Math 275: Geometry of Convex Optimization

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Homework # 3, due Tuesday, October 5

1. Draw the dual body P^{Δ} when the convex body P equals
   (a) ... the intersection of two circles in the plane,
   (b) ... the intersection of a circle and a square in the plane,
   (c) ... a cylinder in 3-space (i.e. a soda can).

   Explore different configurations. What happens when the origin moves?

2. Let P be a regular pentagon in the plane. Write P as a linear projection of a spectrahedron that is defined over the field $\mathbb{Q}$ of rational numbers. In other words, show that P is a *spectrahedral shadow* over $\mathbb{Q}$.

3. The convex hull of the group SO(4) is a convex body $\mathcal{O}$ in the space of real 4×4-matrices. Demonstrate how `Bermeja` can be used to test membership in $\mathcal{O}$. What can you tell me about the boundary of $\mathcal{O}$?

4. Prove or disprove: Every face of a spectrahedron is an exposed face.

5. Prove or disprove: Every homogeneous hyperbolic polynomial is the determinant of a symmetric matrix of linear forms.

6. Study the equation $X\dot{Y} = 0$ when X and Y are unknown symmetric 4×4-matrices. This constraint translates into 10 bilinear equations in 20 unknown matrix entries. Decompose the algebraic variety defined by these 10 equations into its irreducible components. What is the dimension of each component? How do you know that it is irreducible?

7. The *analytic center* of a spectrahedron is the symmetric matrix in its interior that has maximal determinant. Compute the analytic center of the 3-dimensional spectrahedron

$$\begin{pmatrix} x & z+1 & x+y+z \\ z+1 & y & x-y \\ x+y+z & x-y & 1-x-y \end{pmatrix} \succeq 0.$$

Determine the values x^*, y^* and z^* for the optimal matrix as floating point numbers. Make sure that you have at least twenty accurate digits. If this is possible, write x^*, y^* and z^* in terms of radicals over $\mathbb{Q}$.

8. Let X denote the *Veronese surface* in 5-dimensional projective space that has the parametric representation $(1 : x : y : x^2 : xy : y^2)$. Compute the conormal variety $\mathrm{CN}(X)$ and the dual variety X^*. Verify the Biduality Theorem $(X^*)^* = X$ for this example.