The Convex Hull of a Space Curve

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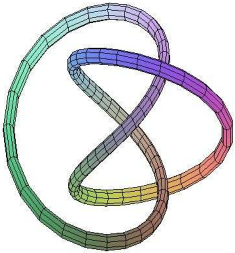
(joint work with Kristian Ranestad)



The Mathematics of Klee & Grünbaum:
100 Years in Seattle
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Convex Hull of a Trigonometric Curve

$$\{\, (\cos(\theta),\ \sin(2\theta),\ \cos(3\theta)\,) \in \mathbb{R}^3 \ :\ \theta \in [0, 2\pi] \,\}$$
$$= \{\ (x, y, z) \in \mathbb{R}^3\ :\ x^2 - y^2 - xz \ =\ z - 4x^3 + 3x \ =\ 0\ \}$$

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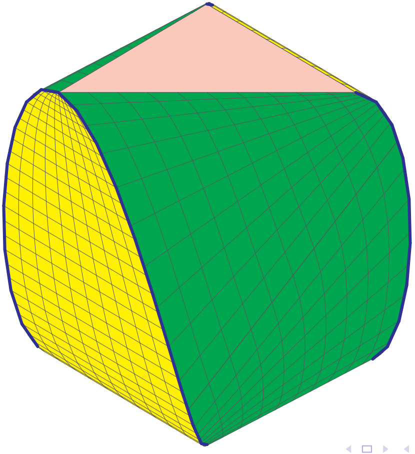
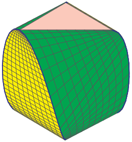


Boundary Surface Patches



The yellow surface has degree 3 and it equals $z - 4x^3 + 3x \ = \ 0$.

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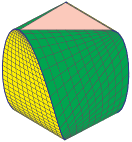

The yellow surface has degree 3 and it equals $z - 4x^3 + 3x \;=\; 0$.

The green surface has degree **16** and its defining polynomial is

$$
\begin{gathered}
1024x^{16} - 12032x^{14}y^2 + 52240x^{12}y^4 - 96960x^{10}y^6 + 56160x^8y^8 + 19008x^6y^{10} + 1296x^4y^{12} + 6144x^{15}z \\
-14080x^{13}y^2z - 72000x^{11}y^4z + 149440x^9y^6z + 79680x^7y^8z + 7488x^5y^{10}z + 15360x^{14}z^2 + 36352x^{12}y^2z^2 \\
+151392x^{10}y^4z^2 + 131264x^8y^6z^2 + 18016x^6y^8z^2 + 20480x^{13}z^3 + 73216x^{11}y^2z^3 + 105664x^9y^4z^3 + 23104x^7y^6z^3 \\
+15360x^{12}z^4 + 41216x^{10}y^2z^4 + 16656x^8y^4z^4 + 6144x^{11}z^5 + 6400x^9y^2z^5 + 1024x^{10}z^6 - 26048x^{14} - 135688x^{12}y^2 \\
+178752x^{10}y^4 + 124736x^8y^6 - 210368x^6y^8 + 792x^4y^{10} + 5184x^2y^{12} + 432y^{14} - 77888x^{13}z + 292400x^{11}y^2z \\
+10688x^9y^4z - 492608x^7y^6z - 67680x^5y^8z + 21456x^3y^{10}z + 2592xy^{12}z - 81600x^{12}z^2 - 65912x^{10}y^2z^2 \\
-464256x^8y^4z^2 - 192832x^6y^6z^2 + 31488x^4y^8z^2 + 6552x^2y^{10}z^2 - 40768x^{11}z^3 - 194400x^9y^2z^3 - 196224x^7y^4z^3 \\
+14912x^5y^6z^3 + 8992x^3y^8z^3 - 20800x^{10}z^4 - 84088x^8y^2z^4 - 7360x^6y^4z^4 + 7168x^4y^6z^4 - 12480x^9z^5 - 9680x^7y^2z^5 \\
+3264x^5y^4z^5 - 2624x^8z^6 + 760x^6y^2z^6 + 64x^7z^7 + 189649x^{12} + 104700x^{10}y^2 - 568266x^8y^4 + 268820x^6y^6 \\
+118497x^4y^8 - 42984x^2y^{10} - 432y^{12} + 62344x^{11}z - 592996x^9y^2z + 421980x^7y^4z + 377780x^5y^6z - 79748x^3y^8z \\
-18288xy^{10}z + 104620x^{10}z^2 + 56876x^8y^2z^2 + 480890x^6y^4z^2 - 12440x^4y^6z^2 - 51354x^2y^8z^2 - 936y^{10}z^2 \\
+35096x^9z^3 + 181132x^7y^2z^3 + 73800x^5y^4z^3 - 52792x^3y^6z^3 - 3780xy^8z^3 - 6730x^8z^4 + 52596x^6y^2z^4 \\
-19062x^4y^4z^4 - 5884x^2y^6z^4 + y^8z^4 + 6008x^7z^5 + 2516x^5y^2z^5 - 4324x^3y^4z^5 + 4xy^6z^5 + 2380x^6z^6 \\
-1436x^4y^2z^6 + 6x^2y^4z^6 - 152x^5z^7 + 4x^3y^2z^7 + x^4z^8 - 305250x^{10} + 313020x^8y^2 + 174078x^6y^4 \\
-291720x^4y^6 + 74880x^2y^8 + 84400x^9z + 278676x^7y^2z - 420468x^5y^4z + 20576x^3y^6z + 40704xy^8z \\
-25880x^8z^2 - 76516x^6y^2z^2 - 148254x^4y^4z^2 + 77840x^2y^6z^2 + 5248y^8z^2 - 29808x^7z^3 - 49388x^5y^2z^3 \\
+23080x^3y^4z^3 + 14560xy^6z^3 + 14420x^6z^4 - 7852x^4y^2z^4 + 9954x^2y^4z^4 + 568y^6z^4 + 848x^5z^5 + 92x^3y^2z^5 \\
+1164xy^4z^5 - 984x^4z^6 + 724x^2y^2z^6 - 2y^4z^6 + 112x^3z^7 - 4xy^2z^7 - 2x^2z^8 + 140625x^8 - 270000x^6y^2 \\
+172800x^4y^4 - 36864x^2y^6 - 75000x^7z + 36000x^5y^2z + 46080x^3y^4z - 24576xy^6z - 12500x^6z^2 \\
+49200x^4y^2z^2 - 19968x^2y^4z^2 - 4096y^6z^2 + 15000x^5z^3 - 10560x^3y^2z^3 - 3072xy^4z^3 \\
-2250x^4z^4 - 1872x^2y^2z^4 + 768y^4z^4 - 520x^3z^5 + 672xy^2z^5 + 204x^2z^6 - 48y^2z^6 - 24xz^7 + z^8.
\end{gathered}
$$

Basic Definitions

Let C be a compact real algebraic curve in $\mathbb{R}^3$ and $\bar{C}$ its Zariski closure of C in $\mathbb{CP}^3$. We define the *degree* and *genus* of C by way of the complex projective curve $\bar{C} \subset \mathbb{CP}^3$:

$$d = \mathrm{degree}(C) := \mathrm{degree}(\bar{C}) \text{ and } g = \mathrm{genus}(C) := \mathrm{genus}(\bar{C}).$$

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The convex hull $\mathrm{conv}(C)$ of the real algebraic curve C is a compact, convex, semi-algebraic subset of $\mathbb{R}^3$, and its boundary $\partial\mathrm{conv}(C)$ is a pure 2-dimensional semi-algebraic subset of $\mathbb{R}^3$.

Let K be the subfield of $\mathbb{R}$ over which the curve C is defined. The *algebraic boundary* of $\mathrm{conv}(C)$ is the K-Zariski closure of $\partial\mathrm{conv}(C)$ in $\mathbb{C}^3$. The algebraic boundary is denoted $\partial_a\mathrm{conv}(C)$. This complex surface is usually reducible and reduced. Its defining polynomial in $K[x, y, z]$ is unique up to scaling.

Degree Formula for Smooth Curves

Theorem. Let C be a general smooth compact curve of degree d and genus g in $\mathbb{R}^3$. The algebraic boundary $\partial_a \mathrm{conv}(C)$ of its convex hull is the union of the edge surface of degree $2(d-3)(d+g-1)$ and the tritangent planes of which there are $8\binom{d+g-1}{3} - 8(d+g-4)(d+2g-2) + 8g - 8$.

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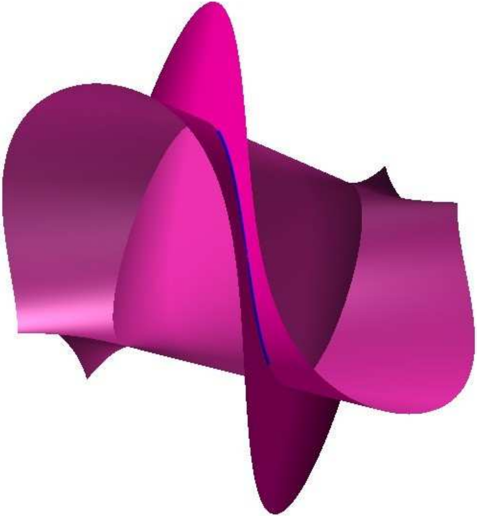
A plane H in $\mathbb{CP}^3$ is a *tritangent plane* of $\bar{C}$ if H is tangent to $\bar{C}$ at three points. We count these using *De Jonquières' formula*.

Given points $p_1, p_2 \in C$, their secant line $L = \mathrm{span}(p_1, p_2)$ is a *stationary bisecant* if the tangent lines of C at p_1 and p_2 lie in a common plane. The *edge surface* of C is the union of all stationary bisecant lines. Its degree was determined by Arrondo *et al.* (2001).

Example. If $d = 4$ and $g = 0$ then the two numbers are 6 and 0.

Smooth Rational Quartic Curve



The edge surface of the curve $(\cos(\theta), \sin(\theta) + \cos(2\theta), \sin(2\theta))$ is irreducible of degree six. $d = 4, g = 0$

Edge Surface of an Elliptic Curve

The intersection $C = Q_1 \cap Q_2$ of two general quadratic surfaces is an *elliptic curve*: it has genus $g = 1$ and degree $d = 4$. The edge surface of C has degree 8. It is the union of four quadratic cones.

Proof: The pencil of quadrics $Q_1 + tQ_2$ contains four singular quadrics, corresponding to the four real roots t_1, t_2, t_3, t_4 of $f(t) = \det(Q_1 + tQ_2)$. The stationary bisecants to C are the rulings of these cones. The defining polynomial of $\partial_a \mathrm{conv}(C)$ is

$$\prod_{i=1}^{4} (Q_1 + t_i Q_2)(x, y, z) \;=\; \mathrm{resultant}_t\big(f(t),\, (Q_1 + tQ_2)(x, y, z) \big).$$

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Conclusion: The edge surface of a curve $C \subset \mathbb{R}^3$ can have multiple components even if $\bar{C} \subset \mathbb{CP}^3$ is smooth and irreducible.

Conjecture: At most one of these components is not a cone.

Trigonometric Curves

A *trigonometric polynomial* of degree d is an expression of the form

$$f(\theta) \;=\; \sum_{j=1}^{d/2} \alpha_j \cos(j\theta) \;+\; \sum_{j=1}^{d/2} \beta_j \sin(j\theta) \;+\; \gamma.$$

Here d is even. A *trigonometric space curve* of degree d is a curve parametrized by three such trigonometric polynomials:

$$C \;=\; \big\{ \big(f_1(\theta), f_2(\theta), f_3(\theta)\big) \in \mathbb{R}^3 \,:\, \theta \in [0, 2\pi] \big\}.$$

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For general $\alpha_j, \beta_j, \gamma \in \mathbb{R}$, the curve $\bar{C} \subset \mathbb{CP}^3$ is smooth and $g = 0$. We get a rational parametrization by the change of coordinates

$$\cos(\theta) \;=\; \frac{1-t^2}{1+t^2} \quad \text{and} \quad \sin(\theta) \;=\; \frac{2t}{1+t^2}.$$

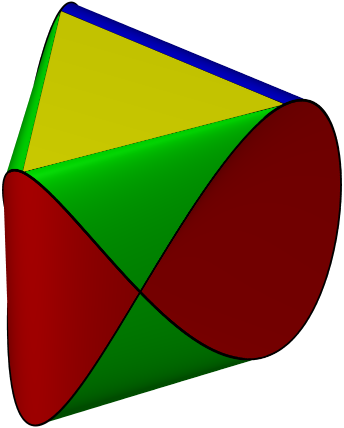
Substituting into the right hand side of the equation

$$\begin{pmatrix} \cos(j\theta) & \sin(j\theta) \\ -\sin(j\theta) & \cos(j\theta) \end{pmatrix} \;=\; \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}^{j},$$

this expresses $\cos(j\theta)$ and $\sin(j\theta)$ as rational functions in t.

Rational Sextic Curves

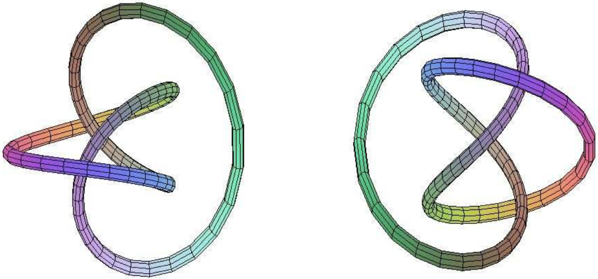
Fix $d = 6, g = 0$. The algebraic boundary of the convex hull of a *general* trigonometric curve of degree 6 consists of 8 tritangent planes and an irreducible edge surface of degree 30. For special curves, these degrees drop and the edge surface degenerates...



$$\mathrm{conv}\{(\cos(\theta), \cos(2\theta), \sin(3\theta)) : 0 \leq \theta \leq 2\pi\}$$

Morton's Curve

$$C \quad : \quad \theta \quad \mapsto \quad \frac{1}{2 - \sin(2\theta)} \big(\cos(3\theta), \sin(3\theta), \cos(2\theta) \big)$$



Freedman (1980) whether every knotted curve in $\mathbb{R}^3$ must have a tritangent plane. Morton (1991) showed that the answer is NO.

Curves with Singularities

Theorem. The edge surface of a general irreducible space curve of degree d, geometric genus g, with n ordinary nodes and k ordinary cusps, has degree $2(d-3)(d+g-1) - 2n - 2k$. The cone of bisecants through each cusp has degree $d-2$ and is a component of the edge surface.

Here the singularity is called *ordinary* if no plane in $\mathbb{CP}^3$ intersects the curve with multiplicity more than 4.

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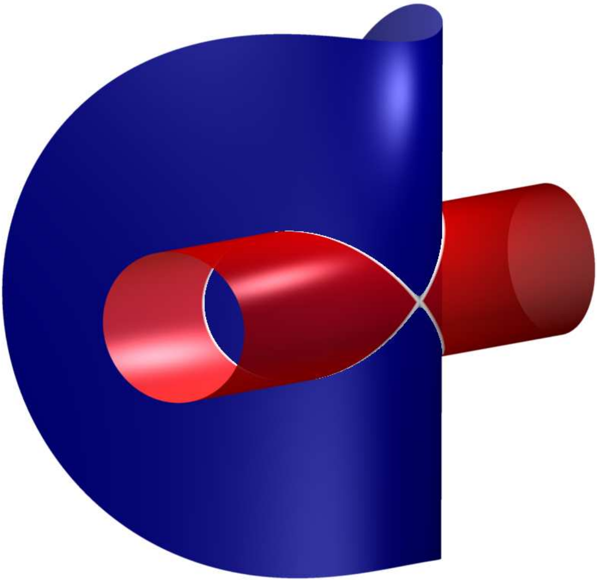
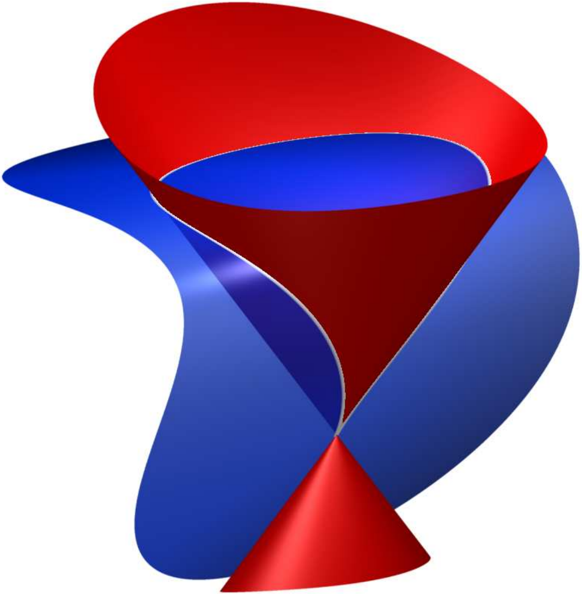
Example. $(d = 4, g = 0, n + k = 1)$
Consider a rational quartic curve with one ordinary singular point. The edge surface has degree 4. It is the union of two quadric cones whose intersection equals the curve. If the singularity is an ordinary cusp then one of the two quadrics has its vertex at the cusp.

Rational Quartic with an Ordinary Node



Rational Quartic with an Ordinary Cusp



Extension to Arbitrary Varieties

Let X be a compact real algebraic variety in $\mathbb{R}^n$, whose complexification $\bar{X} \subset \mathbb{CP}^n$ is smooth. For $k \in \mathbb{Z}_+$ let $X^{[k]}$ be the Zariski closure in $(\mathbb{CP}^n)^\vee$ of the set of all hyperplanes that are tangent to $\bar{X}$ at k regular points that span a $(k-1)$-flat. Thus $X^{[1]} = X^*$ is the *dual variety*. Consider the inclusions

$$X^{[n]} \subseteq \cdots \subseteq X^{[2]} \subseteq X^{[1]} \subseteq (\mathbb{CP}^n)^\vee.$$

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$$X^{[n]} \subseteq \cdots \subseteq X^{[2]} \subseteq X^{[1]} \subseteq (\mathbb{CP}^n)^\vee.$$

For small k, the dual variety $(X^{[k]})^*$ is the *k-th secant variety* of X. Let $r(X)$ be the minimal integer k such that the k-th secant variety of X has codimension ≤ 1. The condition $k \geq r(X) \geq \lceil \frac{n}{\dim(X)+1} \rceil$ is necessary for $(X^{[k]})^*$ to be a hypersurface.

Theorem

The algebraic boundary of the convex body $P = \mathrm{conv}(X)$ can be computed by projective biduality using the formula

$$\partial_a(P) \quad \subseteq \quad \bigcup_{k=r(X)}^{n} (X^{[k]})^*.$$

Curves Revisited

Plane Curves: If X is a non-convex curve in $\mathbb{R}^2$ of degree d then

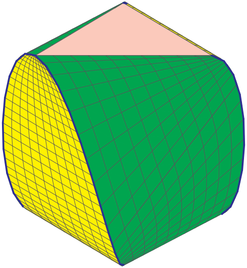
$$\partial_a(P) \;=\; (X^{[1]})^* \cup (X^{[2]})^* \;=\; X \;\cup\; (X^{[2]})^*.$$

For smooth X, the classical Plücker formulas determine the number of (complex) bitangent lines. Hence, $\partial_a(P)$ is a curve of degree

$$d + \deg(X^{[2]}) \;=\; d + \frac{(d-3)(d-2)\,d\,(d+3)}{2}.$$

Space Curves: If $n = 3$, $\dim(X) = 1$, and $r(X) = 2$ then

$$\partial_a(P) \;=\; (X^{[2]})^* \cup (X^{[3]})^*.$$



Surfaces in 3-Space

Let X be a general smooth compact surface in $\mathbb{R}^3$. Then

$$\partial_a(P) \;=\; (X^{[1]})^* \cup (X^{[2]})^* \cup (X^{[3]})^* \;=\; X \cup (X^{[2]})^* \cup (X^{[3]})^*,$$

Suppose $\deg(X) = d$. Following classical work by Salmon, Piene and Vainsencher (1970s) give the following formulas for the degree of the curve $X^{[2]}$, its dual surface $(X^{[2]})^*$, and the finite set $X^{[3]}$:

$$\begin{aligned}
\deg(X^{[2]}) &= \frac{d(d-1)(d-2)(d^3-d^2+d-12)}{2},\\
\deg\big((X^{[2]})^*\big) &= d(d-2)(d-3)(d^2+2d-4),
\end{aligned}$$

$$\begin{aligned}
\deg(X^{[3]}) &= \deg\big((X^{[3]})^*\big) = \\
&= \frac{d^9-6d^8+15d^7-59d^6+204d^5-339d^4+770d^3-2056d^2+1920d}{6}
\end{aligned}$$

Of course, the degree of $\partial_a(X)$ is much smaller for singular X ...

A Sextic Surface From Herwig Hauser's Gallery

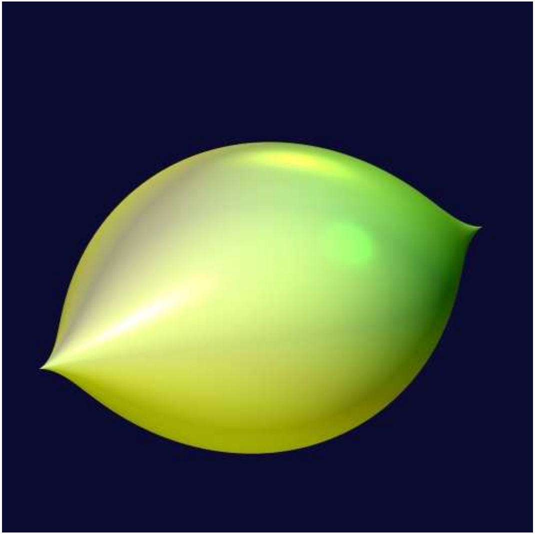


Figure: The Zitrus surface $x^2 + y^2 + (z^2 - 1)^3 = 0$

THE END: Four Pairwise Touching Circles

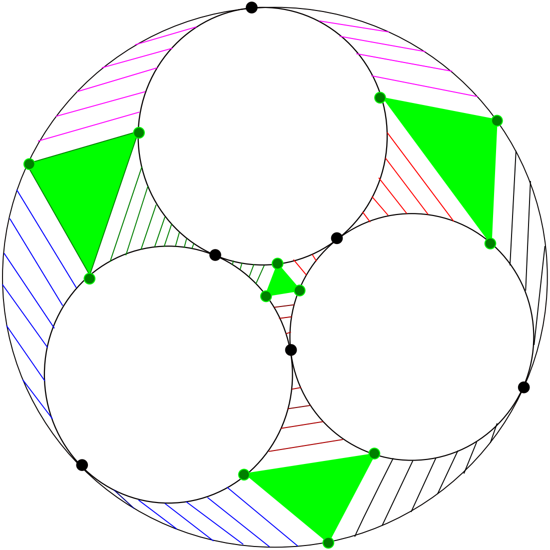


Figure: Schlegel diagram of the convex hull of 4 pairwise touching circles