Math 275: Geometry of Convex Optimization

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Homework # 4, due Tuesday, October 19

1. Consider the problem of maximizing a linear function $u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4$ subject to the following Hankel matrix PSD constraint:
$$\begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix} \succeq 0.$$
   - Formulate the dual SDP problem.
   - Write the KKT equations and solve them.
   - Determine the algebraic degree and the optimal value function.
   - Discuss the role of Hankel matrices in polynomial optimization.

2. Find an explicit homogeneous polynomial of degree four in four variables that is non-negative on $\mathbb{R}^4$ but is not a sum of squares.

3. The cubic polynomial
$$f(x,y) \quad = \quad 12x^2y - 5x^2 - 9y^2 + 1$$
defines a smooth curve (check this) that bounds a spectrahedron. Construct real symmetric 3×3-matrices A and B such that
$$f(x,y) \quad = \quad \det(\mathrm{Id}_3 + Ax + By),$$
where Id_3 is the identity matrix. How unique is your solution?

4. The *elliptope* $\mathcal{E}_4$ is the set of correlation matrices of size 4×4, that is, positive semidefinite symmetric 4×4-matrices whose diagonal entries are all 1. Describe all faces of the six-dimensional spectrahedron $\mathcal{E}_4$.