SPECTRAHEDRA

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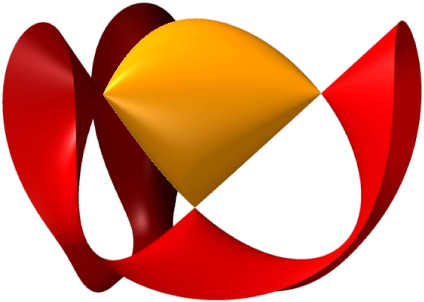

GAeL Lecture I on Convex Algebraic Geometry
Coimbra, Portugal, Monday, June 7, 2010

Positive Semidefinite Matrices

For a real symmetric $n \times n$-matrix A the following are equivalent:

- All n eigenvalues of A are positive real numbers.
- All 2^n principal minors of A are positive real numbers.
- Every non-zero vector $x \in \mathbb{R}^n$ satisfies $x^T A \cdot x > 0$.

A matrix A is *positive definite* if it satisfies these properties, and it is *positive semidefinite* if the following equivalent properties hold:

- All n eigenvalues of A are non-negative real numbers.
- All 2^n principal minors of A are non-negative real numbers.
- Every vector $x \in \mathbb{R}^n$ satisfies $x^T A \cdot x \geq 0$.

The set of all positive semidefinite $n \times n$-matrices is a convex cone of full dimension $\binom{n+1}{2}$. It is closed and semialgebraic.

The interior of this cone consists of all positive definite matrices.

Semidefinite Programming

A *spectrahedron* is the intersection of the cone of positive semidefinite matrices with an affine-linear space. Its algebraic representation is a linear combination of symmetric matrices

$$A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_m A_m \ \succeq \ 0 \qquad (*)$$

Engineers call this is a *linear matrix inequality*.

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Semidefinite programming is the computational problem of maximizing a linear function over a spectrahedron:

$$\text{Maximize } \ c_1 x_1 + c_2 x_2 + \cdots + c_m x_m \ \text{ subject to } \ (*)$$

Example: *The smallest eigenvalue of a symmetric matrix* A *is the solution of the SDP* Maximize x subject to $A - x \cdot \mathrm{Id} \succeq 0$.

Convex Polyhedra

Linear programming is semidefinite programming for diagonal matrices. If $A_0, A_1, \ldots, A_m$ are diagonal $n \times$n-matrices then

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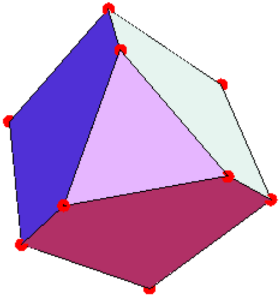
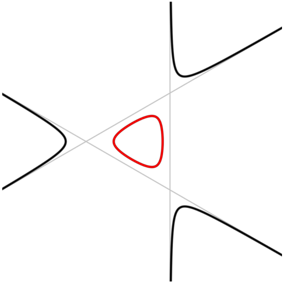
translates into a system of n linear inequalities in the m unknowns.
A spectrahedron defined in this manner is a convex polyhedron:



Pictures in Dimension Two

Here is a picture of a spectrahedron for $m = 2$ and $n = 3$:



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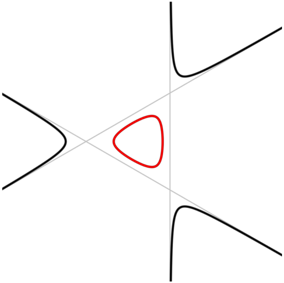
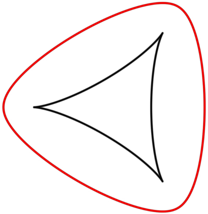


Duality is important in both optimization and projective geometry:



Example: Multifocal Ellipses

Given m points $(u_1, v_1), \ldots, (u_m, v_m)$ in the plane $\mathbb{R}^2$, and a radius $d > 0$, their *m-ellipse* is the convex algebraic curve

$$\left\{(x, y) \in \mathbb{R}^2 \,:\, \sum_{k=1}^{m} \sqrt{(x-u_k)^2 + (y-v_k)^2} \,=\, d \right\}.$$

The 1-ellipse and the 2-ellipse are algebraic curves of degree 2.

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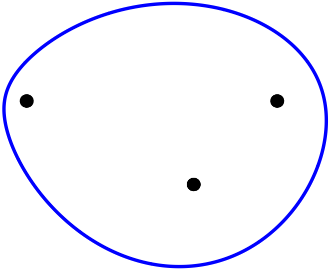
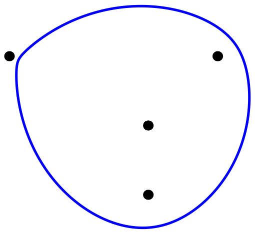
The 3-ellipse is an algebraic curve of degree 8:

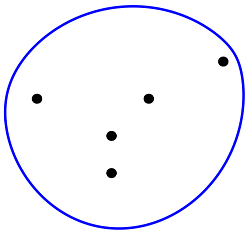


2, 2, 8, 10, 32, ...

The 4-ellipse is an algebraic curve of degree 10:

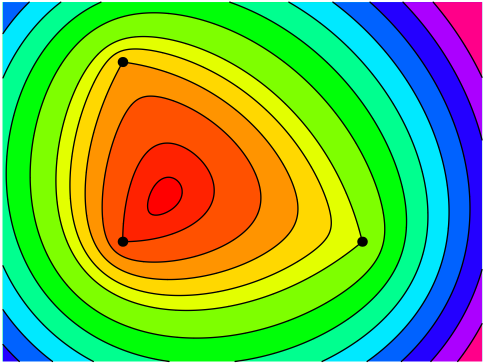


The 5-ellipse is an algebraic curve of degree 32:



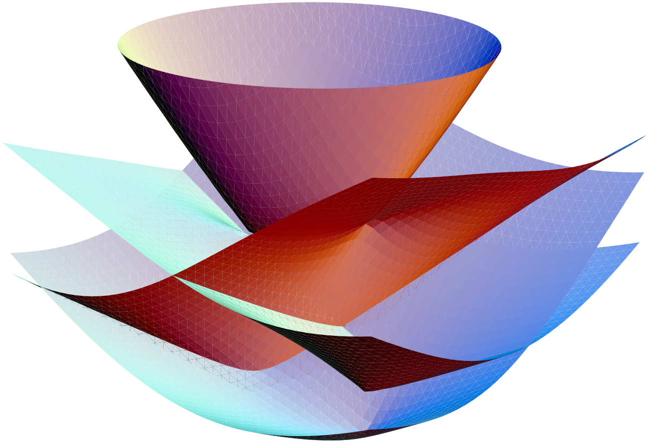
Concentric Ellipses

What is the algebraic degree of the m-ellipse?
How to write its equation?



What is the smallest radius d for which the m-ellipse is non-empty? How to compute the Fermat-Weber point?

3D View



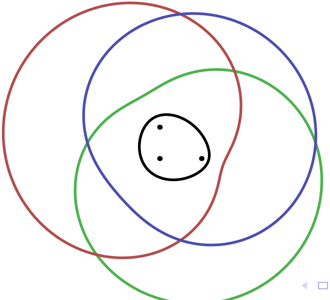
$$\mathcal{C} \;=\; \left\{ (x,y,d) \in \mathbb{R}^3 \;:\; \sum_{k=1}^{m} \sqrt{(x-u_k)^2 + (y-v_k)^2} \;\leq\; d \right\}.$$

Ellipses are Spectrahedra

The 3-ellipse with foci $(0,0),(1,0),(0,1)$ has the representation

$$\begin{bmatrix} d+3x-1 & y-1 & y & 0 & y & 0 & 0 & 0 \\ y-1 & d+x-1 & 0 & y & 0 & y & 0 & 0 \\ y & 0 & d+x+1 & y-1 & 0 & 0 & y & 0 \\ 0 & y & y-1 & d-x+1 & 0 & 0 & 0 & y \\ y & 0 & 0 & 0 & d+x-1 & y-1 & y & 0 \\ 0 & y & 0 & 0 & y-1 & d-x-1 & 0 & y \\ 0 & 0 & y & 0 & y & 0 & d-x+1 & y-1 \\ 0 & 0 & 0 & y & 0 & y & y-1 & d-3x+1 \end{bmatrix}$$

The ellipse consists of all points (x,y) where this symmetric 8×8-matrix is positive semidefinite. Its boundary is a curve of degree eight:



2, 2, 8, 10, 32, 44, 128, ...

Theorem: *The polynomial equation defining the m-ellipse has degree* 2^m *if m is odd and degree* $2^m - \binom{m}{m/2}$ *if m is even. We express this polynomial as the determinant of a symmetric matrix of linear polynomials. Our representation extends to weighted m-ellipses and m-ellipsoids in arbitrary dimensions .....*

[J. Nie, P. Parrilo, B.St.: Semidefinite representation of the k-ellipse, in *Algorithms in Algebraic Geometry*, I.M.A. Volumes in Mathematics and its Applications, 146, Springer, New York, 2008, pp. 117-132]

In other words, m-ellipses and m-ellipsoids are spectrahedra.
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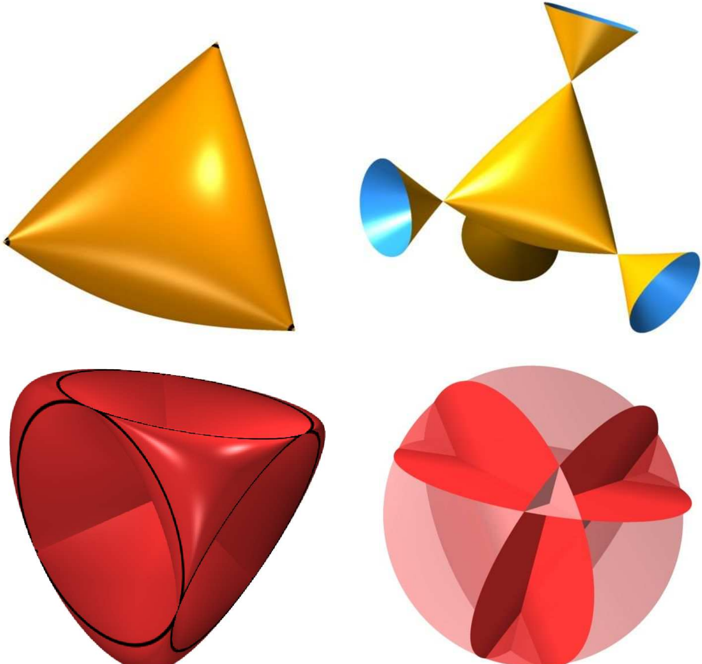
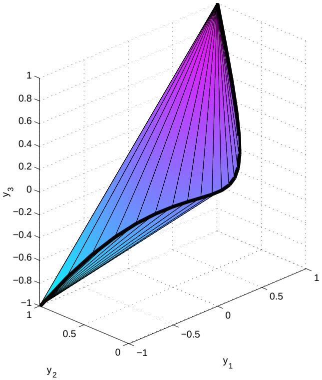
Let's now look at some spectrahedra in dimension three.
Our next picture shows the typical behavior for $m = 3$ and $n = 3$.

A Spectrahedron and its Dual



Non-Linear Convex Hull Computation

Input : $\left\{(t, t^2, t^3) \in \mathbb{R}^3 : -1 \leq t \leq 1\right\}$



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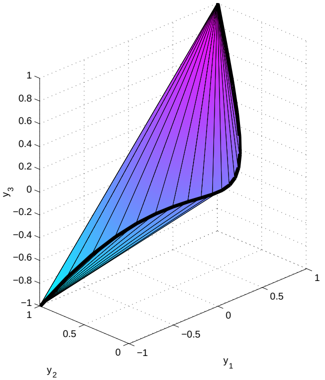


The convex hull of the moment curve is a spectrahedron.

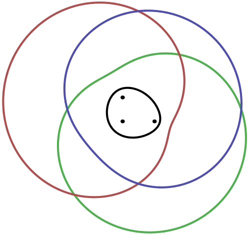
Output : $\begin{pmatrix} 1 & x \\ x & y \end{pmatrix} \pm \begin{pmatrix} x & y \\ y & z \end{pmatrix} \succeq 0$

Characterization of Spectrahedra

A convex hypersurface of degree d in $\mathbb{R}^n$ is *rigid convex* if every line passing through its interior meets (the Zariski closure of) that hypersurface in d real points.

Theorem (Helton–Vinnikov (2006))

Every spectrahedron is rigid convex. The converse is true for $n = 2$.



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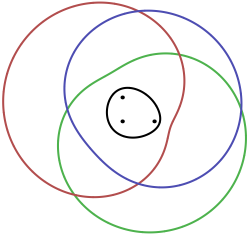


Open problem: Is every compact convex basic semialgebraic set $\mathcal{S}$ the projection of a spectrahedron in higher dimensions?

Theorem (Helton–Nie (2008))

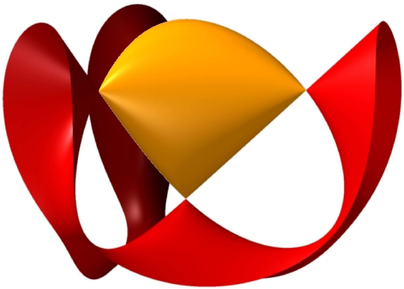
The answer is yes if the boundary of $\mathcal{S}$ is "sufficiently smooth".

Questions about 3-Dimensional Spectrahedra

What are the edge graphs of spectrahedra in $\mathbb{R}^3$?

How can one define their *combinatorial types*?

Is there an analogue to Steinitz' Theorem for polytopes in $\mathbb{R}^3$?



Consider 3-dimensional spectrahedra whose boundary is an irreducible surface of degree n. Can such a spectrahedron have $\binom{n+1}{3}$ isolated singularities in its boundary? How about $n = 4$?

Minimizing Polynomial Functions

Let $f(x_1, \ldots, x_m)$ be a polynomial of even degree $2d$.
We wish to compute the global minimum x^* of $f(x)$ on $\mathbb{R}^m$.

This optimization problem is equivalent to

Maximize λ such that $f(x) - \lambda$ is non-negative on $\mathbb{R}^m$.

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Empirically, the optimal value of the SDP almost always agrees with the global minimum. In that case, the optimal matrix of the dual SDP has rank one, and the optimal point x^* can be recovered from this. **How to reconcile this with Blekherman's results?**

SOS Programming: A Univariate Example

Let $m = 1$, $d = 2$ and $f(x) = 3x^4 + 4x^3 - 12x^2$. Then

$$f(x) - \lambda \quad = \quad \begin{pmatrix} x^2 & x & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & \mu - 6 \\ 2 & -2\mu & 0 \\ \mu - 6 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}$$

Our problem is to find (λ, μ) such that the 3×3-matrix is positive semidefinite and λ is maximal.

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Our problem is to find (λ, μ) such that the 3×3-matrix is positive semidefinite and λ is maximal. The optimal solution of this SDP is

$$(\lambda^*, \mu^*) \; = \; (-32, -2).$$

Cholesky factorization reveals the SOS representation

$$f(x) - \lambda^* \quad = \quad \left((\sqrt{3}\,x - \frac{4}{\sqrt{3}}) \cdot (x+2)\right)^2 \; + \; \frac{8}{3}(x+2)^2.$$

We see that the global minimum is $x^* \; = \; -2$.
This approach works for many polynomial optimization problems.

My Favorite Spectrahedron

Consider the intersection of the cone of 6×6 PSD matrices with the 15-dimensional linear space consisting of all Hankel matrices

$$H = \begin{pmatrix} \lambda_{400} & \lambda_{220} & \lambda_{202} & \lambda_{310} & \lambda_{301} & \lambda_{211} \\ \lambda_{220} & \lambda_{040} & \lambda_{022} & \lambda_{130} & \lambda_{121} & \lambda_{031} \\ \lambda_{202} & \lambda_{022} & \lambda_{004} & \lambda_{112} & \lambda_{103} & \lambda_{013} \\ \lambda_{310} & \lambda_{130} & \lambda_{112} & \lambda_{220} & \lambda_{211} & \lambda_{121} \\ \lambda_{301} & \lambda_{121} & \lambda_{103} & \lambda_{211} & \lambda_{202} & \lambda_{112} \\ \lambda_{211} & \lambda_{031} & \lambda_{013} & \lambda_{121} & \lambda_{112} & \lambda_{022} \end{pmatrix}.$$

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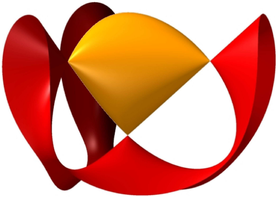
Dual to this intersection is the projection

$$\mathrm{Sym}_2(\mathrm{Sym}_2(\mathbb{R}^3)) \;\rightarrow\; \mathrm{Sym}_4(\mathbb{R}^3)$$

taking a 6×6-matrix to the ternary quartic it represents. Its image is a cone whose algebraic boundary is a *discriminant* of degree 27.

Conclusion

Spectrahedra and their geometry deserve to be studied in their own right, independently of their important uses in applications.



A true understanding of these convex bodies will require the integration of three different areas of mathematics:

- Convexity
- Algebraic Geometry
- Optimization Theory

Please join the SIAM Special Interest Group on Algebraic Geometry