Math 55, **Final Exam**
Thursday, May 10, 7:00pm–10:00pm

This exam is closed book. Do not use any books, notes or electronic devices. Please write in a blue note book, with your name, the name of your GSI and your section time on the front. Each of the 10 *problems is worth* 10 *points, for a total of* 100 *points. Answers without justification will receive no credit.*

(1) Suppose $n \geq 1$ is an integer.
  (a) How many functions are there from the set $\{1, \ldots, n\}$ to the set $\{1, 2, 3\}$?
  (b) How many of the functions in part (a) are injective?
  (c) How many of the functions in part (a) are surjective?

(2) Let G be a simple graph with at least two vertices. Show that if G is not connected, then the complementary graph $\overline{G}$ is connected.

(3) Suppose each of the three vertices of a triangle is colored at random either black or white, with equal probability for each color. What is the expected number of edges of the triangle that have both endpoints of the same color?

(4) Give an example of a relation R such that its transitive closure R^* satisfies $R^* = R \cup R^2 \cup R^3$, but $R^* \neq R \cup R^2$.

(5) Recall that the Fibonacci numbers are defined by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-2} + f_{n-1}$ for $n \geq 2$. Prove that

$$f_0 f_1 + f_1 f_2 + \cdots + f_{2n-1} f_{2n} \;=\; f_{2n}^2$$

for any positive integer n.

(6) Find the set of all solutions x to the system of congruences

$$x \equiv 2 \pmod{6} \quad \text{and} \quad x \equiv 3 \pmod{9}.$$

(7) Suppose we randomly choose nonnegative integers x_1, x_2, x_3, and x_4 that solve the equation $x_1 + x_2 + x_3 + x_4 = 8$. Here we assume that each solution has an equal probability of being chosen. Given that at least one of x_1 and x_2 is equal to 1, what is the probability that $x_1 = 1$.

(8) Solve the recurrence relation

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = 0$, and $a_2 = 7$.

(9) Give a recursive definition of the set of bit strings that have the same number of zeros and ones.

(10) Use generating functions to determine the number of different ways 10 identical balloons can be given to four children if each child receives at least two balloons. *(Note: The children are distinguishable)*