon,m)}$ are collections of subsets such that

$$X \subset \bigcup_{\ell} B_\ell, \quad X \subset \bigcup_r C_r,$$

and $\mathrm{diam}_{d_n}(B_\ell) \leq \varepsilon$ $\quad \forall \ell, r$.
$\mathrm{diam}_{d_m}(C_r) \leq \varepsilon$

Define the sets $(A_{\ell r})_{\substack{1 \leq \ell \leq D_\varphi(\varepsilon, n) \\ 1 \leq r \leq D_\varphi(\varepsilon, m)}}$

as $A_{\ell r} := B_\ell \cap \varphi^{-n}(C_r)$.

That is, $x \in A_{\ell r}$ if $x \in B_\ell$ AND $\varphi^n(x) \in C_r$.

Roughly speaking, ℓ encodes what happens to the trajectory $\varphi^k(x)$ for $0 \leq k < n$ and r encodes what happens for $n \leq k < n+m$.

We have $X \subset \bigcup_{\ell, r} A_{\ell r}$, so we are done once we prove that

$\forall x, y \in A_{\ell r}$, $d_{n+m}(x,y) \leq \varepsilon$.

But here

$$d_{n+m}(x,y) = \max\left(d(x,y), \dots, d\left(\varphi^{n+m-1}(x), \varphi^{n+m-1}(y)\right)\right)$$

$$= \max\left(d_n(x,y), d_m\left(\varphi^n(x), \varphi^n(y)\right)\right)$$

and $d_n(x,y) \leq \varepsilon$ (as $x,y \in B_e$)

and $d_m(\varphi^n(x), \varphi^n(y)) \leq \varepsilon$ (as $\varphi^n(x), \varphi^n(y) \in C_r$).

$\square$

We now want to describe the <u>asymptotics</u> of $D_\varphi(\varepsilon, n)$ as $n \to \infty$.

I.e. how many "different at resolution ε" trajectories of length n are there?

We are hoping for something like

$$D_\varphi(\varepsilon, n) \sim e^{h_\varepsilon n} \quad \text{for some } h_\varepsilon \geq 0.$$

So we should define

$$h_\varepsilon := \lim_{n \to \infty} \frac{\log D_\varphi(\varepsilon, n)}{n}.$$

Here the limit exists since $\log D_\varphi(\varepsilon, n)$ is subadditive (by Lemma 2) and by

Fekete's Lemma Assume $a_n \in \mathbb{R}$ is a subadditive sequence, i.e.

$$a_{n+m} \leq a_n + a_m \quad \forall n, m \geq 1.$$

Then $$\lim_{n\to\infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n}.$$

Proof Certainly $\liminf_{n\to\infty} \frac{a_n}{n} \geq \inf_n \frac{a_n}{n}$.

So it suffices to show that

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \inf_m \frac{a_m}{m}.$$

For that it is enough to show that $\forall m$,

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_m}{m}$$

Take large n & write $n = q \cdot m + r$, $0 \leq r < m$, $q \geq 0$.

Then $a_n \leq q \cdot a_m + a_r$.

So $\frac{a_n}{n} \leq \frac{q \cdot a_m + a_r}{n} \leq$

$$\leq \frac{qm}{n} \cdot \frac{a_m}{m} + \frac{a_r}{n}.$$

But as $n \to \infty$, we have $\frac{a_r}{n} \to 0$

(only finitely many options for r)

and $\frac{qm}{n} \to 1$. So

$\limsup_{n \to \infty} \frac{a_n}{n} \leq \frac{a_m}{m}$ as needed. $\square$

So now we have defined

$$h_\varepsilon := \lim_{n \to \infty} \frac{1}{n} \log D_\varphi(\varepsilon, n).$$

Of course this can depend on ε,
e.g. if $\operatorname{diam}(X) \leq \varepsilon$ then $h_\varepsilon = 0$
(as $D_\varphi(\varepsilon, n) = 1$)

Note that h_ε is a <u>decreasing</u> function of $\varepsilon > 0$:

if $\varepsilon_1 < \varepsilon_2$ then $\forall n$

$$D_\varphi(\varepsilon_1, n) \geq D_\varphi(\varepsilon_2, n)$$

and thus $h_{\varepsilon_1} \geq h_{\varepsilon_2}$.

<u>Defn</u> The <u>topological entropy</u> of φ is defined as

$$h_{top}(\varphi) = \lim_{\varepsilon \to 0+} h_\varepsilon = \sup_{\varepsilon > 0} h_\varepsilon,$$

i.e. $h_{top}(\varphi) = \lim_{\varepsilon \to 0+} \lim_{n \to \infty} \frac{1}{n} \log D_\varphi(\varepsilon, n)$.

<u>Note:</u> $h_{top}(\varphi)$ does not change if we replace d by an equivalent metric (exercise)

Lemma 3 ① If φ^k, $k \geq 1$, is k-th iterate of φ, then

$$h_{top}(\varphi^k) = k \cdot h_{top}(\varphi)$$

② If φ is a homeomorphism (i.e. φ^{-1} exists & is continuous) then $h_{top}(\varphi^{-1}) = h_{top}(\varphi)$.

Proof ① Assume that $B \subset X$ is such that $\operatorname{diam}_{d_{nk,\varphi}}(B) \leq \varepsilon$, i.e. $\forall j = 0, 1, \dots, nk-1$, we have $\forall x, y \in B$

$$d(\varphi^j(x), \varphi^j(y)) \leq \varepsilon.$$

Then $\forall l = 0, \dots, n-1$ we have $\forall x, y \in B$

$$d(\varphi^{kl}(x), \varphi^{kl}(y)) \leq \varepsilon \quad (\text{putting } j = kl).$$

So $\operatorname{diam}_{d_{n,\varphi^k}}(B) \leq \varepsilon$.

This shows $D_{\varphi^k}(\varepsilon, n) \leq D_\varphi(\varepsilon, nk)$.

Thus $\frac{1}{n}\log D_{\varphi^k}(\varepsilon,n) \le \frac{k}{nk}\log D_\varphi(\varepsilon,nk)$

which (taking $n\to\infty$) gives

$$h_\varepsilon(\varphi^k) \le k\, h_\varepsilon(\varphi)$$

and thus (taking $\varepsilon\to 0$)

$$h_{top}(\varphi^k) \le k\cdot h_{top}(\varphi).$$

On the other hand, if we fix $\varepsilon>0$ then there exists $\tilde\varepsilon>0$ such that $\forall x,y\in X$,

$$d(x,y)\le\tilde\varepsilon \Rightarrow d(\varphi^r(x),\varphi^r(y))\le\varepsilon \quad \forall r=0,\dots,k-1$$

(by uniform continuity of φ).

If $B\subset X$ has $\mathrm{diam}_{d_{n,\varphi^k}} B\le\tilde\varepsilon$,

i.e. $\forall x,y\in B$ $\forall \ell=0,\dots,n-1$

$$d(\varphi^{k\ell}(x),\varphi^{k\ell}(y))\le\tilde\varepsilon,$$

Then $\forall x, y \in B$ $\forall j = 0, 1, \dots, nk-1$ we get $d(\varphi^j(x), \varphi^j(y)) \leq \varepsilon$ by writing $j = k\ell + r$, $0 \leq r < k$, $0 \leq \ell < n$ and using that $d(\varphi^{k\ell}(x), \varphi^{k\ell}(y)) \leq \tilde{\varepsilon}$.

It follows that

$$\operatorname{diam}_{d_{nk,\varphi}}(B) \leq \varepsilon.$$

Thus $D_\varphi(\varepsilon, nk) \leq D_{\varphi^k}(\tilde{\varepsilon}, n)$ and (taking $\frac{1}{n}\log(\dots)$ & $n \to \infty$)

$$k \cdot h_\varepsilon(\varphi) \leq h_{\tilde{\varepsilon}}(\varphi^k)$$

which gives (taking sup over $\varepsilon > 0$)

$$k \cdot h_{top}(\varphi) \leq h_{top}(\varphi^k).$$

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(2) Assume now that φ is a homeomorphism. We show that

$$h_{top}(\varphi^{-1}) \leq h_{top}(\varphi)$$

which is enough since we can apply the same argument to φ^{-1}.

It is enough to prove that $\forall \varepsilon > 0$, n we have

$$D_{\varphi^{-1}}(\varepsilon, n) \leq D_\varphi(\varepsilon, n).$$

Let $N := D_\varphi(\varepsilon, n)$ and $B_1, \dots, B_N \subset X$ be such that $X \subset \bigcup_\ell B_\ell$ and $\operatorname{diam}_{d_{n,\varphi}}(B_\ell) \leq \varepsilon$.

Put $\tilde{B}_\ell := \varphi^{n-1}(B_\ell)$. Then $X \subset \bigcup_\ell \tilde{B}_\ell$ and $\operatorname{diam}_{d_{n,\varphi^{-1}}}(\tilde{B}_\ell) \leq \varepsilon$:

indeed, if $\tilde{x}, \tilde{y} \in \tilde{B}_\ell$ then $\tilde{x} = \varphi^{n-1}(x)$, $\tilde{y} = \varphi^{n-1}(y)$ for some $x, y \in B_\ell$ and $\operatorname{diam}_{d_{n,\varphi}}(B_\ell) \leq \varepsilon \Rightarrow d(\varphi^j(x), \varphi^j(y)) \leq \varepsilon$ $\forall j = 0, \dots, n-1$

$\Rightarrow d(\varphi^{-k}(\tilde{x}), \varphi^{-k}(\tilde{y})) \leq \varepsilon$ $\forall k = 0, \dots, n-1$

as $\varphi^{-k}(\tilde{x}) = \varphi^{n-1-k}(x) \dots \Rightarrow d_{n,\varphi^{-1}}(\tilde{x}, \tilde{y}) \leq \varepsilon$.

It follows that $D_{\varphi^{-1}}(\varepsilon, n) \leq N$ and finishes the proof. $\square$

For flows, entropy is defined similarly to maps: if $\varphi^t : X \circlearrowleft$ is a flow then we consider the metrics $d_T(x,y) = \sup_{0 \leq t < T} d(\varphi^t(x), \varphi^t(y))$, and we can define $D_\varphi(\varepsilon, T)$ and $h_{top}(\varphi)$.

§9.2 Examples of entropy computation

① $X = S^1 = \mathbb{R}/\mathbb{Z}$, $\varphi(x) = x + r \mod \mathbb{Z}$ for some fixed $r \in \mathbb{R}$:

We have $d_n(x,y) = \max_{j=0}^{n-1} d(\varphi^j(x), \varphi^j(y)) = d(x,y)$ (where d is the standard metric on S^1)

Since $d(\varphi^j(x), \varphi^j(y)) = d(x,y)$ $\forall j$.

Thus $D_\varphi(\varepsilon, n)$ is independent of n, so $h_\varepsilon = 0$ $\forall \varepsilon > 0$, and thus $\boxed{h_{top}(\varphi) = 0}$.

This works any time φ is an <u>isometry</u>.

② $X = S^1$, $\varphi(x) = 2x \bmod \mathbb{Z}$. | 18.118 9-17

We use the following

Lemma If $\varepsilon \le \frac{1}{4}$ and $x, y \in S^1$ then $d_n(x,y) \le \varepsilon \iff d(x,y) \le \frac{\varepsilon}{2^{n-1}}$.

Proof We need to show

$$d(x,y) \le \frac{\varepsilon}{2^{n-1}} \iff \forall j = 0, \dots, n-1,\ d(\varphi^j(x), \varphi^j(y)) \le \varepsilon.$$

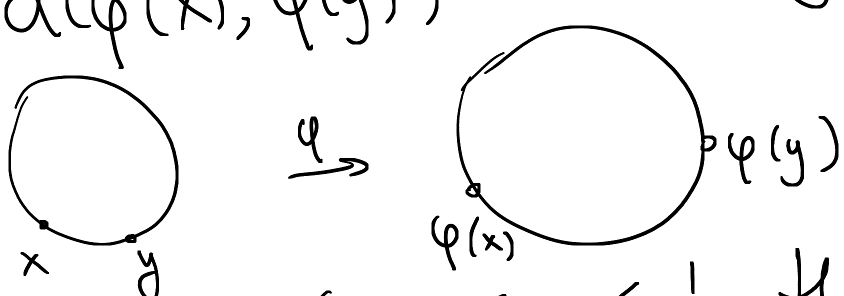
$\Rightarrow$: follows from the Lipschitz bound

$d(\varphi(x), \varphi(y)) \le 2\, d(x,y)$ $\forall\, x, y \in S^1$.

$\Leftarrow$: we have $\forall x, y \in S^2$:

if $d(x,y) \le \frac{1}{4}$ then

$$d(\varphi(x), \varphi(y)) = 2\, d(x,y):$$



Thus, if $d_n(x,y) \le \varepsilon \le \frac{1}{4}$ then

$d(\varphi^j(x), \varphi^j(y)) = 2^j d(x,y)$ $\forall j = 0, \dots, n-1$ (by induction on j)

and thus $d(x,y) \le \frac{\varepsilon}{2^{n-1}}$. $\square$

Given the lemma, we see that $\forall \varepsilon \le \frac{1}{4}$, $\forall n$, we have

$D_\varphi(\varepsilon, n) =$ minimal number N of sets $B_1, \dots, B_N$ s.t. $S^1 \subset \bigcup_\ell B_\ell$

and $\mathrm{diam}_d(B_\ell) \le \frac{\varepsilon}{2^{n-1}}$.

diameter w.r.t. the usual metric on $\mathbb{R}/\mathbb{Z}$.

This gives $D_\varphi(\varepsilon, n) = \left\lceil \frac{2^{n-1}}{\varepsilon} \right\rceil$

Then $h_\varepsilon = \lim_{n \to \infty} \frac{\log D_\varphi(\varepsilon, n)}{n} = \log 2$

So $\boxed{h_{top}(\varphi) = \log 2}$

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③ <u>Hyperbolic toral automorphisms:</u>

$X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, $\varphi(x) = Ax \mod \mathbb{Z}^2$,

$A \in SL(2, \mathbb{Z})$ hyperbolic.

Let λ, λ^{-1} be the eigenvalues of A, with $|\lambda| > 1$.

Fix some $x \in X$ and $n \geq 0$, as well as small enough $\varepsilon > 0$.

We want to understand the ball
$$\{y \in X \mid d_n(x,y) \leq \varepsilon\}.$$
Certainly, it's contained in the small ball $B_d(x,\varepsilon)$, so we can locally identify $\mathbb{T}^2$ with $\mathbb{R}^2$ here.

Then we can define the stable/unstable distances $d_s(x,y)$, $d_u(x,y)$ as follows:

if $v_u, v_s \in \mathbb{R}^2$ are unit length vectors s.t. $Av_u = \lambda v_u$, $Av_s = \lambda^{-1} v_s$, then write $x - y = a_u v_u + a_s v_s$ for some small $a_u, a_s \in \mathbb{R}$ and put

$$d_u(x,y) := |a_u|$$
$$d_s(x,y) := |a_s|.$$

Similarly to the Lemma in Example ②, we have

Lemma If ε is small enough then $\exists C$ (independent of ε) such that $\forall x, y \in X$ $\forall n \geq 0$ we have

ⓐ $d_u(x,y) \leq \dfrac{\varepsilon}{C \cdot |\lambda|^n}$, $d_s(x,y) \leq \dfrac{\varepsilon}{C}$

$$\Downarrow$$

$$d_n(x,y) \leq \varepsilon$$

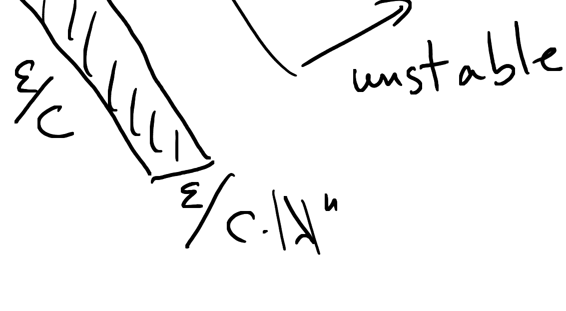
and

ⓑ $d_n(x,y) \leq \varepsilon \Rightarrow$

$\Rightarrow d_u(x,y) \leq \dfrac{C\varepsilon}{|\lambda|^n}$, $d_s(x,y) \leq C\varepsilon.$

Proof: ⓐ Let's look at the set of

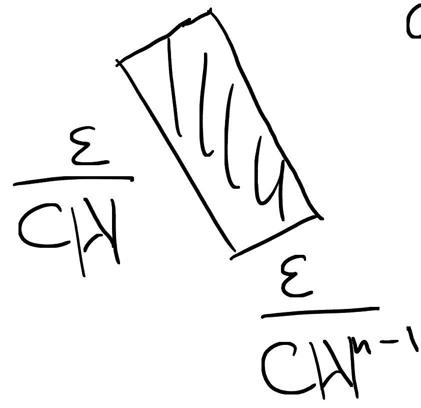
$y \in X$: $d_u(x,y) \leq \dfrac{\varepsilon}{C \cdot |\lambda|^n}$, $d_s(x,y) \leq \dfrac{\varepsilon}{C}$: $(*)$



Looks like a rectangle.

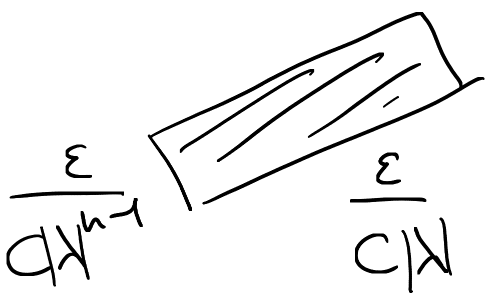
Apply φ to this rectangle $\to$ another rectangle

$\to$ we set



and so on up until φ^{n-1} (this set)

is yet another rectangle.



(Note: no wrapping around the torus as ε is small)

Each of these sets has diameter $\leq \varepsilon$ if C is large enough.

So if y satisfies $(*)$ then

$d_n(x,y) \leq \varepsilon$.

(b) Assume now that $d_n(x,y) \leq \varepsilon$.

Then we can see (similarly to the pictures in part (a), and using that ε is small) that for $j = 0, \dots, n-2$

$$d_u(\varphi^{j+1}(x), \varphi^{j+1}(y)) = |\lambda| \cdot d_u(\varphi^j(x), \varphi^j(y))$$
$$d_s(\varphi^{j+1}(x), \varphi^{j+1}(y)) = |\lambda|^{-1} \cdot d_s(\varphi^j(x), \varphi^j(y)).$$

Indeed, write $x-y$ as a linear combination of v_u, v_s and apply the linear map A. (No wrapping around since ε is small).

So then: $d_u(\varphi^{n-1}(x), \varphi^{n-1}(y)) = |\lambda|^{n-1} \cdot d_u(x,y)$

And $d_u(\varphi^{n-1}(x), \varphi^{n-1}(y)) \leq C d(\varphi^{n-1}(x), \varphi^{n-1}(y))$
$\leq C\varepsilon$.

So $d_u(x,y) \leq \dfrac{C\varepsilon}{|\lambda|^n}$.

And $d_s(x,y) \leq C d(x,y) \leq C\varepsilon$.

$\square$

Now, $D_\varphi(\varepsilon, n)$ is the minimal number N of sets $B_1, \dots, B_N$ s.t. $X \subset \bigcup_\ell B_\ell$ and

$$x, y \in B_\ell \Rightarrow d_n(x,y) \le \varepsilon.$$

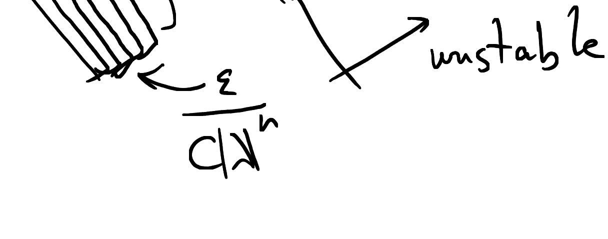
- Each such B_ℓ is contained in a rectangle (for any $x \in B_\ell$)

$$\{y : d_u(x,y) \le \frac{C\varepsilon}{|\lambda|^n}, d_s(x,y) \le C\varepsilon\}$$

which has area $\le \dfrac{C' \cdot \varepsilon^2}{|\lambda|^n}$.

So $\boxed{N \ge \dfrac{|\lambda|^n}{C' \cdot \varepsilon^2}}$

- On the other hand, if C' is large enough then we can find $\boxed{N \le \dfrac{C'|\lambda|^n}{\varepsilon^2}}$ rectangles $B_1, \dots, B_N$, $X \subset \bigcup_\ell B_\ell$ such that $\operatorname{diam}_{d_u}(B_\ell) \le \dfrac{\varepsilon}{C|\lambda|^n}$, $\operatorname{diam}_{d_s}(B_\ell) \le \dfrac{\varepsilon}{C}$



and thus $\operatorname{diam}_{d_n}(B_\ell) \le \varepsilon$.

So, $\frac{|\lambda|^n}{C'\varepsilon^2} \le D_\varphi(\varepsilon,n) \le \frac{C'|\lambda|^n}{\varepsilon^2}$.

Then $h_\varepsilon = \lim_{n\to\infty} \frac{\log D_\varphi(\varepsilon,n)}{n} = \log|\lambda|$.

Thus $\boxed{h_{top}(\varphi) = \log|\lambda|.}$

4. Geodesic flow $\varphi^t : X \circlearrowleft$ where $X = SM$ and (M,g) is a <u>hyperbolic</u> surface.

A similar argument to 3 shows that $\boxed{h_{top}(\varphi) = 1}$ where we use that the expansion/ /contraction rate of the flow on the unstable/stable spaces is equal to 1: $d\varphi^t \cdot U_\pm = e^{\mp t} U_\pm$.

We finish this subsection with the following

Fact: if $\varphi: X \circlearrowleft$ is an Anosov map and

$\mathcal{N}(T) = \#$ of periodic points of φ of period $\leq T$

then $\mathcal{N}(T) \underset{T \to \infty}{\sim} e^{h_{top}(\varphi) \cdot T}$,

i.e. $\lim\limits_{T \to \infty} \frac{1}{T} \log \mathcal{N}(T) = h_{top}(\varphi)$.

A similar statement holds for Anosov flows

We won't give a proof; see [Katok-Hasselblatt, §18.5]