

# A perturbative approximation for the nonlinear Schrödinger equation

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**Presentation of the problem and statement of result.** Our goal is to approximate the flow of the nonlinear Schrödinger equation with a small delta function potential, which may be either attractive or repulsive,

$$i\partial_t u + \frac{1}{2}\partial_x^2 u + u|u|^2 - q\delta_0(x)u = 0,$$

by the free nonlinear flow

$$i\partial_t u_1 + \frac{1}{2}\partial_x^2 u_1 + u_1|u_1|^2 = 0.$$

We will assume the initial condition is the same in each case and is given:

$$u(x, 0) = u_1(x, 0) = u_0(x).$$

We will do this by first estimating the  $H_x^1$  norm of the difference  $w(x, t) \stackrel{\text{def}}{=} u(x, t) - u_1(x, t)$  as a function of time. An iteration of our estimate will allow us to conclude that for any  $\delta > 0$ ,

$$\|w\|_{L_{[0, mT]}^\infty H_x^1} \leq |q|^{1-\delta} \quad \text{for} \quad m \leq \delta \log_A(1/|q|).$$

Here  $T$  is a small positive number, denoting a unit-sized time interval, and the base of the logarithm,  $A$ , is a parameter dependent on  $u_1$  and on  $T$ . I would like to thank Justin Holmer and Maciej Zworski for suggesting this problem and for helping me solve it.

**Differential and integral equations for the error.** We begin by writing a differential equation for  $w$  in terms of  $u_1$ :

$$\begin{aligned} i\partial_t w + \frac{1}{2}\partial_x^2 w &= u_1|u_1|^2 - u|u|^2 + q\delta_0(x)u \\ &= u_1|u_1|^2 - (w + u_1)|w + u_1|^2 + q\delta_0(x)(w + u_1) \\ &= -w|w|^2 - 2u_1|w|^2 - \overline{u_1}w^2 - 2|u_1|^2 w - u_1^2 \overline{w} + q\delta_0(x)w + q\delta_0(x)u_1. \end{aligned} \quad (1)$$

We rewrite this as an integral equation, using the free linear Schrödinger propagator, regarding the right hand side as a forcing term. Although  $w(x, 0) = 0$ , we include a term for the initial condition, because when we iterate our estimate this will be nonzero.

$$\begin{aligned} w(x, t) &= e^{it\partial_x^2/2}w(x, t_0) \\ &+ \int_{t_0}^t e^{i(t-t')\partial_x^2/2} \left( -2|u_1|^2 w - u_1^2 \overline{w} - 2u_1|w|^2 - \overline{u_1}w^2 - w|w|^2 + q\delta_0(x)w + q\delta_0(x)u_1 \right) (t') dt'. \end{aligned} \quad (2)$$

**A Gagliardo-Nirenberg-Sobolev Inequality.** In one dimension, the  $L^\infty$  norm is controlled by the  $H^1$  norm. Indeed, suppose  $\phi \in H^1$ . Then  $\phi$  is absolutely continuous, and

$$\begin{aligned} |\phi(x_0)|^2 &= \frac{1}{2} \left( \int_{-\infty}^{x_0} \frac{d}{dx} |\phi(x)|^2 dx - \int_{x_0}^{\infty} \frac{d}{dx} |\phi(x)|^2 dx \right) \leq \int_{-\infty}^{\infty} |\phi'(x)| |\phi(x)| dx \\ &\leq \|\phi'\|_{L^2} \|\phi\|_{L^2} \leq \|\phi\|_{H^1}^2. \end{aligned}$$

From this we conclude

$$\|\phi\|_{L^\infty} \leq \|\phi\|_{H^1} \quad (3)$$

**An estimate of the error in  $L^2$ .** To obtain this, we will estimate the  $L_t^\infty L_x^2$  norms of the terms in the right hand side of (2), one by one. Here we use  $L_t^\infty$  as shorthand for  $L_{[t_0, t_0+T]}^\infty$ , where  $T$  is a fixed time, the length of each iteration. This will be chosen later in such a way as to make it possible to subtract the undesirable terms from the right hand side of our estimate, and to absorb them into the left hand side.

The initial condition term is straightforward, due to the unitarity of the propagator.

$$\left\| e^{it\partial_x^2/2} w(x, t_0) \right\|_{L_t^\infty L_x^2} = \|w(x, t_0)\|_{L_x^2}$$

The five terms in the integral which do not involve a  $\delta$ -function all follow the same pattern. The distinction between  $w$  and  $\bar{w}$ , and between  $u_1$  and  $\bar{u}_1$ , will not be important here, so we will treat a representative term of the following form, where  $k \in \{1, 2, 3\}$ .

$$m \left\| \int_{t_0}^t e^{i(t-t')\partial_x^2/2} u_1^k(t') w^{3-k}(t') dt' \right\|_{L_t^\infty L_x^2} \leq mT \|u_1^k(x, t') w^{3-k}(x, t')\|_{L_t^\infty L_x^2}.$$

At this point we pull out two of the factors from the integral in  $x$ , estimating them by their  $L_t^\infty L_x^\infty$  norms, and then use our estimate for  $L^\infty$  computed in (3). This allows us to control all three factors by their  $H^1$  norms:

$$\leq mT \|u_1\|_{L_t^\infty H_x^1}^k \|w\|_{L_t^\infty H_x^1}^{3-k}.$$

Finally, the  $\delta$ -function terms cause the most difficulty. They can be controlled either by using a Strichartz estimate, or by using local smoothing. We will follow the latter route, using  $v$  as a stand-in for either  $w$  or  $u_1$ , because the two terms are treated in the same fashion. We will argue by duality, so let  $\phi$  have  $\hat{\phi} \in C_0^\infty(\mathbb{R} \setminus 0)$ , with  $\|\phi\|_{L^2} = 1$ .

$$|q| \int_x \int_{t_0}^t e^{i(t-t')\partial_x^2/2} \delta_0(x) v(0, t') dt' \bar{\phi}(x) dx = |q| \int_{t_0}^t \int_x \left[ e^{i(t-t')\partial_x^2/2} \delta_0 \right] (x) \bar{\phi}(x) dx v(0, t') dt'$$

To justify the interchange of the order of integration, we write out the inner integral explicitly as follows:  $\int_{t_0}^t e^{i(t-t')\partial_x^2/2}\delta_0(x)v(0,t')dt' = c \int_{t_0}^t \frac{v(0,t')}{\sqrt{t-t'}} e^{ix^2/2(t-t')} dt'$ . This is an absolutely and uniformly convergent integral in  $t'$ , because the supremum norm of  $v$  is controlled by its  $H^1$  norm. Meanwhile  $\phi$  guarantees that the integral in  $x$  will be absolutely convergent as well. We now use the unitarity of the propagator:

$$\begin{aligned} &= |q| \int_{t_0}^t \overline{(e^{-i(t-t')\partial_x^2/2}\phi)(0)} v(0,t') dt' \\ &= |q| \int \overline{(e^{-i(t-t')\partial_x^2/2}\phi)(0)} \chi_{[t_0,t]}(t') v(0,t') dt' \\ &= \frac{|q|}{2\pi} \int_{t'} \left( \int_{\xi} e^{i(t-t')\xi^2/2} \overline{\hat{\phi}(\xi)} d\xi \right) \chi_{[t_0,t]}(t') v(0,t') dt'. \end{aligned}$$

Here we use Parseval's theorem, Fourier transforming  $t'$  to  $\tau$ . Then we multiply and divide by  $\tau^{1/4}$ , and apply Cauchy-Schwarz in such a way as to exploit the local smoothing:

$$\begin{aligned} &\leq \left\| \frac{|\tau|^{1/4}|q|}{2\pi} \int_{t'} e^{-it'\tau} \int_{\xi} e^{i(t-t')\xi^2/2} \overline{\hat{\phi}(\xi)} d\xi dt' \right\|_{L^2_{\tau}} \cdot \left\| |\tau|^{-1/4} \int e^{-it'\tau} \chi_{[t_0,t]}(t') v(0,t') dt' \right\|_{L^2_{\tau}} \\ &= \text{I} \quad \cdot \quad \text{II}. \end{aligned}$$

At this point we treat each term separately. For the first term we use a change of variables to re-express the integral in  $\xi$  as an inverse Fourier transform  $u \rightarrow t'$ .

$$\begin{aligned} \text{I} &= \frac{|\tau|^{1/4}|q|}{2\pi} \int_{t'} e^{-it'\tau} \int_{\xi} e^{i(t-t')\xi^2/2} \overline{\hat{\phi}(\xi)} d\xi dt' \\ &= \sum_{\pm} \frac{\pm|\tau|^{1/4}|q|}{2\pi} \int_{t'} e^{-it'\tau} \int_{u=0}^{\infty} e^{i(t-t')u} \overline{\hat{\phi}(\pm\sqrt{2u})} (2u)^{-1/2} du dt'. \end{aligned}$$

We now have an inverse Fourier transform followed by a Fourier transform

$$= \sum_{\pm} \pm|\tau|^{1/4}|q| \mathcal{F}\mathcal{F}^{-1} \left( \chi_{[0,\infty)}(u) \overline{\hat{\phi}(\pm\sqrt{2u})} (2u)^{-1/2} \right).$$

Because of the way we have chosen  $\phi$ , both Fourier transforms take place in Schwartz space. The characteristic function truncates both summands to the region  $\tau > 0$ :

$$= \frac{|q|}{\sqrt{2}} \sum_{\pm} \pm\tau^{-1/4} \chi_{[0,\infty)}(\tau) \overline{\hat{\phi}(\pm\sqrt{2\tau})}.$$

Now when we take the  $L^2$  norm we can use the change of variable  $u = \sqrt{2\tau}$ . The factor of  $\tau^{1/4}$  which we included from the beginning provides us with just the necessary Jacobian. This means we have

$$\text{I} = \left\| \frac{|\tau|^{1/4}|q|}{2\pi} \int_{t'} e^{-it'\tau} \int_{\xi} e^{i(t-t')\xi^2/2} \overline{\hat{\phi}(\xi)} d\xi dt' \right\|_{L^2_{\tau}} \leq c|q| \|\phi\|_{L^2} = c|q|.$$

We now treat the second term using Sobolev embedding. Recall that the Fourier transform of a radial function is radial, and that a distribution which is homogeneous of degree  $a$  has a Fourier transform homogeneous of degree  $-d-a$ . This means that the Fourier transform of

$|\tau|^{-1/4}$  is proportional to  $|t'|^{-3/4}$ . Furthermore the Fourier transform changes multiplication to convolution, so that we have

$$\text{II} = \left\| |\tau|^{-1/4} \int e^{-it'\tau} \chi_{[t_0, t]}(t') v(0, t') dt' \right\|_{L_t^2} = c \left\| |t'|^{-3/4} * \chi_{[t_0, t]}(t') v(0, t') \right\|_{L_{t'}^2}.$$

This term can be treated using a Hardy-Littlewood-Sobolev inequality, which we state here without proof:  $\| |x|^{-3/4} * f \|_{L^2} \leq \| f \|_{L^{4/3}}$ :

$$\leq c \left\| \chi_{[t_0, t]}(t') v(0, t') \right\|_{L_{t'}^{4/3}}.$$

Here we again use the supremum estimate followed by the  $H^1$  estimate.

$$\leq c T^{3/4} \| v \|_{L_t^\infty L_x^\infty} \leq c T^{3/4} \| v \|_{L_t^\infty H_x^1}.$$

Combining these two terms gives us an estimate by duality on the  $L^2$  norm we seek:

$$|q| \left\| \int_{t_0}^t e^{i(t-t')\partial_x^2/2} \delta_0(x) v(0, t') dt' \right\|_{L_t^\infty L_x^2} \leq \text{I} \cdot \text{II} \leq c |q| T^{3/4} \| v \|_{L_t^\infty H_x^1}.$$

Finally, we combine our estimates for each of the individual terms.

$$\begin{aligned} \| w(x, t) \|_{L_t^\infty L_x^2} &\leq \| w(x, t_0) \|_{L_x^2} + 3T \| u_1 \|_{L_t^\infty H_x^1}^2 \| w \|_{L_t^\infty H_x^1} + 3T \| u_1 \|_{L_t^\infty H_x^1} \| w \|_{L_t^\infty H_x^1}^2 + T \| w \|_{L_t^\infty H_x^1}^3 \\ &\quad + c |q| T^{3/4} \| w \|_{L_t^\infty H_x^1} + c |q| T^{3/4} \| u_1 \|_{L_t^\infty H_x^1}. \end{aligned} \quad (4)$$

All the terms but the first can be made small by choosing  $T$  small. The first term tracks the contribution of the initial condition to the growth of  $w$ .

**An estimate of the error in  $H^1$ .** We use energy techniques to deduce the  $H^1$  estimate from the  $L^2$  estimate, reviewing first how the conserved energy is obtained for the free nonlinear Schrödinger equation:

$$\begin{aligned} 0 &= i \partial_t u_1 + \frac{1}{2} \partial_x^2 u_1 + u_1 |u_1|^2 \\ 0 &= \int_x i \partial_t u_1 \partial_t \bar{u}_1 + \frac{1}{2} \partial_x^2 u_1 \partial_t \bar{u}_1 + u_1 |u_1|^2 \partial_t \bar{u}_1 dx. \end{aligned}$$

We take the real part and integrate by parts in the second term.

$$0 = 0 - \frac{1}{4} \partial_t \int |\partial_x u_1|^2 dx + \frac{1}{4} \partial_t \int |u_1|^4 dx.$$

This shows us that

$$E(t) \stackrel{\text{def}}{=} \int |\partial_x u_1|^2 dx - \int |u_1|^4 dx = \text{const.}$$

For  $w$  we will not have conserved energy, but we will be able to control the rate at which the energy increases. We use the same technique, multiplying the differential equation (1)

by  $\partial_t \bar{w}$  and integrating in  $x$ :

$$\int_x \left( i \partial_t w + \frac{1}{2} \partial_x^2 w + w|w|^2 + 2u_1|w|^2 + \bar{u}_1 w^2 + 2|u_1|^2 w + u_1^2 \bar{w} - q \delta_0(x) w - q \delta_0(x) u_1 \right) \partial_t \bar{w} dx = 0.$$

As before we proceed term by term, keeping only the real part. In each case, our goal is to put all the factors of  $w$  under the time derivative; we will ultimately integrate in  $t$ , and remove those derivatives by integrating by parts. The first three terms follow the free case:

$$\begin{aligned} \operatorname{Re} \left( i \int_x |\partial_t w|^2 dx \right) &= 0 \\ \frac{1}{2} \operatorname{Re} \int \partial_x^2 w \partial_t \bar{w} dx &= \frac{1}{4} \partial_t \int |\partial_x w|^2 dx \\ \operatorname{Re} \int w|w|^2 \partial_t \bar{w} dx &= \frac{1}{4} \partial_t \int |w|^4 dx. \end{aligned}$$

We treat the quadratic terms together:

$$\begin{aligned} \operatorname{Re} \int (2u_1|w|^2 + \bar{u}_1 w^2) \partial_t \bar{w} dx &= \operatorname{Re} \int (u_1|w|^2 + \bar{u}_1 w^2/2 + \bar{u}_1|w|^2 + u_1 \bar{w}^2/2) \partial_t \bar{w} dx \\ &= \frac{1}{2} \int u_1 \partial_t (|w|^2 \bar{w}) + \bar{u}_1 \partial_t (|w|^2 \bar{w}) dx \\ &= \int (\operatorname{Re} u_1) \partial_t (|w|^2 \bar{w}) dx. \end{aligned}$$

The linear terms we treat one by one:

$$\begin{aligned} \operatorname{Re} \int 2|u_1|^2 w \partial_t \bar{w} dx &= \int |u_1|^2 \partial_t |w|^2 dx \\ \operatorname{Re} \int u_1^2 \bar{w} \partial_t \bar{w} dx &= \frac{1}{2} \operatorname{Re} \int u_1^2 \partial_t (\bar{w}^2) dx. \end{aligned}$$

Finally we treat the delta terms:

$$\begin{aligned} -q \operatorname{Re} \int \delta_0(x) w \partial_t \bar{w} dx &= \frac{-q}{2} \partial_t |w(0, t)|^2 \\ -q \operatorname{Re} \int \delta_0(x) u_1 \partial_t \bar{w} dx &= -q \operatorname{Re} (u_1(0, t) \partial_t \bar{w}(0, t)). \end{aligned}$$

Now we put everything together,

$$\begin{aligned} \frac{1}{4} \partial_t \int_x |\partial_x w|^2 dx &= - \int_x \left( \frac{1}{4} \partial_t |w|^4 + (\operatorname{Re} u_1) \partial_t (|w|^2 \bar{w}) + \frac{1}{2} \operatorname{Re} (u_1^2 \partial_t (\bar{w}^2)) \right) dx \\ &\quad + \frac{q}{2} \partial_t |w(0, t)|^2 + q \operatorname{Re} (u_1(0, t) \partial_t \bar{w}(0, t)), \end{aligned}$$

and integrate in  $t$ :

$$\begin{aligned} \frac{1}{4} \|\partial_x w\|_{L_x^2}^2(t) &= \frac{1}{4} \|\partial_x w\|_{L_x^2}^2(t_0) - \frac{1}{4} \|w\|_{L_x^4}^4(t) + \frac{1}{4} \|w\|_{L_x^4}^4(t_0) - \int_{t_0}^t \int_x (\operatorname{Re} u_1) \partial_{t'} (|w|^2 \bar{w}) dx dt' \\ &\quad - \operatorname{Re} \frac{1}{2} \int_{t_0}^t \int_x u_1^2 \partial_{t'} (\bar{w}^2) dx dt' + \frac{q}{2} |w(0, t)|^2 - \frac{q}{2} |w(0, t_0)|^2 \\ &\quad + q \operatorname{Re} \int_{t_0}^t u_1(0, t') \partial_{t'} \bar{w}(0, t') dt'. \end{aligned}$$

We wish to take the  $L_t^\infty$  norm, and estimate all the  $w$  terms on the right in terms of  $L_t^\infty H_x^1$ . We proceed as usual term by term, treating first the terms that require no integration by parts. The first term tracks the contribution of the initial condition to the growth of  $w$ , and we leave it untouched:

$$\|\partial_x w\|_{L_x^2}^2(t_0) \leq \|\partial_x w\|_{L_x^2}^2(t_0).$$

These terms will be small because they are quartic in  $w$ :

$$\|w\|_{L_x^4}^4(t_0) - \|w\|_{L_x^4}^4(t) = 2 \|w\|_{L_{t,x}^\infty}^2 \|w\|_{L_t^\infty L_x^2}^2 \leq 2 \|w\|_{L_t^\infty H_x^1}^4.$$

Finally these terms are small because  $|q|$  is small:

$$q |w(0, t)|^2 - q |w(0, t_0)|^2 \leq 2|q| \|w\|_{L_t^\infty H_x^1}^2.$$

We now treat the first of the remaining three terms, using integration by parts.

$$\begin{aligned} \int_{t_0}^t \int_x (\operatorname{Re} u_1) \partial_{t'} (|w|^2 \bar{w}) dx dt' &= \int_x (\operatorname{Re} u_1) (|w|^2 \bar{w}) dx \Big|_{t'=t_0}^t - \int_{t_0}^t \int_x |w|^2 \bar{w} \partial_{t'} (\operatorname{Re} u_1) dx dt' \\ &\leq \int_x (\operatorname{Re} u_1) (|w|^2 \bar{w}) dx \Big|_{t'=t_0}^t + T \|\partial_t u_1\|_{L_{t,x}^\infty} \|w\|_{L_t^\infty H_x^1}^3. \end{aligned}$$

The boundary term will be small because it is cubic:

$$\int_x (\operatorname{Re} u_1) (|w|^2 \bar{w}) dx \Big|_{t'=t_0}^t \leq \|u_1\|_{L_{t,x}^\infty} \|w\|_{L_t^\infty L_x^2}^3.$$

For the next term we will exploit the fact that it is an  $L^2$  norm in  $x$  which appears, and not an  $H^1$  norm. This allows us to apply the  $L^2$  estimate derived in the previous section.

$$\begin{aligned} \operatorname{Re} \int_{t_0}^t \int_x u_1^2 \partial_{t'} (\bar{w}^2) dx dt' &= \operatorname{Re} \int_x u_1^2 \bar{w}^2 dx \Big|_{t'=t_0}^t - \operatorname{Re} \int_{t_0}^t \int_x \partial_{t'} (u_1^2) \bar{w}^2 dx dt' \\ &\leq \left( \|u_1\|_{L_{t,x}^\infty}^2 + T \|\partial_t u_1\|_{L_{t,x}^\infty}^2 \right) \|w\|_{L_t^\infty L_x^2}^2. \end{aligned}$$

The last term will be small because  $|q|$  is small.

$$q \operatorname{Re} \int_{t_0}^t u_1(0, t') \partial_{t'} \bar{w}(0, t') dt' \leq |q| \left( \|u_1\|_{L_{t,x}^\infty} + T \|\partial_t u_1\|_{L_{t,x}^\infty} \right) \|w\|_{L_{t,x}^\infty}.$$

All this put together gives us the following estimate for  $\partial_x w$ :

$$\begin{aligned} \|\partial_x w\|_{L_t^\infty L_x^2}^2 &\leq \|\partial_x w\|_{L_x^2}^2(t_0) + 2\|w\|_{L_t^\infty H_x^1}^4 + 2|q|\|w\|_{L_t^\infty H_x^1}^2 + \left(\|u_1\|_{L_{t,x}^\infty} + T\|\partial_t u_1\|_{L_{t,x}^\infty}\right) \|w\|_{L_t^\infty L_x^2}^3 \\ &\quad + \left(\|u_1\|_{L_{t,x}^\infty}^2 + T\|\partial_t u_1\|_{L_{t,x}^\infty}^2\right) \|w\|_{L_t^\infty L_x^2}^2 + |q| \left(\|u_1\|_{L_{t,x}^\infty} + T\|\partial_t u_1\|_{L_{t,x}^\infty}\right) \|w\|_{L_{t,x}^\infty}. \end{aligned}$$

We combine this with the  $L^2$  estimate (4), which we repeat here for the reader's convenience:

$$\begin{aligned} \|w\|_{L_t^\infty L_x^2} &\leq \|w\|_{L_x^2}(t_0) + 3T\|u_1\|_{L_t^\infty H_x^1}^2 \|w\|_{L_t^\infty H_x^1} + 3T\|u_1\|_{L_t^\infty H_x^1} \|w\|_{L_t^\infty H_x^1}^2 + T\|w\|_{L_t^\infty H_x^1}^3 \\ &\quad + c|q|T^{3/4}\|w\|_{L_t^\infty H_x^1} + c|q|T^{3/4}\|u_1\|_{L_t^\infty H_x^1}. \end{aligned}$$

Added together these give, using  $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ ,

$$\begin{aligned} \|w\|_{L_t^\infty H_x^1} &\leq \|w\|_{H_x^1}(t_0) + 3T\|u_1\|_{L_t^\infty H_x^1}^2 \|w\|_{L_t^\infty H_x^1} + 3T\|u_1\|_{L_t^\infty H_x^1} \|w\|_{L_t^\infty H_x^1}^2 + T\|w\|_{L_t^\infty H_x^1}^3 \\ &\quad + c|q|T^{3/4}\|w\|_{L_t^\infty H_x^1} + c|q|T^{3/4}\|u_1\|_{L_t^\infty H_x^1} + \sqrt{2}\|w\|_{L_t^\infty H_x^1}^2 + \sqrt{2|q|}\|w\|_{L_t^\infty H_x^1} \\ &\quad + \sqrt{\|u_1\|_{L_{t,x}^\infty} + T\|\partial_t u_1\|_{L_{t,x}^\infty}} \|w\|_{L_t^\infty H_x^1}^{3/2} + \left(\|u_1\|_{L_{t,x}^\infty} + \sqrt{T}\|\partial_t u_1\|_{L_{t,x}^\infty}\right) \|w\|_{L_t^\infty L_x^2} \\ &\quad + \sqrt{|q|}\sqrt{\|u_1\|_{L_{t,x}^\infty} + T\|\partial_t u_1\|_{L_{t,x}^\infty}} \|w\|_{L_{t,x}^\infty}^{1/2}. \end{aligned}$$

We now plug the  $L^2$  estimate (4) into the next to last term, and introduce the notation  $M_u \stackrel{\text{def}}{=} \max\left(\|u_1\|_{L_{t,x}^\infty} + \sqrt{T}\|\partial_t u_1\|_{L_{t,x}^\infty}, \sqrt{\|u_1\|_{L_{t,x}^\infty} + T\|\partial_t u_1\|_{L_{t,x}^\infty}}\right)$  in order to simplify the expression:

$$\begin{aligned} &\leq \|w\|_{H_x^1}(t_0) + 3T\|u_1\|_{L_t^\infty H_x^1}^2 \|w\|_{L_t^\infty H_x^1} + 3T\|u_1\|_{L_t^\infty H_x^1} \|w\|_{L_t^\infty H_x^1}^2 + T\|w\|_{L_t^\infty H_x^1}^3 \\ &\quad + c|q|T^{3/4}\|w\|_{L_t^\infty H_x^1} + c|q|T^{3/4}\|u_1\|_{L_t^\infty H_x^1} + \sqrt{2}\|w\|_{L_t^\infty H_x^1}^2 + \sqrt{2|q|}\|w\|_{L_t^\infty H_x^1} \\ &\quad + M_u \left[ \|w\|_{L_t^\infty H_x^1}^{3/2} + \|w\|_{L_x^2}(t_0) + 3T\|u_1\|_{L_t^\infty H_x^1}^2 \|w\|_{L_t^\infty H_x^1} + 3T\|u_1\|_{L_t^\infty H_x^1} \|w\|_{L_t^\infty H_x^1}^2 \right. \\ &\quad \left. + T\|w\|_{L_t^\infty H_x^1}^3 + c|q|T^{3/4}\|w\|_{L_t^\infty H_x^1} + c|q|T^{3/4}\|u_1\|_{L_t^\infty H_x^1} + \sqrt{|q|}\|w\|_{L_{t,x}^\infty}^{1/2} \right] \end{aligned}$$

We now choose  $|q|$  and  $T$  sufficiently small (both will depend on  $u_1$  and on  $c$ , but not on  $w$  in any way) that

$$\begin{aligned} \|w\|_{L_t^\infty H_x^1} &\leq (1 + M_u)\|w\|_{H_x^1}(t_0) + |q|M_u\|w\|_{L_t^\infty H_x^1}^{1/2} + \frac{1}{10}\|w\|_{L_t^\infty H_x^1} + M_u\|w\|_{L_t^\infty H_x^1}^{3/2} + \sqrt{2}\|w\|_{L_t^\infty H_x^1}^2 \\ &\quad + \frac{1}{10}\|w\|_{L_t^\infty H_x^1}^3 + c|q|T^{3/4}\|u_1\|_{L_t^\infty H_x^1}. \end{aligned}$$

Further shrinking  $T$  and  $|q|$ , we will be able to absorb all the  $\|w\|_{L_t^\infty H_x^1}$  terms into the left hand side, and obtain

$$\|w\|_{L_t^\infty H_x^1} \leq 2(1 + M_u)\|w\|_{H_x^1}(t_0) + 2c|q|T^{3/4}\|u_1\|_{L_t^\infty H_x^1}.$$

Iterating this with a starting point  $t_0 = 0$  (recall from the statement of the problem that  $\|w\|_{H_x^1}(0) = 0$ ) gives

$$\begin{aligned} \|w\|_{L_{[0,mT]}^\infty H_x^1} &\leq \left[(2 + 2M_u)^{m+1} + \dots + (2 + 2M_u)\right] c|q|T^{3/4}\|u_1\|_{L_t^\infty H_x^1} \\ &\leq ((2 + 2M_u)^{m+2} - (2 + 2M_u))c|q|T^{3/4}\|u_1\|_{L_t^\infty H_x^1}, \end{aligned}$$

for any integer  $m$  such that the right hand side does not exceed  $\min \left[ \frac{1}{(10M_u)^2}, \frac{1}{10\sqrt{2}} \right]$ . Another way to put this is that

$$\|w\|_{L^\infty_{[0,mT]} H_x^1} \leq A^m |q|,$$

where  $A$  is a constant depending on  $c, u_1$ , and  $T$ . In order to have

$$\|w\|_{L^\infty_{[0,mT]} H_x^1} \leq |q|^{1-\delta},$$

we need

$$m \leq \delta \log_A(1/|q|).$$

This concludes the proof.