§4. Hyperbolic dynamics

§4.1. Hyperbolic maps

Here we discuss a general concept of a smooth map being hyperbolic on a set, which can be thought of as a far-reaching generalization of hyperbolic toral automorphisms studied in §3.2.

Let X be a C^∞ manifold and $\varphi : X \to X$ be a C^∞ diffeomorphism.

We can define $\varphi^n : X \to X$ for $n \in \mathbb{Z}$.

Denote by $d\varphi^n$ the differential: for $x \in X$

$$d\varphi^n(x) : T_x X \to T_{\varphi^n(x)} X$$

($T_x X$, $T_{\varphi^n(x)} X$: tangent spaces to X)

Defn Let $K \subset X$ be a compact φ-invariant set (i.e. $\varphi(K) = K$).

We say that φ is <u>hyperbolic</u> on K, or that K is a <u>hyperbolic set</u> for φ, if we can define for each $x \in K$ a <u>stable/unstable decomposition</u>

$$T_x X = E_u(x) \oplus E_s(x)$$

where $E_u(x), E_s(x) \subset T_x X$ are subspaces (of dimensions constant in x)

such that

- E_u, E_s are <u>φ-invariant</u>: $\forall x \in K$
$d\varphi(x) E_u(x) = E_u(\varphi(x))$, $d\varphi(x) E_s(x) = E_s(\varphi(x))$
- $d\varphi^n$ contracts on E_s as $n \to \infty$:
$\exists C > 0$, $0 < \lambda < 1$ s.t. $\forall x \in K$, $\forall n \geq 0$
$\forall v \in E_s(x)$, $|d\varphi^n(x) v| \leq C \lambda^n |v|$

- $d\varphi^{-n}$ contracts on E_u as $n \to \infty$:

$\exists C > 0$, $0 < \lambda < 1$ s.t. $\forall x \in K$, $\forall n \geq 0$

$\forall v \in E_u(x)$, $\boxed{|d\varphi^{-n}(x)v| \leq C\lambda^n |v|}$.

<u>Remarks</u> ① $|\cdot|$ above is defined using any fixed Riemannian metric on X. The constant C depends on the choice of the metric but the constant λ does not.

② Instead of $T: X \to X$ enough to have $T: U \to V$ where $U, V \subset X$ are some open sets containing K.
And we only need T to be C^1.

③ Later we see that E_u, E_s depend continuously on x.
But this dependence is typically <u>not C^∞</u> (not even C^2)

§4.2. EXAMPLES

Example 1: hyperbolic toral automorphisms

$X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$,

$\varphi(x) = Ax \mod \mathbb{Z}^2$

where $A \in SL(2,\mathbb{Z})$ is hyperbolic.

Then φ is <u>hyperbolic</u> <u>on the entire X</u>:

Let λ, λ^{-1} be the eigenvalues of A, with $|\lambda|<1$ (true since A is hyperbolic)

Note that $\lambda \in \mathbb{R}$.

Let $V_\lambda, V_{\lambda^{-1}}$ be the eigenspaces

Note $\dim V_\lambda = \dim V_{\lambda^{-1}} = 1$,

$V_\lambda \oplus V_{\lambda^{-1}} = \mathbb{R}^2$.

Now, it remains to put $\forall x \in \mathbb{T}^2$,

$E_u(x) = V_{\lambda^{-1}}$, $E_s(x) = V_\lambda$

<u>DEFN</u>: $\varphi : X \circlearrowleft$ is an <u>ANOSOV MAP</u> iff the entire X is a hyperbolic set for φ

Example 2: Hyperbolic periodic orbits

Assume $\gamma = \{x_0, x_1, \ldots, x_{r-1}\}$ is a periodic orbit of $\varphi : X \circlearrowleft$ with minimal period r, i.e.

$x_0 \in X$, $\varphi^r(x_0) = x_0$

and $x_0, x_1 = \varphi(x_0), \ldots, x_{r-1} = \varphi^{r-1}(x_0)$ are all distinct.

Turns out that

γ is a hyperbolic set for φ

$\Updownarrow$

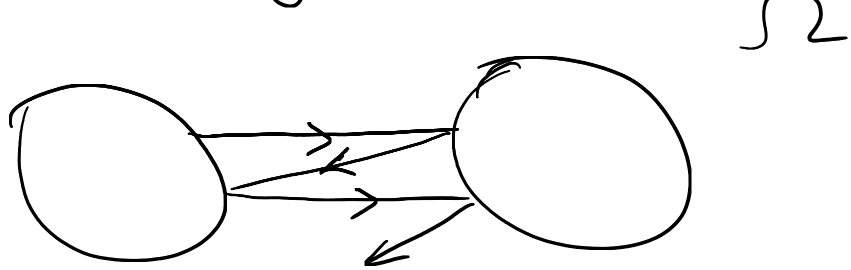
the map $d\varphi^r(x_0) : T_{x_0}X \circlearrowleft$ has no eigenvalues on the unit circle in $\mathbb{C}$.

In this case, γ is called a hyperbolic periodic orbit of φ

Example 3: Billiards

Consider a domain $\Omega \subset \mathbb{R}^2$
(not necessarily bounded)
with C^∞ boundary $\partial\Omega$.
We study the behavior of a particle bouncing off $\partial\Omega$:

e.g.



This can be done using the **billiard ball map**.

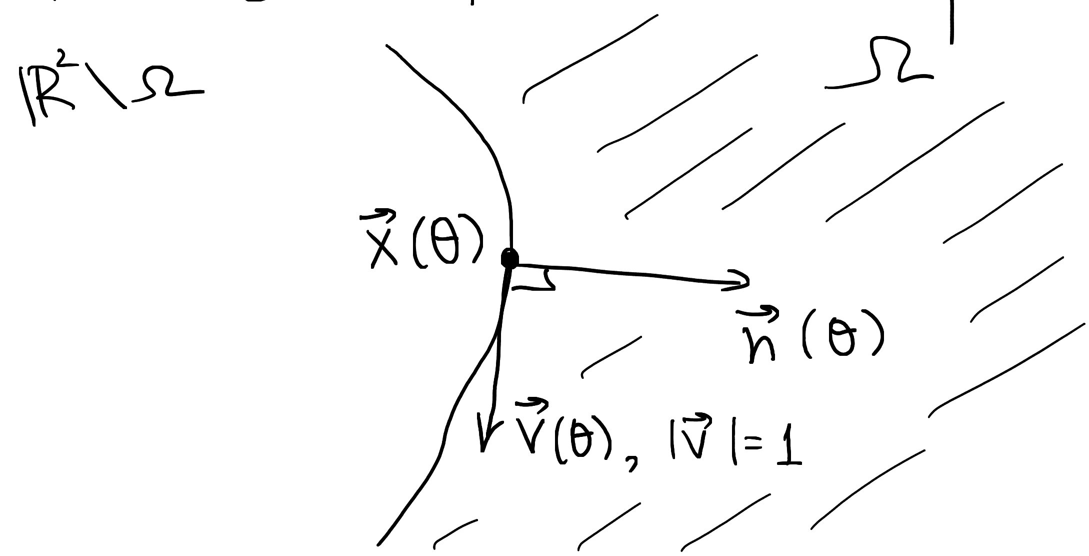
To define it, parametrize $\partial\Omega$ (locally) by $\theta \in \mathbb{R}$ and denote

$\vec{x} : \partial\Omega \to \mathbb{R}^2$ the parametrization map

$\vec{v} := \partial_\theta \vec{x} : \partial\Omega \to \mathbb{R}^2$ the velocity vector

We assume that $|\vec{v}| = 1$ everywhere
(unit speed parametrization)

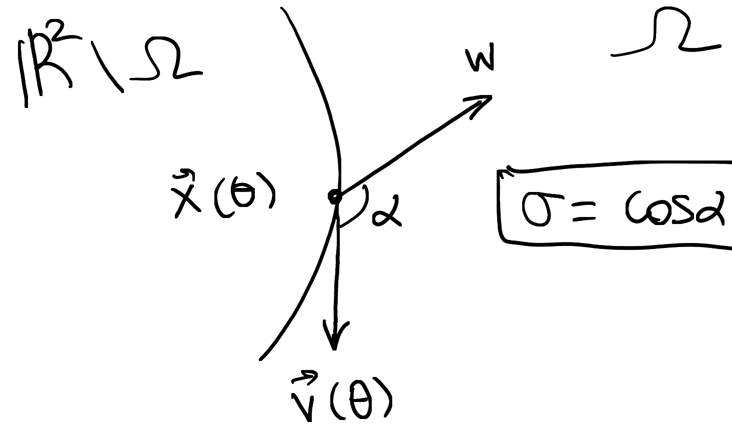
Define the <u>normal vector</u> $\vec{n}(\theta)$ obtained by rotating $\vec{v}(\theta)$ CCW by angle $\pi/2$ and assume that $\vec{n}(\theta)$ points <u>into</u> Ω.



The billiard ball map acts on a subset of the <u>phase space</u>

$$X = \{ (\theta, \vec{w}) \mid \theta \in \partial\Omega,\ \vec{w} \in \mathbb{R}^2,\ |\vec{w}|=1,\ \underbrace{\langle \vec{w}, \vec{n}(\theta) \rangle}_{\mathbb{R}^2 \text{ inner product}} > 0 \}$$

Can identify $X \simeq \partial\Omega \times (-1,1)$ by coordinates (θ, σ) where $\boxed{\sigma = \vec{w} \cdot \vec{v}(\theta)}$



$\sigma = \cos\alpha$

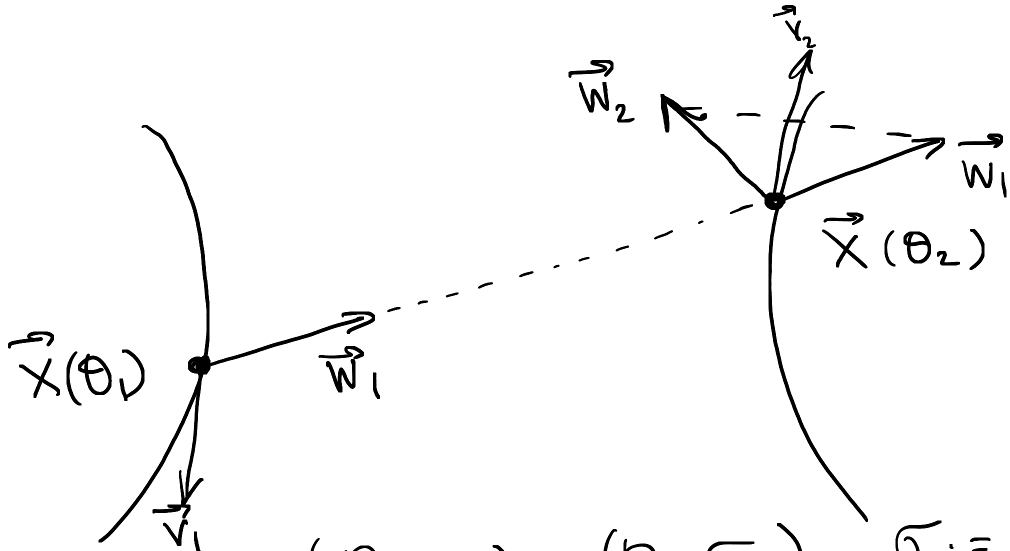
(We do not want to include the glancing case $\sigma = \pm 1$ since it is very difficult to handle.)

Now define the billiard ball map

φ: (open subset of X) $\to$ (open subset of X)

as follows: $\varphi(\theta_1, \sigma_1) = (\theta_2, \sigma_2)$

where the ray $\vec{x}(\theta_1) + t\vec{w}_1$, $t > 0$ intersects $\partial\Omega$ for the first time at the point $\vec{x}(\theta_2)$ and $\vec{w}_2$ is defined by reflecting $\vec{w}_1$ across $\partial\Omega$ at the point θ_2:



$$\varphi(\theta_1, \sigma_1) = (\theta_2, \sigma_2), \quad \sigma_j = \langle \vec{w}_j, \vec{v}_j \rangle$$

To understand φ, it is convenient to introduce the <u>generating function</u>

$$\Phi : \partial\Omega \times \partial\Omega \longrightarrow \mathbb{R},$$

$$\Phi(\theta_1, \theta_2) = |\vec{x}(\theta_1) - \vec{x}(\theta_2)|.$$

Then $\boxed{\varphi(\theta_1, \sigma_1) = (\theta_2, \sigma_2)}$

$\Updownarrow$ (at least "locally")

$$\boxed{\sigma_1 = -\partial_{\theta_1} \Phi(\theta_1, \theta_2), \quad \sigma_2 = \partial_{\theta_2} \Phi(\theta_1, \theta_2)}$$

Indeed, $-\partial_{\theta_1} \Phi(\theta_1, \theta_2) = \left\langle \vec{v}(\theta_1), \dfrac{\vec{x}(\theta_2) - \vec{x}(\theta_1)}{|\vec{x}(\theta_2) - \vec{x}(\theta_1)|} \right\rangle$

$$\partial_{\theta_2} \Phi(\theta_1, \theta_2) = \left\langle \vec{v}(\theta_2), \frac{\vec{x}(\theta_2) - \vec{x}(\theta_1)}{|\vec{x}(\theta_2) - \vec{x}(\theta_1)|} \right\rangle$$

Now, this can be used to compute the differential $d\varphi$ (see e.g. Dyatlov, Notes on Hyperbolic Dynamics, §5.2)

Here we just compute $d\varphi(\theta,\sigma)$ for a trajectory orthogonal to $\partial\Omega$

Note: for general $(\theta_1,\sigma_1)\in X$ and $(\theta_2,\sigma_2)=\varphi(\theta_1,\sigma_1)$, we can differentiate the equations

$$\sigma_1=-\partial_{\theta_1}\Phi(\theta_1,\theta_2(\theta_1,\sigma_1))$$

$$\sigma_2(\theta_1,\sigma_1)=\partial_{\theta_2}\Phi(\theta_1,\theta_2(\theta_1,\sigma_1))$$

in θ_1 and σ_1 to get

$$0=-\partial^2_{\theta_1}\Phi-\partial_{\theta_1\theta_2}\Phi\cdot\partial_{\theta_1}\theta_2$$

$$1=-\partial_{\theta_1\theta_2}\Phi\cdot\partial_{\sigma_1}\theta_2$$

$$\partial_{\theta_1}\sigma_2=\partial_{\theta_1\theta_2}\Phi+\partial^2_{\theta_2}\Phi\cdot\partial_{\theta_1}\theta_2$$

$$\partial_{\sigma_1}\sigma_2=\partial^2_{\theta_2}\Phi\cdot\partial_{\sigma_1}\theta_2.$$

From here we get

$$\partial_{\sigma_1}\theta_2 = -\frac{1}{\partial_{\theta_1\theta_2}\Phi}, \quad \partial_{\theta_1}\theta_2 = -\frac{\partial^2_{\theta_1}\Phi}{\partial_{\theta_1\theta_2}\Phi},$$

$$\partial_{\theta_1}\sigma_2 = \partial_{\theta_1\theta_2}\Phi - \frac{\partial^2_{\theta_1}\Phi \cdot \partial^2_{\theta_2}\Phi}{\partial_{\theta_1\theta_2}\Phi}$$

$$\partial_{\sigma_1}\sigma_2 = -\frac{\partial^2_{\theta_2}\Phi}{\partial_{\theta_1\theta_2}\Phi}.$$

Thus we have written down

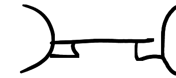
$$d\varphi(\theta_1,\sigma_1) = \begin{pmatrix} \partial_{\theta_1}\theta_2 & \partial_{\sigma_1}\theta_2 \\ \partial_{\theta_1}\sigma_2 & \partial_{\sigma_1}\sigma_2 \end{pmatrix}$$

in terms of the second derivatives of Φ at (θ_1,θ_2).

<u>Note:</u> we always have

$$\boxed{\det d\varphi(\theta_1,\sigma_1) = 1.}$$

Now let us compute the 2nd derivatives of Φ when $\sigma_1 = \sigma_2 = 0$: (trajectory orthogonal to $\partial\Omega$)



<u>1st derivatives:</u>

$$\partial_{\theta_1}\Phi(\theta_1,\theta_2) = \frac{\langle \vec{x}(\theta_1) - \vec{x}(\theta_2), \vec{v}(\theta_1)\rangle}{\Phi(\theta_1,\theta_2)}$$

$$\partial_{\theta_2}\Phi(\theta_1,\theta_2) = \frac{\langle \vec{x}(\theta_2) - \vec{x}(\theta_1), \vec{v}(\theta_2)\rangle}{\Phi(\theta_1,\theta_2)}$$

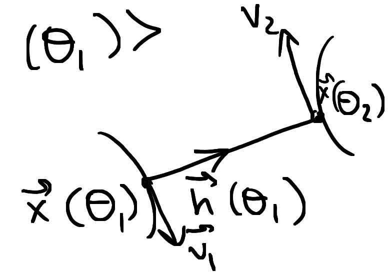
We assume that $\sigma_1 = \sigma_2 = 0$, i.e.

$$\boxed{\vec{x}(\theta_1) - \vec{x}(\theta_2) \perp \vec{v}(\theta_1), \vec{v}(\theta_2).} \qquad (*)$$

Differentiate again & then use $(*)$ to get

$$\partial^2_{\theta_1}\Phi(\theta_1,\theta_2) = \frac{\langle \vec{v}(\theta_1), \vec{v}(\theta_1)\rangle}{\Phi(\theta_1,\theta_2)} + \frac{\langle \vec{x}(\theta_1) - \vec{x}(\theta_2), \partial_\theta\vec{v}(\theta_1)\rangle}{\Phi(\theta_1,\theta_2)}$$

$$= \frac{1}{\Phi} - \langle \vec{n}(\theta_1), \partial_\theta \vec{v}(\theta_1)\rangle$$



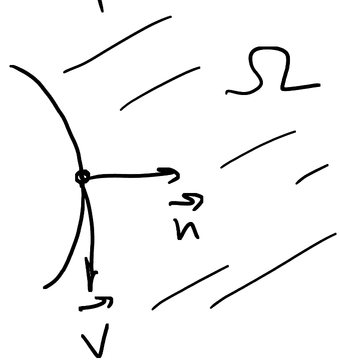
since $\vec{n}(\theta_1) = \dfrac{\vec{x}(\theta_2) - \vec{x}(\theta_1)}{\Phi(\theta_1,\theta_2)}$

Similarly $\partial^2_{\theta_2}\Phi(\theta_1,\theta_2) = \dfrac{1}{\Phi} - \langle \vec{n}(\theta_2), \partial_\theta\vec{v}(\theta_2)\rangle$

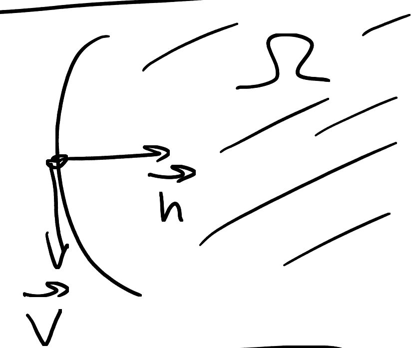
And $\partial_{\theta_1\theta_2}\Phi(\theta_1,\theta_2) = -\dfrac{\langle \vec{v}(\theta_1), \vec{v}(\theta_2)\rangle}{\Phi} = \dfrac{1}{\Phi}.$

We now introduce the <u>curvature</u> of $\partial\Omega$, $K(\theta)$, by putting

$$\boxed{\partial_\theta \vec{v}(\theta) = K(\theta)\vec{n}(\theta)}$$



$\boxed{K<0}$



$\boxed{K>0}$

Then (assuming $\sigma_1 = \sigma_2 = 0$
i.e. $\partial_{\theta_1}\Phi(\theta_1,\theta_2) = \partial_{\theta_2}\Phi(\theta_1,\theta_2) = 0$)

$$\partial^2_{\theta_1}\Phi(\theta_1,\theta_2) = \frac{1}{\Phi} - K(\theta_1)$$

$$\partial^2_{\theta_2}\Phi(\theta_1,\theta_2) = \frac{1}{\Phi} - K(\theta_2)$$

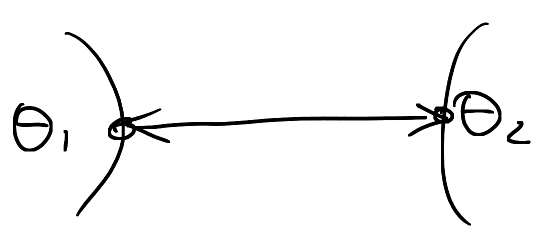
$$\partial_{\theta_1,\theta_2}\Phi = \frac{1}{\Phi}$$

This finally gives
(again, when $\sigma_1 = \sigma_2 = 0$)

$$d\varphi(\theta_1, \sigma_1) = \begin{pmatrix} K_1 \Phi - 1 & -\Phi \\ K_1 + K_2 - K_1 K_2 \Phi & K_2 \Phi - 1 \end{pmatrix}$$

where $K_j = K(\theta_j)$ are the curvatures of $\partial\Omega$ at $\vec{x}(\theta_1)$, $\vec{x}(\theta_2)$
and $\Phi = |\vec{x}(\theta_1) - \vec{x}(\theta_2)|$
is the distance between bounces.

But now, if $\sigma_1 = \sigma_2 = 0$, i.e.
$\varphi(\theta_1, 0) = (\theta_2, 0)$, then
$\varphi(\theta_2, 0) = (\theta_1, 0)$ as well
so $(\theta_1, 0)$ and $(\theta_2, 0)$ form
a <u>period 2 periodic trajectory</u>
for the billiard
ball map φ:



When is this trajectory hyperbolic?

Need to compute

$$A = d\varphi^2(\theta_1, 0) = d\varphi(\theta_2, 0)\, d\varphi(\theta_1, 0)$$

where

$$d\varphi(\theta_1, 0) = \begin{pmatrix} K_1\Phi - 1 & -\Phi \\ K_1 + K_2 - K_1K_2\Phi & K_2\Phi - 1 \end{pmatrix}$$

$$d\varphi(\theta_2, 0) = \begin{pmatrix} K_2\Phi - 1 & -\Phi \\ K_1 + K_2 - K_1K_2\Phi & K_1\Phi - 1 \end{pmatrix}.$$

We have $\boxed{\det A = 1}$

So (see Lemma in §3.2)

the trajectory is hyperbolic $\Leftrightarrow$ $\boxed{|\operatorname{tr} A| > 2}$

We compute

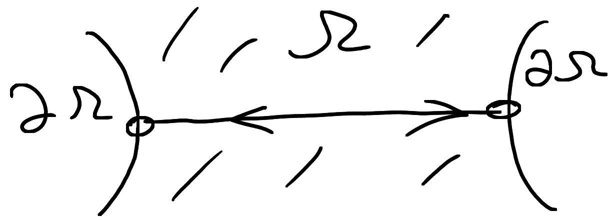
$$A = \begin{pmatrix} 1 + 2\Phi(K_1K_2\Phi - K_1 - K_2) & \text{sth.} \\ \text{sth.} & 1 + 2\Phi(K_1K_2\Phi - K_1 - K_2) \end{pmatrix}$$

So the period 2 orbit is hyperbolic $\iff |\operatorname{tr} A| > 2$

$\iff |1 + 2\Phi(K_1 K_2 \Phi - K_1 - K_2)| > 1.$

$\iff \boxed{(1 - K_1\Phi)(1 - K_2\Phi) \notin [0,1]}$

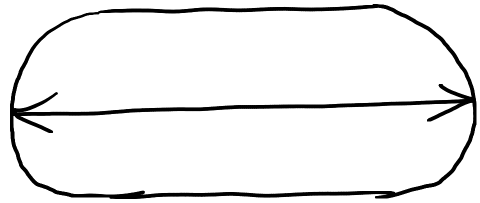
If the boundary is strictly concave, then $K_1, K_2 < 0$ & the orbit is hyperbolic:



In the opposite case, assume for simplicity $\boxed{K_1 = K_2 = 1}$

Then the orbit is hyperbolic $\iff \Phi > 2$.

not hyperbolic



is hyperbolic
(Bunimovich stadium)

§ 4.3. Hyperbolic flows

Assume now that

- X is a manifold
- V a C^∞ vector field on X
- $\varphi^t = e^{tV} : X \circlearrowleft$ the <u>flow</u> of V.
- Assume also that φ^t has no fixed points: $\forall x \in X$, $V(x) \neq 0$.

Defn. Let $K \subset X$ be a compact φ^t-invariant set (i.e. $\varphi^t(K) = K$ $\forall t \in \mathbb{R}$)
We say φ^t is <u>hyperbolic</u> on K, if for each $x \in K$ we have the

<u>flow/stable/unstable decomposition</u>

$$T_x X = E_0(x) \oplus E_u(x) \oplus E_s(x)$$

where $E_0(x) = \mathbb{R}\, V(x)$ (the flow direction)
and $E_u(x)$, $E_s(x)$ are subspaces of $T_x X$ (of constant dimension) and:

- E_u, E_s are φ-invariant: for all $x \in K$ and $t \in \mathbb{R}$

$$d\varphi^t(x) E_u(x) = E_u(\varphi^t(x))$$
$$d\varphi^t(x) E_s(x) = E_s(\varphi^t(x))$$

- $d\varphi^t$ contracts on E_s as $t \to \infty$ and on E_u as $t \to -\infty$:

$\exists C > 0, \ \theta > 0$ s.t. $\forall x \in K, v \in T_x M$

$$|d\varphi^t(x) v| \le C e^{-\theta|t|} |v|, \begin{cases} v \in E_s(x), & t \ge 0 \\ v \in E_u(x), & t \le 0. \end{cases}$$

(same 3 Remarks as for hyperbolic maps

Hyperbolic closed ~~orbit~~ T: $x_0 \in X$, $\varphi^T(x_0) = x_0$ for some $T > 0$

$\gamma = \{ \varphi^t(x_0) \mid 0 \le t \le T \}$ is a hyperbolic set for $\varphi^t \iff d\varphi^T(x_0): T_{x_0}X \to T_{x_0}X$ has a simple eigenvalue 1 and no other eigenvalues on the unit circle.

DEFN: φ^t is an ANOSOV FLOW iff the whole X is a hyperbolic set for φ^t.

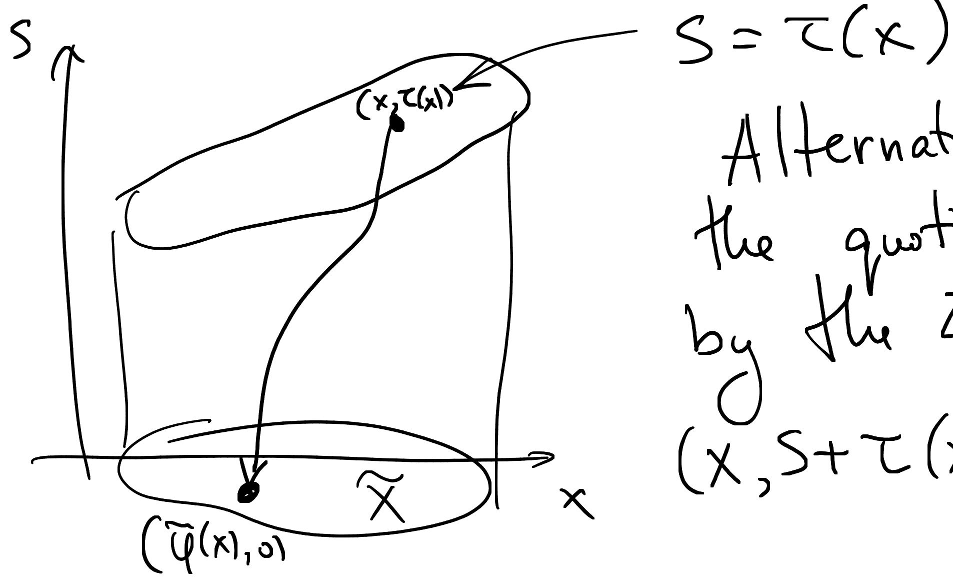
Another example: Suspensions

Assume $\tilde{X}$ is a manifold and $\tilde{\varphi}: \tilde{X} \to \tilde{X}$ is a diffeomorphism. Fix also a C^∞ function (called the roof function)

$$\tau: \tilde{X} \to (0, \infty).$$

Define X as the manifold obtained from the cylinder $\{(x,s) \mid x \in \tilde{X}, 0 \leq s \leq \tau(x)\}$ by gluing the 2 ends by the rule

$$(x, \tau(x)) \sim (\tilde{\varphi}(x), 0):$$



Alternatively, X is the quotient of $\tilde{X}_x \times \mathbb{R}_s$ by the $\mathbb{Z}$-action of the map

$$(x, s+\tau(x)) \mapsto (\tilde{\varphi}(x), s)$$

Define now $V = \partial_s$, a vector field on X. Then it's flow is $\varphi^t(x,s) = (x, s+t)$.

Thm. Assume that $\tilde{K} \subset \tilde{X}$ is a hyperbolic set for $\tilde{\varphi}$. Then $K := \{(x,s) \mid x \in \tilde{K}\} \subset X$ is a hyperbolic set for φ^t.

Proof. If τ is constant, e.g. $\tau \equiv 1$, this is straightforward: if $\tilde{E}_u, \tilde{E}_s$ are the unstable/stable spaces of $\tilde{\varphi}$, then $\forall x \in K$ put

$$E_u(x,s) = \{(v,0) \mid v \in \tilde{E}_u(x)\}$$
$$E_s(x,s) = \{(v,0) \mid v \in \tilde{E}_s(x)\}$$
$$E_0(x,s) = \mathbb{R}\partial_s$$

and these give the flow/unstable/stable decomposition for φ^t.

In general need more work with cones, see e.g. [KH, Proposition 17.4.5] $\square$

In particular, if
$\tilde{X}$ is a compact manifold and
$\tilde{\varphi}: \tilde{X} \to \tilde{X}$ is an Anosov map,
then $\forall$ roof fn. $\tau \in C^\infty(\tilde{X}; (0, \infty))$
the suspended flow $\varphi^t: X \to X$
is an Anosov flow.