MANY CHEERFUL FACTS

presents

Why Can't We Count the Reals?

a talk by Kenny Easwaran

1:10 pm - 2:00 on Wednesday, November 16th, in room 1015.

We all know (ever since Cantor) that there are more reals than there are natural numbers, so the reals are said to be uncountable. If we remember a bit more, we'll recall that there are in fact lots and lots of uncountable sets that are different sizes. We call each such size a cardinality. Given the Axiom of Choice, we can always compare whether one set is bigger than another, so we can order all the infinite cardinalities, starting with $\aleph_0$ as the smallest one (the number of natural numbers), and continuing with $\aleph_1$, $\aleph_2$, and so on. (We have to use some set-theoretic trickery to number them all, because we run out of natural numbers pretty quickly, but never run out of infinite cardinalities).

Once Cantor discovered all this, he conjectured the Continuum Hypothesis, namely that the number of reals is $\aleph_1$. This problem was considered important enough that Hilbert made it the first of his 23 problems in 1900. But in 1963, Paul Cohen invented the method of forcing, and was able to show that if the standard axioms (ZFC) of set theory are consistent, then it has models where there are exactly $\aleph_\alpha$ reals for almost any $\alpha > 0$, and thus CH is undecidable from ZFC.

In this talk I will describe the method of forcing and show how to construct the relevant models of set theory. Along the way I'll also (cheerfully) explain what logicians mean when they talk about Boolean algebras, ultrafilters, and models of set theory. You may find it less than cheerful to hear how set theorists abuse your notational friends 'ℕ' and 'ℝ' though.

I am the very model of a modern Major General,
I've information vegetable, animal, and mineral,
I know the kings of England, and I quote the fights historical
From Marathon to Waterloo, in order categorical;
I'm very well acquainted, too, with matters mathematical,
I understand equations, both the simple and quadratical,
About binomial theorem I'm teeming with a lot o' news,
With many cheerful facts about the square of the hypotenuse!

- Gilbert & Sullivan $P \circ P$