Math 55, **First Midterm Exam**
SOLUTIONS

(1) This problem is # 31 in Section 1.5 on page 67 of Rosen's book. We use the methods explained in the subsection *Negating Nested Quantifiers* on pages 63-64 to solve these.

(a) The negation is obtained by switching the quantifiers: $\exists x \forall y \exists z \neg T(x,y,z)$.
(b) Negation takes disjunction into conjunction: $\exists x \forall y \neg P(x,y) \wedge \exists x \forall y \neg Q(x,y)$.
(c) Conjunction becomes disjunction: $\exists x \forall y \left(\neg P(x,y) \vee \forall z \neg R(x,y,z)\right)$.
(d) We use that $p \to q$ means $\neg p \vee q$ to find the negation $\exists x \forall y\, (P(x,y) \wedge \neg Q(x,y))$.

(2) We use the method described in the proof of the Chinese Remainder Theorem. We set $m_1 = 7$, $m_2 = 8$, and $m_3 = 9$, as well as $M_1 = 8 \cdot 9 = 72$, $M_2 = 7 \cdot 9 = 63$ and $M_3 = 7 \cdot 8 = 56$. For $i = 1, 2, 3$ we need to compute an inverse y_i of M_i modulo m_i.

We can take $y_1 = 4$ because $4 \cdot 72 \equiv 4 \cdot 2 \equiv 1 \pmod 7$, we can take $y_2 = -1$ because $(-1) \cdot 63 \equiv (-1)(-1) \equiv 1 \pmod 8$, and we can take $y_3 = 5$ because $5 \cdot 56 \equiv 5 \cdot 2 \equiv 1 \pmod 9$. The desired solution is found to be

$$1 y_1 M_1 + 3 y_2 M_2 + 2 y_3 M_3 \;=\; 1 \cdot 4 \cdot 72 \;+\; 3 \cdot (-1) \cdot 63 \;+\; 2 \cdot 5 \cdot 56 \;=\; 288 - 189 + 560 \;=\; 659.$$

If we subtract $504 = 7 \cdot 8 \cdot 9$ then we obtain the smallest positive integer solution $n = \mathbf{155}$.

(3) The largest amount of postage which cannot be formed with 5-cent and 6-cent stamps is 29 cents. This can be found by trial and error, or by the computing *Frobenius number* as $5 \cdot 6 - 5 - 6 = 19$. Below 19 cents, precisely the following positive integers can be expressed:

5, 6, 10, 11, 12, 15, 16, 17, 18.

Our conjecture states: *Every amount of postage of* 20 *cents or more can be formed using just* 5*-cent and* 6*-cent stamps.* The **proof** is analogous to that in Example 4 on page 337:

We will use *strong induction* to prove this result. Let $P(n)$ be the statement that postage of n cents can be formed using 5-cent and 6-cent stamps.

Basis step: The propositions $P(20), P(21), P(22), P(23)$ and $P(24)$ are true because $20 = 5{+}5{+}5{+}5$, $21 = 5{+}5{+}5{+}6$, $22 = 5{+}5{+}6{+}6$, $23 = 5{+}6{+}6{+}6$, and $24 = 6{+}6{+}6{+}6$.

Inductive step: The inductive hypothesis is the statement $P(j)$ is true for $20 \leq j \leq k$ where k is an integer with $k \geq 24$. Assuming this to be the case, we need to show that $P(k+1)$ is true. Using the inductive hypothesis we can assume that $P(k-4)$ is true because $k-4$ must be at least 20. Hence we can form postage of $k-4$ cents using 5-cent and 6-cent stamps. Just add one extra 5-cent stamp to the solution for $k-4$ cents and get postage for $k+1$ cents. Hence $P(k+1)$ is true.

Because we have completed the basis step and the inductive step of a strong induction proof, we know by strong induction that $P(n)$ is true for all integers $n \geq 20$.

(4) We use *Fermat's Little Theorem* (Theorem 3 in Section 4.4 on page 281) which states that $a^{p-1} \equiv 1 \pmod{p}$ for all primes p and all integers a not divisible by p.

(a) By Fermat's Little Theorem, we have $19^{12} \equiv 1 \pmod{13}$, and therefore

$$19^{145} = 19^{12\cdot 12+1} = (19^{12})^{12} \cdot 19 \equiv 1^{12} \cdot 19 = 19 \equiv \mathbf{6} \pmod{13}.$$

(b) By Fermat's Little Theorem, we have $(-12)^6 \equiv 1 \pmod{7}$. We also note that $50 \equiv 1 \pmod{7}$. We conclude $(-12)^{36} \cdot 50^{19} \equiv ((-12)^6)^6 \cdot 1^{19} \equiv \mathbf{1} \pmod{7}$.

(5) This problem is # 11 in Section 2.5 on page 176 of Rosen's book. The set $\mathbf{Z}^+$ of all positive integers is countably infinite. We also use the fact that any closed interval $[a, b]$, where a and b are real numbers satisfying $a < b$, is an uncountable set.

(a) Let $A = [0, 1]$ and $B = [1, 2]$. Then $A \cap B = \{1\}$, which is a finite set.

(b) Let $A = [0, 1] \cup \mathbf{Z}^+$ and $B = [1, 2] \cup \mathbf{Z}^+$. Then $A \cap B = \mathbf{Z}^+$ is countably infinite.

(c) Let $A = [0, 1]$ and $B = [0, 1]$. Then $A \cap B = [0, 1]$ is uncountable.