Math 55: Discrete Mathematics

UC Berkeley, Spring 2012
Homework # 9, due Wednesday, April 11

8.1.**5** *How many ways are there to pay a bill of* 17 *pesos using a currency with coins of values of* 1 *peso,* 2 *pesos,* 5 *pesos, and* 10 *pesos, and with bills with values of* 5 *pesos and* 10 *pesos?*

Let a_n denote the number of ways to pay a bill of n pesos. Then $a_n = 0$ for $n < 0$, and $a_0 = 1$, and we have the recurrence relation

$$a_n \;=\; a_{n-1} + a_{n-2} + 2 \cdot a_{n-5} + 2 \cdot a_{n-10} \qquad \text{for } n \geq 1.$$

This recurrence relation gives the following sequence:

n	1	2	3	4	5	6	7	8
a_n	1	2	3	5	10	17	31	54

8	9	10	11	12	13	14	15	16	17
54	95	171	302	539	955	1694	3011	5343	9494

Hence there are 9494 ways to pay a bill of 17 pesos.

8.1.**8** a) *Find a recurrence relation for the number of bit strings of length n that contain three consecutive* 0*s.*

b) *What are initial conditions?*

c) *How many bit strings of length seven contain three consecutive* 0*s?*

Let a_n denote the number of such strings of length n.

a) Consider a string of length $n \geq 3$ that contains three consecutive 0s. Such a string either ends with 1, or with 10, or with 100, or with 000. In the first case, there are a_{n-1} possibilities. In the second case, there are a_{n-2} possibilities. In the third case, there are a_{n-3} possibilities. And, in the fourth case, there are 2^{n-3} possibilities. Hence the recurrence relation is

$$a_n \;=\; a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3} \qquad \text{for } n \geq 3.$$

b) The initial conditions are $a_0 = a_1 = a_2 = 0$. Note that $a_3 = 1$.

c) The recurrence gives the sequence of positive integers

$$0, 0, 0, 1, 3, 8, 20, 47, 107, 238, 520, 1121, 2391, \ldots.$$

Hence there are $a_7 = 47$ bit strings of length seven that contain three consecutive 0s.

8.1.**12** *a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.*

b) What are the initial conditions?

c) In how many ways can this person climb a flight of eight stairs?

Let S_n denote the number of ways of climbing the stairs.

a) Let $n \geq 3$. The last step either was a single step, for which there are S_{n-1} possibilities, or a double step, for which there are S_{n-2} possibilities. The recurrence is $S_n = S_{n-1} + S_{n-2}$ for $n \geq 3$.

b) We have $S_1 = 1$ and $S_2 = 2$. You can take two stairs either directly or by taking stair at a time.

c) The recurrence gives the Fibonacci sequence

$$1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots.$$

Hence there are $S_8 = 34$ ways to climb a flight of eight stairs.

8.1.**20** *A bus driver pays all tolls, using only nickels and dimes, by throwing one coin at a time into the mechanical toll collector.*

a) Find the recurrence relation for the number of different ways the bus driver can pay a toll of n cents (where the order in which the coins are used matters).

b) In how many ways can the driver pay a toll of 45 cents?

Let c_n denote the number of ways of paying n cents.

a) Consider the coin that was paid last. That coin either is a nickel, for which there are c_{n-5} possibilities, or it is a dime, for which there are c_{n-10} possibilities. Hence the recurrence relation is $c_n = c_{n-5} + c_{n-10}$ for $n \geq 11$. Also, the initial conditions are $c_1 = c_2 = c_3 = c_4 = c_6 = c_7 = c_8 = c_9 = 0$, $c_5 = 1$, and $c_{10} = 2$.

b) This recurrence generates the interspersed Fibonacci sequence

$$0,0,0,0,1,0,0,0,0,2,0,0,0,0,3,0,0,0,0,5,0,0,0,0,8,$$
$$0,0,0,0,13,0,0,0,0,21,0,0,0,0,34,0,0,0,0,55,\ldots$$

Hence there are $c_{45} = 55$ ways for the bus driver to pay toll.

8.1.**30** *a) Write out all the ways the product $x_0 \cdot x_1 \cdot x_2 \cdot x_3 \cdot x_4$ can be parenthesized to determine the order of multiplication.*

b) Use the recurrence relation developed in Example 5 to calculate C_n, the number of ways to parenthesize the product of five numbers. Verify that you listed the correct number of ways in part (a).

c) Check your result in part (b) by finding C_4, using the closed formula for C_n in the solution of Example 5.

a) There are 14 ways to parenthesize a product of five factors:

$$(((x_0 \cdot x_1) \cdot x_2) \cdot x_3) \cdot x_4\,,\ \ ((x_0 \cdot (x_1 \cdot x_2)) \cdot x_3) \cdot x_4\,,$$
$$((x_0 \cdot x_1) \cdot (x_2 \cdot x_3)) \cdot x_4\,,\ \ (x_0 \cdot ((x_1 \cdot x_2) \cdot x_3)) \cdot x_4\,,$$
$$(x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))) \cdot x_4\,,\ \ (x_0 \cdot x_1 \cdot x_2) \cdot (x_3 \cdot x_4)\,,$$
$$(x_0 \cdot x_1 \cdot x_2) \cdot (x_3 \cdot x_4)\,,\ \ (x_0 \cdot x_1) \cdot ((x_2 \cdot x_3) \cdot x_4)\,,$$
$$(x_0 \cdot x_1) \cdot (x_2 \cdot (x_3 \cdot x_4))\,,\ \ x_0 \cdot (((x_1 \cdot x_2) \cdot x_3) \cdot x_4)\,,$$
$$x_0 \cdot ((x_1 \cdot (x_2 \cdot x_3)) \cdot x_4)\,,\ \ x_0 \cdot ((x_1 \cdot x_2) \cdot (x_3 \cdot x_4))\,,$$
$$x_0 \cdot (x_1 \cdot ((x_2 \cdot x_3) \cdot x_4))\,,\ \ x_0 \cdot (x_1 \cdot (x_2 \cdot (x_3 \cdot x_4))).$$

b) From the recurrence on page 507 we get

$$C_4 = C_0 \cdot C_3 + C_1 \cdot C_2 + C_2 \cdot C_1 + C_3 \cdot C_0 = 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1 = 14.$$

c) The closed formula for the Catalan numbers is $C_n = \binom{2n}{n}/(n+1)$. Substituting $n = 4$, we confirm $C_4 - \binom{8}{4}/5 = 70/5 = 14$.

8.2.**2** *Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.*

a) $a_n = 3a_{n-2}$

b) $a_n = 3$

c) $a_n = a_{n-1}^2$

d) $a_n = a_{n-1} + a_{n-3}$

e) $a_n = a_{n-1}/n$

f) $a_n = a_{n-1} + a_{n-2} + n + 3$

g) $a_n = 4a_{n-2} + 5a_{n-4} + 9a_{n-7}$

a) This is a linear homogeneous recurrence relation with constant coefficients of degree 2.

b) This recurrence is not homogeneous.

c) This recurrence is not linear.

d) This is a linear homogeneous recurrence relation with constant coefficients of degree 3.

e) This recurrence does not have constant coefficients.

f) This recurrence is not homogeneous.

g) This is a linear homogeneous recurrence relation with constant coefficients of degree 7.

8.2.**8** *A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the previous two years.*

a) Find a recurrence relation for $\{L_n\}$ *where* L_n *is the number of lobsters caught in year* n*, under the assumption of this model.*

b) Find L_n *if* $100,000$ *lobsters were caught in year* 1 *and* $200,000$ *lobsters were caught in year* 2*.*

a) Since L_n is the average of L_{n-1} and L_{n-2}, the recurrence is

$$L_n \;=\; \frac{1}{2}L_{n-1} + \frac{1}{2}L_{n-2}.$$

b) The characteristic polynomial $x^2 - x/2 - 1/2 = \frac{1}{2}(2x+1)(x-1)$ has the roots 1 and $-1/2$. Hence there exist real constants c_1 and c_2 such that $L_n = \gamma_1 + \gamma_2(-1/2)^n$. The initial conditions imply the linear relations $\gamma_1 + \gamma_2 = 100000$ and $\gamma_1 - \gamma_2/2 = 200000$. The solution is $\gamma_1 = 500000/3$ and $\gamma_2 = -200000/3$, and we conclude

$$c_n \quad = \quad \frac{500000}{3} + \frac{200000 \cdot (-1/2)^n}{3}.$$

The second term converges to zero. Thus, the steady state scenario is that $166,667$ lobsters will be caught every year.

8.2.**11** *The Lucas numbers satisfy the recurrence relation*

$$L_n = L_{n-1} + L_{n-2},$$

and the initial conditions $L_0 = 2$ *and* $L_1 = 1$.

a) *Show that* $L_n = f_{n-1} + f_{n+1}$ *for* $n = 2, 3, \ldots,$ *where* f_n *is the nth Fibonacci number.*

b) *Find an explicit formula for the Lucas numbers.*

a) We define a new sequence $\{M_n\}$ by setting $M_0 = 2$ and $M_n = f_{n-1} + f_{n+1}$ for $n \geq 1$. Then $M_1 = f_0 + f_2 = 1$ and $M_2 = f_1 + f_3 = 1 + 2 = 3$. The two sequences $\{M_n\}$ and $\{L_n\}$ satisfy the same recurrence, and they satisfy the same initial conditions. Therefore they must be equal.

b) The roots of the characteristic polynomial are given by the golden ratio, and we find

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{for } n = 0, 1, 2, 3, \ldots.$$

8.2.**18** *Solve the recurrence relation* $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ *with* $a_0 = -5$, $a_1 = 4$ *and* $a_2 = 88$.

The characteristic polynomial equals $x^3 - 6x^2 + 12x - 8 = (x-2)^3$. We know from Theorem 4 on page 519 that $a_n = \gamma_1 2^n + \gamma_2 n 2^n + \gamma_3 n^2 2^n$ for some constants $\gamma_1, \gamma_2, \gamma_3$. By substituting $n = 0, 1, 2$ into this equation, we obtain the linear system

$$\gamma_1 = -5\,,\ \ 2\gamma_1 + 2\gamma_2 + 2\gamma_3 = 4 \ \text{ and } \ \ 4\gamma_1 + 8\gamma_2 + 16\gamma_3 = 88.$$

The solution is $\gamma_1 = -5$, $c_2 = 1/2$, $c_3 = 13/2$, and therefore

$$a_n \quad = \quad -5 \cdot 2^n \; + \; n \cdot 2^{n-1} \; + \; 13 \cdot n^2 \cdot 2^{n-1} \quad \text{for } n \; = \; 0, 1, 2, 3, \ldots.$$

8.2.**46** *Suppose that there are two goats on an island initially. The number of goats on the island doubles every year by natural reproduction, and some goats are either added or removed each year.*

a) *Construct a recurrence relation for number of goats on the island at the start of the nth year, assuming that during each year an extra* 100 *goats are put on the island.*

b) *Solve the recurrence relation from part (a) to find the number of goats on the island at the start of the* n*th year.*

c) *Construct a recurrence relation for number of goats on the island at the start of the* n*th year, assuming that* n *goats are removed during the* n*th year for each* $n \geq 3$*.*

d) *Solve the recurrence relation in part (c) to find the number of goats on the island at the start of the* n*th year.*

a) Let G_n denote the number of goats on the island at the start of the nth year. The recurrence is $G_n = 2G_{n-1} + 100$ for $n = 1, 2, \ldots$.

b) A particular solution to his inhomogenous linear recurrence is $100 \cdot (2^n - 1)$. By Theorem 5 on page 521, the solution must have the form $G_n = 100(2^n - 1) + \gamma 2^n$. Since $G_0 = 2$, we see that $\gamma = 2$, and hence the number of goats equals

$$G_n = 100(2^n - 1) + 2^{n+1} \quad \text{for } n = 0, 1, 2, 3, \ldots.$$

c) The recurrence is $G_n = 2G_{n-1} - n$ for $n = 1, 2, \ldots,$

d) We observe that the recurrence gives the sequence $2, 3, 4, 5, 6, \ldots$. Using mathematical induction, we easily prove that indeed

$$G_n = n + 2 \quad \text{for } n = 0, 1, 2, 3, \ldots.$$

8.4.**6**** *Find a closed form for the generating function for the sequence* $\{a_n\}$ *where*

a) $a_n = -1$ *for all* $n = 0, 1, 2, \ldots$

b) $a_n = 2^n$ *for* $n = 1, 2, 3, 4, \ldots$ *and* $a_0 = 0$

c) $a_n = n - 1$ *for* $n = 0, 1, 2, \ldots$

d) $a_n = 1/(n + 1)!$ *for* $0, 1, 2, \ldots$

e) $a_n = \binom{n}{2}$ *for* $n = 0, 1, 2, \ldots,$

f) $a_n = \binom{10}{n+1}$ *for* $n = 0, 1, 2, \ldots$

The following generating functions represent the desired sequences:

a) $1/(x - 1)$.

b) $2x/(1 - 2x)$

c) $(2x-1)/(1-x)^2$

d) $(e^x-1)/x$.

e) $x^2/(1-x)^3$

f) $((x+1)^{10}-1)/x$

8.4.**8** *For each of these generating functions, provide a closed formula for the sequence it determines.*

a) $(x^2+1)^3$

b) $(3x-1)^3$

c) $1/(1-2x^2)$

d) $x^2/(1-x)^3$

e) $x-1+(1/(1-3x))$

f) $(1+x^3)/(1+x)^3$

g) $x/(1+x+x^2)$

h) $e^{3x^2}-1$

a) $\binom{3}{n/2}$ if n is even, and 0 if n is odd

b) $-(-3)^n\binom{3}{n}$

c) $2^{n/2}$ if n is even, and 0 if n is odd

d) $\binom{n}{3}$

e) 0 for $n=0$, 4 for $n=1$, and 3^n for $n\geq 2$.

f) 1 for $n=0$, and $(-1)^n\cdot 3n$ for $n\geq 1$.

g) 0 if $n\equiv 0 \pmod 3$, 1 if $n\equiv 1 \pmod 3$, -1 if $n\equiv 2 \pmod 3$.

h) $2^{n/2}/(n/2)!$ if n is even and positive, and 0 otherwise

8.4.**14** *Use generating functions to determine the number of different ways* 12 *identical action figures can be given to five children so that each child receives at most three action figures.*

This is the coefficient of x^{12} in $(1+x+x^2+x^3)^5$. It is found to be 35. So, there are 35 ways of distributing the 12 action figures.

8.4.**20** *What is the generating function for the sequence* $\{c_k\}$*, where* c_k *represents the number of ways to make change for* k *pesos using bills worth* 10 *pesos,* 20 *pesos,* 50 *pesos and* 100 *pesos?*

The generating function for making change with these bills is

$$\sum_{k=0}^{\infty} x^k \;=\; \frac{1}{(1-x^{10})(1-x^{20})(1-x^{50})(1-x^{100})}$$

8.4.**22** *Give a combinatorial interpretation of the coefficient of x^6 in the expansion $(1+x+x^2+x^3+\cdots)^n$. Use this interpretation to find this number.*

This is the number of ways of giving n action figures to six children. While the action figures are indistinguishable, the children are very precious and thus we consider them to be distinguishable. More abstractly, that coefficient counts the n-combinations of a set of 6 elements with repetition allowed. The number equals

$$\binom{n+5}{6}.$$

8.4.**36** *Use generating functions to solve the recurrence relation $a_k = a_{k-1} + 2a_{k-2} + 2^k$ with initial conditions $a_0 = 4$ and $a_1 = 12$.*

From the initial conditions and the recurrence relation, we find that the generating function $G(x) = \sum_{k=0}^{\infty} a_k x^k$ satisfies the identity

$$G(x) - 4 - 12x \;=\; x(G(x)-4) \,+\, 2x^2 G(x) \,+\, \frac{1}{1-2x} - 1 - 2x.$$

We solve this equation for $G(x)$ to find

$$G(x) \;=\; \frac{4\cdot(1-3x^2)}{(1-2x)^2(1+x)} \;=\; 4 + 12x + 24x^2 + 56x^3 + 120x^4 + \cdots$$