NOTES ON MEASURE AND INTEGRATION IN LOCALLY COMPACT SPACES

William Arveson

Department of Mathematics
University of California
Berkeley, CA 94720 USA

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Abstract. This is a set of lecture notes which present an economical development of measure theory and integration in locally compact Hausdorff spaces. We have tried to illuminate the more difficult parts of the subject. The Riesz-Markov theorem is established in a form convenient for applications in modern analysis, including Haar measure on locally compact groups or weights on C^*-algebras...though applications are not taken up here. The reader should have some knowledge of basic measure theory, through outer measures and Carathéodory's extension theorem.

Contents

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Introduction

At Berkeley the material of the title is taught in Mathematics 202B, and that discussion normally culminates in some form of the Riesz-Markov theorem. The proof of the latter can be fairly straightforward or fairly difficult, depending on the generality in which it is formulated. One can eliminate the most serious difficulties by limiting the discussion to spaces which are compact or σ-compact, but then one must still deal with differences between Baire sets and Borel sets; one can eliminate all of the difficulty by limiting the discussion to second countable spaces. I have taken both shortcuts myself, but have not been satisfied with the result.

These notes present an approach to the general theory of integration on locally compact spaces that is based on Radon measures. My own experience in presenting alternate approaches has convinced me that Radon measures are the most sensible way to reduce the arbitrariness and the bother involved with doing measure theory in these spaces. We prove the Riesz-Markov theorem in general, in a form appropriate for constructing Haar measure on locally compact groups or for dealing with weights on commutative C^*-algebras.

If I have neglected to mention significant references in the bibliography it is partly because these lecture notes have been dashed off in haste. I apologize to any of my colleagues who may have been abused or offended, in that order.

Finally, I want to thank Cal Moore for pointing out an error in the proof of Proposition 2.1 (the present version has been fixed) and Bob Solovay for supplying the idea behind the example preceding Proposition 1.2.

1. The trouble with Borel sets

Throughout these notes, X will denote a locally compact Hausdorff space. A *Borel* set is a subset of X belonging to the σ-algebra generated by the closed sets of X. A *Baire* set is an element belonging to the σ-algebra generated by the compact G_δs...that is, compact sets K having the form

$$K = U_1 \cap U_2 \cap \dots,$$

where $U_1, U_2, \dots$ is a sequence of open sets in X. We will write $\mathcal{B}$ (resp. $\mathcal{B}_0$) for the σ-algebra of all Borel (resp. Baire) sets.

If the topology of X has a countable base then $\mathcal{B} = \mathcal{B}_0$. It is a good exercise to prove that assertion. In general, however, there is a difference between these two σ-algebras, even when X is compact. At the same time, each of them is more or less inevitable: $\mathcal{B}$ is associated with the topology of X and $\mathcal{B}_0$ is associated with the space $C_0(X)$ of continuous real-valued functions on X which vanish at ∞ (see Proposition 4.1). In these notes we deal mainly with Borel sets and Borel measures. The corresponding results for Baire sets and Baire measures are treated in §4.

The point we want to make is that the trouble with measure and integration in locally compact spaces has little to do with the fact that $\mathcal{B}$ and $\mathcal{B}_0$ are different, and a lot to do with the fact that X can be very large...i.e., very *non*-compact. And one needs the result in general if one wishes to discuss Haar measure on locally compact groups (even commutative ones), or weights on C^*-algebras (especially commutative ones).

In these notes I have taken the approach that I have come to prefer, in which *measure* means *Radon* measure. I have attempted to cast light on the pitfalls that can occur, to avoid verbosity in the mathematics, and especially I have tried to avoid the pitfalls I have stumbled through in the past.

A *Radon measure* is a positive Borel measure

$$\mu : \mathcal{B} \to [0, +\infty]$$

which is finite on compact sets and is *inner regular* in the sense that for every Borel set E we have

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, K \in \mathcal{K}\}$$

$\mathcal{K}$ denoting the family of all compact sets. There is a corresponding notion of outer regularity: a Borel measure μ is *outer regular* on a family $\mathcal{F}$ of Borel sets if for every $E \in \mathcal{F}$ we have

$$\mu(E) = \inf\{\mu(O) : O \supseteq E, O \in \mathcal{O}\},$$

$\mathcal{O}$ denoting the family of all open sets. The following result implies that when X is compact (or even σ-compact) one has the best of it, in that inner and outer regularity are equivalent properties. A set is called *bounded* if it is contained in some compact set, and *σ-bounded* if it is contained in a countable union of compact sets. Every σ-bounded Borel set can obviously be written as a countable union of bounded Borel sets.

Proposition 1.1. *Let μ be a Borel measure which is finite on compact sets. Then the following are equivalent.*

(1) *μ is outer regular on σ-bounded sets.*
(2) *μ is inner regular on σ-bounded sets.*

proof. (1) $\implies$ (2) Suppose first that E is a bounded Borel set, say $E \subseteq L$ where L is compact, and fix $\epsilon > 0$. We have to show that there is a compact set $K \subseteq E$ with $\mu(K) \geq \mu(E) - \epsilon$. But since the relative complement $L \setminus E$ is bounded, we see by outer regularity that there is an open set $O \supseteq L \setminus E$ such that

$$\mu(O) \leq \mu(L \setminus E) + \epsilon.$$

It follows that $K = L \setminus O = L \cap O^c$ is a compact subset of E satisfying

$$\mu(K) = \mu(L) - \mu(L \cap O) \geq \mu(L) - \mu(O) \geq \mu(L) - \mu(L \setminus E) - \epsilon = \mu(E) - \epsilon,$$

as required.

More generally, suppose that

$$E = E_1 \cup E_2 \cup \dots$$

is a countable union of bounded Borel sets E_n. Without loss of generality, we may assume that the sets E_n are disjoint. If some E_n has infinite measure, then by the preceding paragraph we have

$$\sup\{\mu(K) : K \subseteq E_n, K \in \mathcal{K}\} = \mu(E_n) = +\infty.$$

Hence

$$\sup\{\mu(K) : K \subseteq E, K \in \mathcal{K}\} = \mu(E) = +\infty,$$

and we are done. If, on the other hand, $\mu(E_n) < \infty$ for every n, then fixing $\epsilon > 0$ we may find a sequence of compact sets $K_n \subseteq E_n$ with

$$\mu(E_n) \leq \mu(K_n) + \epsilon/2^n.$$

Putting $L_n = K_1 \cup K_2 \cup \cdots \cup K_n$, it is clear that L_n is a compact subset of E for which

$$\mu(L_n) = \sum_{k=1}^{n} \mu(K_k) \geq \sum_{k=1}^{n} (\mu(E_k) - \epsilon/2^k) \geq \sum_{k=1}^{n} \mu(E_k) - \epsilon.$$

Taking the supremum over all n we obtain

$$\sup \mu(L_n) \geq \mu(E) - \epsilon,$$

from which inner regularity follows.

(2) $\implies$ (1) Suppose first that E is a bounded Borel set. Then the closure $\overline{E}$ of E is compact, and a simple covering argument implies that there is a *bounded* open set U such that $\overline{E} \subseteq U$. Set $L = \overline{U}$, and fix $\epsilon > 0$. Then $L \setminus E$ is a bounded Borel set, hence by inner regularity there is a compact set $K \subseteq L \setminus E$ with

$$\mu(K) \geq \mu(L \setminus E) - \epsilon.$$

Put $V = U \setminus K = U \cap K^c$. V is a bounded open set which contains E, and we have

$$\mu(V) = \mu(U \cap K^c) \leq \mu(L \cap K^c) = \mu(L) - \mu(K) \leq \mu(L) - (\mu(L \setminus E) - \epsilon) = \mu(E) - \epsilon.$$

Since ϵ is arbitrary, this shows that μ is outer regular on bounded sets.

In general, suppose $E = \cup_n E_n$, where each E_n is a bounded Borel set. Again, we may assume that the sets E_n are mutually disjoint. Since the assertion 1.1 (1) is trivial when $\mu(E) = +\infty$ we may assume $\mu(E) < +\infty$, and hence $\mu(E_n) < +\infty$ for every n. Fix $\epsilon > 0$. By the preceding paragraph we may find a sequence of open sets $O_n \supseteq E_n$ such that

$$\mu(O_n) \leq \mu(E_n) + \epsilon/2^n.$$

Then E is contained in the union $O = \cup_n O_n$, and we have

$$\mu(O) \leq \sum_n \mu(O_n) \leq \sum_n \mu(E_n) + \epsilon = \mu(E) + \epsilon$$

as required. □

We emphasize that inner and outer regularity are not equivalent properties when X fails to be σ-compact. In order to discuss this phenomenon, we consider the family $\mathcal{R}$ of all σ-bounded Borel sets. Notice that $\mathcal{R}$ contains X if and only if X is σ-compact; and in that case we have $\mathcal{R} = \mathcal{B}$. But if X is not σ-compact then $\mathcal{R}$ is not a σ-algebra but merely a σ-*ring* of subsets of X.

More explicitly, a σ-ring is a nonvoid family $\mathcal{S}$ of subsets of X satisfying the conditions

$$E, F \in \mathcal{S} \implies E \setminus F \in \mathcal{S}$$
$$E_1, E_2, \cdots \in \mathcal{S} \implies \cup_n E_n \in \mathcal{S}.$$

There is a theory of measures defined on σ-rings that is parallel to and generalizes the theory of measures defined on σ-algebras. The 'σ-ring' approach to Baire measures was emphasized and popularized by Paul Halmos [1], who co-invented the name itself.

Remarks. We are now able to make some concrete observations about the degree of arbitrariness that accompanies measure theory in humongous spaces. Assume X is *not* σ-compact, let $\mathcal{R}$ be the σ-ring of all σ-bounded Borel sets and let $\mathcal{R}'$ denote the set of all complements of sets in $\mathcal{R}$,

$$\mathcal{R}' = \{E^c : E \in \mathcal{R}\}.$$

Then $\mathcal{R} \cap \mathcal{R}' = \emptyset$ and $\mathcal{R} \cup \mathcal{R}'$ is the σ-algebra generated by $\mathcal{R}$. This is a σ-algebra of Borel sets, but it is not $\mathcal{B}$ since it does not necessarily contain open sets or closed sets. In any event, we have a convenient partition of this σ-algebra into Borel sets which are either σ-bounded or co-σ-bounded.

It is natural to ask if a "reasonable" measure that is initially defined on $\mathcal{R}$ can be extended uniquely to a measure on the σ-algebra generated by $\mathcal{R}$. The answer is no. Indeed [2, Exercise 9, pp. 258-59] shows that a measure on $\mathcal{R}$ always has an extension but that extensions are not unique. Actually, there is a one-parameter family of extensions of any "reasonable" measure on $\mathcal{R}$. There is a smallest one (the "inner regular" extension) and a largest one (the "outer regular" extension), and there is an arbitrary positive constant involved with each of the others.

A bad apple. Big locally compact spaces can be pathological in subtle ways. For example, let S be an uncountable discrete space, let $\mathbb{R}$ be the Euclidean real line, and let

$$X = S \times \mathbb{R}.$$

X is a locally compact Hausdorff space, being the cartesian product of two such. For every subset $E \subseteq X$ and every $s \in S$, let $E_s \subseteq \mathbb{R}$ be the section of E defined by

$$E_s = \{x \in \mathbb{R} : (s, x) \in E\}.$$

It is easy to show that that E is open iff every section E_s is an open set in $\mathbb{R}$. Similarly, E is compact iff all but a finite number of sections of E are empty and all the remaining sections are compact subsets of $\mathbb{R}$.

It would seem reasonable to guess that if a set $E \subseteq X$ is "locally Borel" in the sense that its intersection with every compact set is a Borel set, then it must be a Borel set (many of us have been so fooled: see [2, Lemma 9, p. 334]). That guess is wrong, as the following example shows. Since a complete discussion of the example would require more information about the Baire hierarchy than we have at hand, we merely give enough details for a persistent reader to complete the argument.

We may as well take S to be the set of all countable ordinals. It follows from the above remarks that $E \cap K$ is a Borel set for every compact set K iff every section of E is a Borel set in $\mathbb{R}$. Here is an example of a non-Borel set $E \subseteq X$ having the latter property. For every countable ordinal $\omega \in S$, let E_ω be a Borel set in $\mathbb{R}$ which belongs to the ω^{th} Baire class but to no properly smaller Baire class. Define

$$E = \{(\omega, x) \in S \times \mathbb{R} : x \in E_\omega\}.$$

Clearly every section of E is a Borel set. To see that E is not a Borel set, suppose that it did belong to the σ-algebra $\mathcal{B}$ generated by the family $\mathcal{O}$ of all open sets of X. Then E would have to belong to some Baire class over $\mathcal{O}$, say to the ω_0^{th} Baire class. It is easy to see that this implies that every section E_λ must belong to the ω_0^{th} Baire class in the real line, contradicting our construction of the sections of E.

Finally, we point out that *finite* Radon measures behave as well as possible, even when the underlying space is huge:

Proposition 1.2. *Every finite Radon measure is both inner and outer regular.*

sketch of proof. Outer regularity of the measure on any Borel set follows from the inner regularity of the measure on the complement of the set, because the measure of any set is finite.

2. How to construct Radon measures

In this section we show how Radon measures can be constructed from certain simpler entities defined on the family $\mathcal{O}$ of all open subsets of X. Let

$$m : \mathcal{O} \to [0, +\infty]$$

be a function having the following properties

$$\text{(A)} \qquad \overline{U} \in \mathcal{K} \implies m(U) < +\infty$$

$$\text{(B)} \qquad U \subseteq V \implies m(U) \leq m(V)$$

$$\text{(C)} \qquad U_1, U_2, \dots \in \mathcal{U} \implies m(\cup_n U_n) \leq \sum_n m(U_n),$$

$$\text{(D)} \qquad U \cap V = \emptyset \implies m(U \cup V) = m(U) + m(V)$$

$$\text{(E)} \qquad m(U) = \sup\{m(V) : V \in \mathcal{O}, \overline{V} \subseteq U, \quad \overline{V} \in \mathcal{K}\}.$$

Notice that A and D together imply that $m(\emptyset) = 0$. If we *start* with a Radon measure μ on $\mathcal{B}$ and define m to be the restriction of μ to $\mathcal{O}$, then such an m obviously has properties A through D, and a simple argument establishes E as well. Conversely, we have

Proposition 2.1. *Any function m defined on the open sets which has properties A through E can be extended uniquely to a Radon measure defined on all Borel sets.*

proof. For uniqueness, let μ be a Radon measure which agrees with m on $\mathcal{O}$. Since Radon measures are obviously determined by their values on compact sets, it suffices to observe that for every compact set $K \subseteq X$, we have

$$\mu(K) = \inf\{m(U) : U \supseteq K, U \in \mathcal{O}\}.$$

Indeed, μ is inner regular by definition of Radon measures, and every compact set is obviously a bounded Borel set. Thus the assertion is an immediate consequence of the equivalence of conditions (1) and (2) in proposition 1.1.

Turning now to existence, we consider the set function μ^* defined on arbitrary subsets $A \subseteq X$ by

$$\mu^*(A) = \inf\{m(U) : U \supseteq A, U \in \mathcal{O}\}.$$

We claim first that μ^* is an outer measure, that is

$$\mu^*(\emptyset) = 0$$
$$A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$$
$$A_1, A_2, \dots \subseteq X \implies \mu^*(\cup_n A_n) \leq \sum_n \mu^*(A_n).$$

The first two properties are obvious. To prove the third, it is clear that we need only consider the case in which $\mu^*(A_n)$ is finite for every $n = 1, 2, \dots$. In that case, fix $\epsilon > 0$ and choose open sets $U_n \supseteq A_n$ with the property that

$$\mu^*(U_n) \leq \mu^*(A_n) + \epsilon/2^n$$

for every n. Then $\cup_n U_n$ is an open set containing $\cup_n A_n$, hence by property C above,

$$\mu^*(\cup_n A_n) \leq m(\cup_n U_n) \leq \sum_n m(U_n) \leq \sum_n \mu^*(A_n) + \epsilon,$$

and the claim follows from the fact that ϵ is arbitrary.

It is apparent from the definition of μ^* that $\mu^*(U) = m(U)$ if U is an open set. We claim next that every open set is measurable; that is, for each open set O we have

$$\mu^*(A) = \mu^*(A \cap O) + \mu^*(A \cap O^c), \tag{2.2}$$

for every subset $A \subseteq X$. To prove 2.2 it suffices to prove the inequality $\geq$, since the opposite one follows from the subadditivity of μ^*. For that, fix A and O. If $\mu^*(A) = +\infty$ then there is nothing to prove, so we may assume that $\mu^*(A)$ (and hence both $\mu^*(A \cap O)$ and $\mu^*(A \cap O^c)$) is finite. Fix $\epsilon > 0$ and choose an open set $U \supseteq A$ so that

$$m(U) \leq \mu^*(A) + \epsilon.$$

We will prove that

$$m(U) \geq m(U \cap O) + \mu^*(U \cap O^c). \tag{2.3}$$

Note that it suffices to prove 2.3, since

$$\mu^*(A \cap O) + \mu^*(A \cap O^c) \leq m(U \cap O) + \mu^*(U \cap O^c) \leq m(U) \leq \mu^*(A) + \epsilon,$$

and ϵ is arbitrary.

In order to prove 2.3, we use property E above to find an open set V whose closure $\overline{V}$ is a compact subset of $U \cap O$ and

$$m(U \cap O) \leq m(V) + \epsilon.$$

Notice that V and $U \cap \overline{V}^c$ are disjoint open sets which are both contained in U. Thus we have by properties B and D

$$m(V) + m(U \cap \overline{V}^c) = m(V \cup (U \cap \overline{V}^c)) \leq m(U).$$

Finally, since $U \cap O^c \subseteq U \cap \overline{V}^c$,

$$\begin{aligned} m(U \cap O) + \mu^*(U \cap O^c) &\leq m(V) + \epsilon + \mu^*(U \cap O^c) \\ &\leq m(V) + m(U \cap \overline{V}^c) + \epsilon \leq m(U) + \epsilon, \end{aligned}$$

and 2.3 follows because ϵ is arbitrary.

By Carathéodory's extension theorem [2, Chapter 12, §2] the restriction of μ^* to the σ-algebra of μ^*-measurable sets is a measure; hence the restriction of μ^* to the σ-algebra of Borel sets is a measure $\overline{\mu}$ satisfying $\overline{\mu}(O) = m(O)$ for every open set O.

Notice that $\overline{\mu}$ is finite on bounded sets by property A, and $\overline{\mu}$ is outer regular by the definition of μ^*. So if X is σ-compact, then $\overline{\mu}$ is already a Radon measure by proposition 1.1.

If X is not σ-compact then $\overline{\mu}$ is not necessarily a Radon measure, and we must modify it as follows. Let $\mathcal{M}$ denote the σ-ring of all σ-bounded Borel sets. We define a new set function $\underline{\mu}$ on Borel sets as follows:

$$\underline{\mu}(E) = \sup\{\overline{\mu}(B) : B \in \mathcal{M}, B \subseteq E\}.$$

It is a fact that $\underline{\mu}$ is countably additive. Granting that for a moment, notice that $\underline{\mu}$ is a Radon measure. Indeed, Proposition 1.1 implies that $\overline{\mu}(B) = \underline{\mu}(B)$ for every $B \in \mathcal{M}$, that $\overline{\mu}$ is inner regular on $\mathcal{M}$, and thus by its definition $\underline{\mu}$ must be inner regular on all Borel sets. We also have $\underline{\mu}(O) = m(O)$ for every open set O. To see that, fix O and choose an open set V whose closure is a compact subset of O. Then we have

$$m(O) = \overline{\mu}(O) \geq \underline{\mu}(O) \geq \underline{\mu}(V).$$

Since V is a σ-bounded open set we have $\underline{\mu}(V) = \overline{\mu}(V) = m(V)$, and hence

$$m(O) \geq \underline{\mu}(O) \geq m(V)$$

for all such V. After taking the sup over V and using property (E) above, we find that $m(O) = \underline{\mu}(O)$. We may conclude in this case that $\underline{\mu}$ is a Radon measure which agrees with m on open sets.

It remains to check that $\underline{\mu}$ is countably additive. For that, let $E_1, E_2, \dots$ be a sequence of mutually disjoint Borel sets, and let B be a σ-bounded subset of $\cup_n E_n$. Set $B_n = B \cap E_n$. Then $B_1, B_2, \dots$ are mutually disjoint σ-bounded Borel sets, hence

$$\overline{\mu}(B) = \sum_{n=1}^{\infty} \overline{\mu}(B_n) \leq \sum_{n=1}^{\infty} \underline{\mu}(E_n).$$

By taking the supremum over all such B we obtain

$$\underline{\mu}(\cup_n E_n) \leq \sum_{n=1}^{\infty} \underline{\mu}(E_n).$$

To prove the opposite inequality it suffices to consider the case in which $\underline{\mu}(E_n)$ is finite for every n. In this case, for each positive number ϵ we can find a σ-bounded Borel set $B_n \subseteq E_n$ such that

$$\overline{\mu}(B_n) \geq \underline{\mu}(E_n) - \epsilon/2^n.$$

By summing on n we obtain $\overline{\mu}(B) \geq \sum_n \underline{\mu}(E_n) - \epsilon$ where $B = \cup_n B_n$. Since B is a σ-bounded subset of $\cup_n E_n$ we find that

$$\underline{\mu}(\cup_n E_n) \geq \overline{\mu}(B) \geq \sum_{n=1}^{\infty} \underline{\mu}(E_n) - \epsilon,$$

and the countable additivity of $\underline{\mu}$ follows because ϵ is arbitrary. □

Remark. There are other routes to the construction of Borel measures which begin with a set function m defined on the family $\mathcal{K}$ of all *compact* sets. Such entities m are called *contents*, and are in a sense dual to set functions obeying the properties A–E that we have used. The interested reader is referred to [1, sections 53–54] and [2, chapter 13, section 3].

3. Measures and linear functionals

Let $C_c(X)$ be the space of all continuous functions $f : X \to \mathbb{R}$ which have *compact support* in the sense that the set $supp(f) = \overline{\{x \in X : f(x) \neq 0\}}$ is compact. $\operatorname{supp}(f)$ is called the *support* of the function f. $C_c(X)$ is an algebra of functions, and is in fact an *ideal* in the algebra $C(X)$ of all continuous functions $f : X \to \mathbb{R}$, in the sense that

$$f \in C_c(X),\ g \in C(X) \implies fg \in C_c(X).$$

If X is compact then $C_c(X) = C(X)$. If X is not compact, then $C_c(X)$ is sup-norm dense in the algebra $C_0(X)$ of all continuous real functions which vanish at ∞.

The Riesz-Markov theorem gives a useful and concrete description of positive linear functionals

$$\Lambda : C_c(X) \to \mathbb{R},$$

that is, linear functionals Λ which are *positive* in the sense that

$$f \geq 0 \implies \Lambda(f) \geq 0, \qquad f \in C_c(X).$$

For example, let μ be a Radon measure on $\mathcal{B}$. Since compact sets have finite μ-measure, it follows that every function in $C_c(X)$ belongs to $L^1(X, \mathcal{B}, \mu)$ and we can define $\Lambda : C_c(X) \to \mathbb{R}$ by

$$\Lambda(f) = \int_X f\, d\mu, \qquad f \in C_c(X). \tag{3.1}$$

Λ is a positive linear functional on $C_c(X)$. The following lemma shows how certain values of the measure μ can be recovered directly from Λ.

Lemma 3.2. *Suppose that μ and Λ are related by 3.1. Then for every open set $U \subseteq X$ we have*

$$\mu(U) = \sup\{\Lambda(f) : 0 \leq f \leq 1, f \in C_c(X), supp(f) \subseteq U\}.$$

proof. The inequality $\geq$ is clear from the fact that if $0 \leq f \leq 1$ and $\operatorname{supp}(f) \subseteq U$ then

$$\chi_U(x) \geq f(x) \qquad \text{for every } x \in X,$$

and after integrating this inequality we obtain

$$\mu(U) \geq \int_X f\, d\mu = \Lambda(f).$$

For the opposite inequality, let K be an arbitrary compact subset of U. We may find a bounded open set V satisfying

$$K \subseteq V \subseteq \overline{V} \subseteq U.$$

By Tietze's extension theorem, there is a continuous function f satisfying $0 \leq f \leq 1$, $f = 1$ on K, and $f = 0$ on the complement of V. Thus the support of f is contained in $\overline{V} \subseteq U$, and since $\chi_K \leq f$ we may integrate the latter inequality to obtain

$$\mu(K) \leq \Lambda(f) \leq \sup\{\Lambda(f) : 0 \leq f \leq 1, f \in C_c(X), \operatorname{supp}(f) \subseteq U\}.$$

The desired inequality follows from inner regularity after taking the sup over K. $\square$

The following theorem of Riesz and Markov asserts that 3.1 gives the the most general example of a positive linear functional on $C_c(X)$.

Theorem 3.3 (Riesz-Markov). *Let Λ be a positive linear functional on $C_c(X)$. Then there is a unique Radon measure μ such that*

$$\Lambda(f) = \int_X f\, d\mu, \qquad f \in C_c(X).$$

Remarks. I should point out that in spite of the fact that this formulation of the Riesz-Markov theorem is the one I happen to prefer, it is not the only reasonable one. See [1],[2] for others. The connection between linear functionals and Baire measures will be described in section 4 below.

proof of Theorem 3.3. The uniqueness of μ is a direct consequence of Lemma 3.2 and the results of section 1. Indeed, Lemma 3.2 implies that the values of μ on open sets are determined by the linear functional Λ, and by proposition 1.1 the value of μ on any compact set K obeys

$$\mu(K) = \inf\{\mu(U) : U \supseteq K, U \in \mathcal{O}\}.$$

Finally, since for an arbitrary Borel set E we have

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, K \in \mathcal{K}\},$$

it follows that the $\mu(E)$ is uniquely determined by the linear functional Λ.

For existence, we define a number $m(U) \in [0, +\infty]$ for every open set U by

$$m(U) = \sup\{\Lambda(f) : f \in C_c(X), 0 \leq f \leq 1, \operatorname{supp}(f) \subseteq U\}.$$

We will show first that m satisfies the hypostheses A–E of proposition 2.1, and hence defines a Radon measure $\mu : \mathcal{B} \to [0, +\infty]$ by way of

$$\mu(U) = m(U), \qquad U \in \mathcal{O}.$$

We then show that Λ is truly integration against this Radon measure μ. We will require the following result on the existence of partitions of unity.

Lemma 3.4. *Let $\{O_\alpha : \alpha \in I\}$ be an open cover of a compact subset $K \subseteq X$. Then there is a finite set $\phi_1, \phi_2, \ldots, \phi_n$ of real continuous functions on X and there is a finite subset $\alpha_1, \alpha_2 \ldots, \alpha_n \in I$ satisfying*

(i) $$0 \leq \phi_k \leq 1$$

(ii) $$supp(\phi_k) \subseteq O_{\alpha_k}$$

(iii) $$\sum_k \phi_k = 1 \quad on\ K.$$

This is a standard result whose proof can be found in [2, proposition 9.16]. Let us establish the properties A–E of section 2. For A, let U be a bounded open set. Since the closure of U is compact, a simple covering argument shows that we may find another bounded open set V which contains the closure of U. By Tietze's extension theorem, there is a continuous function $g : X \to \mathbb{R}$ such that

$$\begin{aligned} &0 \leq g \leq 1, \\ &g = 1 \quad \text{on } \overline{U} \\ &g = 0 \quad \text{on the complement of } V. \end{aligned}$$

Any function $f \in C_c(X)$ satisfying $0 \leq f \leq 1$ and $\operatorname{supp}(f) \subseteq U$ must also satisfy $0 \leq f \leq g$ and hence $\Lambda(f) \leq \Lambda(g)$. It follows that

$$m(U) = \sup\{\Lambda(f) : 0 \leq f \leq 1, f \in C_c(X)\} \leq \Lambda(g) < +\infty$$

and property A follows.

Property B is obvious. For property C, choose open sets $U_1, U_2, \ldots,$ put $U = \cup_n U_n$, and choose $f \in C_c(X)$ with $0 \leq f \leq 1$ and such that f is supported in U. By Lemma 3.4, we may find an integer n and continuous functions $\phi_1, \phi_2, \ldots, \phi_n$ taking values in the unit interval, such that

$$\begin{gathered} \operatorname{supp}(\phi_k) \subseteq U_k, \quad \text{and} \\ \phi_1 + \phi_2 + \ldots \phi_n = 1 \quad \textit{on } \operatorname{supp}(f). \end{gathered}$$

It follows that $f = \sum_k \phi_k f$, and hence

$$\Lambda(f) = \sum_{k=1}^{n} \Lambda(\phi_k f) \leq \sum_{k=1}^{n} m(U_k) \leq \sum_{k=1}^{\infty} m(U_k).$$

Property C follows by taking the supremum over all f in the preceding line.

To establish D we prove only the inequality $\geq$, since the opposite one is a consequence of C. Let U and V be disjoint open sets and let f and g be two functions in $C_c(X)$ satisfying $0 \leq f, g \leq 1$, $\operatorname{supp}(f) \subseteq U$, and $\operatorname{supp}(g) \subseteq V$. Since $U \cap V = \emptyset$ we have $0 \leq f + g \leq 1$ and $supp(f + g) \subseteq U \cup V$. Hence

$$\Lambda(f) + \Lambda(g) = \Lambda(f + g) \leq m(U \cup V),$$

and the inequality $m(U) + m(V) \leq m(U \cup V)$ follows after taking the supremum over f and g.

For property E, choose $f \in C_c(X)$ satisfing $0 \leq f \leq 1$ and $\operatorname{supp}(f) \subseteq U$. Another simple covering argument on the compact subset $\operatorname{supp}(f) \subseteq U$ shows that we can find an open set V having compact closure such that

$$\operatorname{supp}(f) \subseteq V \subset \overline{V} \subseteq U.$$

Since $\operatorname{supp}(f) \subseteq V$ we have

$$\Lambda(f) \leq m(V) \leq \sup\{m(V) : \overline{V} \subset U, \overline{V} \in \mathcal{K}\}$$

and now property E follows after taking the sup over f.

Using proposition 2.1 we may conclude that there is a Radon measure μ on $\mathcal{B}$ which agrees with m on open sets. It remains to show that

$$\Lambda(f) = \int_X f \, d\mu \tag{3.4}$$

for every $f \in C_c(X)$. Since both sides of 3.4 are linear in f and since $C_c(X)$ is spanned by its nonnegative functions, it suffices to establish 3.4 for the case where $0 \leq f \leq 1$. To this end, fix f and choose a positive number ϵ. We will exhibit a pair of simple Borel functions u, v having the following properties.

$$u \leq f \leq v \tag{3.5}$$

$$\int_X (v-u) \, d\mu \leq \epsilon \tag{3.6}$$

$$\int_X u \, d\mu \leq \Lambda(f) \leq \int_X v \, d\mu + \epsilon \tag{3.7}$$

Note that this will complete the proof. Indeed, we may integrate 3.5 to obtain

$$\int_X u \, d\mu \leq \int_X f \, d\mu \leq \int_X v \, d\mu$$

and since the integrals of u and v are within ϵ of each other by 3.6, we see from 3.7 and the preceding inequality that

$$|\Lambda(f) - \int_X f \, d\mu| \leq 2\epsilon.$$

The result follows since ϵ is arbitrary.

In order to construct u and v, fix a positive integer n and define a sequence of bounded open sets $\mathcal{O}_0 \supseteq \mathcal{O}_1 \supseteq \cdots \supseteq \mathcal{O}_n$ by

$$\mathcal{O}_k = \{x \in X : f(x) > k/n\},$$

for $k = 1, 2, \ldots, n$, and let $\mathcal{O}_0$ be any bounded open set which contains $\operatorname{supp}(f)$. Notice that $\mathcal{O}_n$ is empty and that the closure of $\mathcal{O}_k$ is contained in $\mathcal{O}_{k-1}$ for $k = 1, 2, \ldots, n$. Define u and v as follows

$$u = \frac{1}{n} \sum_{k=1}^{n} c_k$$

$$v = \frac{1}{n} \sum_{k=1}^{n} c_{k-1},$$

where c_k denotes the characteristic function of $\mathcal{O}_k$. We have $c_n = 0$ because $\mathcal{O}_n$ is empty, and it apparent that $0 \le u \le v$.

We will show that if n is sufficiently large then the conditions 3.5–3.7 are satisfied. For 3.5, notice that f, u, v all vanish outside $\mathcal{O}_0$, and that if $x \in \mathcal{O}_{k-1} \setminus \mathcal{O}_k$ for $k = 1, 2, \ldots, n$ then

$$u(x) = \frac{1}{n}\sum_{i=1}^{k-1} 1 = \frac{k-1}{n} < f(x) \le \frac{k}{n} = \frac{1}{n}\sum_{i=0}^{k-1} 1 = v(x),$$

which proves 3.5. For 3.6 we have $v - u = \frac{1}{n}c_0$; hence

$$\int (v-u)\, d\mu = \frac{1}{n}\mu(\mathcal{O}_0)$$

which is smaller than ϵ provided n is sufficiently large.

To prove 3.7 we employ a device from [1, §56, p. 246]. Define a sequence $\phi_1, \phi_2, \ldots, \phi_n$ of continuous functions by

$$\begin{aligned}\phi_k &= [(f - \frac{k-1}{n}) \vee 0] \wedge \frac{1}{n} \\ &= [(f - \frac{k-1}{n}) \wedge \frac{1}{n}] \vee 0.\end{aligned}$$

Each ϕ_k vanishes on the complement of $\mathcal{O}_0$, hence $\phi_k \in \mathcal{C}_c(X)$. Moreover, we have

$$\phi_k(x) = \begin{cases} 0 & \text{if } x \notin O_{k-1} \\ f(x) - \frac{k-1}{n} & \text{if } x \in O_{k-1} \setminus O_k \\ \frac{1}{n} & \text{if } x \in O_k. \end{cases} \tag{3.8}$$

Clearly, $0 \le \phi_k \le \frac{1}{n}$. Noting that for $x \in O_{k-1} \setminus O_k$,

$$\begin{aligned}&\phi_1(x) = \phi_2(x) = \cdots = \phi_{k-1}(x) = \frac{1}{n} \\ &\phi_k(x) = f(x) - \frac{k-1}{n} \\ &\phi_{k+1}(x) = \phi_{k+2}(x) = \cdots = \phi_n(x) = 0\end{aligned}$$

it follows that $f = \phi_1 + \phi_2 + \ldots \phi_n$.

For convenience, we define $O_{-1} = O_0$. We claim that for every $k = 1, 2, \ldots, n$ we have the inequalities

$$\frac{1}{n}\mu(O_k) \le \Lambda(\phi_k) \le \frac{1}{n}\mu(O_{k-2}), \tag{3.9}$$

To prove the first inequality, choose any function $g \in C_c(X)$ satisfying $0 \le g \le 1$ and $\mathrm{supp}(g) \subseteq O_k$. Since $n\phi_k(x) = 1$ for every $x \in O_k$ we have $0 \le g \le n\phi_k$, and hence

$$\Lambda(g) \le \Lambda(n\phi_k).$$

Taking the supremum over all such g gives

$$\mu(O_k) \leq \Lambda(n\phi_k) = n\Lambda(\phi_k),$$

and the inequality follows after division by n. For the second inequality, notice that for $k \geq 2$, the closed support of $n\phi_k$ is contained in $\overline{O}_{k-1} \subseteq O_{k-2}$. So by definition of $\mu(O_{k-2}) = m(O_{k-2})$ we have

$$n\Lambda(\phi_k) = \Lambda(n\phi_k) \leq \mu(O_{k-2})$$

and we obtain the second inequality after dividing by n. The case $k = 1$ follows similarly from the fact that the closed support of $n\phi_1$ is contained in O_0. Noting that

$$\int_X u\,d\mu = \frac{1}{n}\sum_{k=1}^{n}\mu(O_k) = \frac{1}{n}\sum_{k=1}^{n-1}\mu(O_k), \qquad \text{and}$$
$$\int_X v\,d\mu = \frac{1}{n}\sum_{k=1}^{n}\mu(O_{k-1}) = \frac{1}{n}\sum_{k=0}^{n-1}\mu(O_k)$$

we may sum the inequalities 3.9 from $k = 1$ to n and use $\Lambda(f) = \sum_k \Lambda(\phi_k)$ to obtain

$$\int_X u\,d\mu \leq \Lambda(f) \leq \int_X v\,d\mu + \frac{1}{n}\mu(O_0).$$

Since the last term on the right is less than ϵ when n is large, 3.7 follows. $\square$

4. Baire meets Borel

We have already pointed out that the Borel σ-algebra $\mathcal{B}$ is natural because it is the σ-algebra generated by the topology of X. The following result implies that the Baire σ algebra $\mathcal{B}_0$ has just as strong a claim to inevitability.

Proposition 4.1. *$\mathcal{B}_0$ is the smallest σ-algebra with respect to which the functions in $C_c(X)$ are measurable.*

proof. We show first that every function f in $C_c(X)$ is $\mathcal{B}_0$-measurable. Since the $\mathcal{B}_0$-measurable functions are a vector space and since every function in $C_c(X)$ is a difference of nonnegatives ones, we may assume that $f \geq 0$. Fix $t \in \mathbb{R}$ and consider the set

$$F_t = \{x \in X : f(x) \geq t\}.$$

If $t \leq 0$ then $F_t = X$ belongs to $\mathcal{B}_0$. If $t > 0$ then

$$F_t = \cap_{n=2}^{\infty}\{x \in X : f(x) > t - t/n\}$$

is exhibited as a compact G_δ, hence $F_t \in \mathcal{B}_0$.

Conversely, if $\mathcal{A}$ is any σ-algebra with the property that every function in $C_c(X)$ is $\mathcal{A}$-measurable, then we claim that $\mathcal{A}$ contains every compact G_δ, and hence $\mathcal{A}$ contains $\mathcal{B}_0$. For that, let K be a compact set having the form

$$K = U_1 \cap U_2 \cap \dots$$

where the sets U_n are open. By replacing each U_n with a smaller open set if necessary, we can assume that each U_n is bounded, and hence has compact closure. By Tietze's extension theorem there are continuous functions $f_n : X \to [0, 1]$ with the properties $f_n = 1$ on K and $f_n = 0$ on the complement of U_n. f_n belongs to $C_c(X)$ and is therefore $\mathcal{A}$-measurable. Finally, since for each $x \in X$ we have

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } x \notin K, \end{cases}$$

it follows that the characteristic function of K, being a pointwise limit of a sequence of $\mathcal{A}$-measurable functions, is $\mathcal{A}$-measurable. Hence $K \in \mathcal{A}$. □

Remark. Since the closure of $C_c(X)$ in the sup norm

$$\|f\| = \sup\{|f(x)| : x \in X\}$$

is the algebra $C_0(X)$ of all continuous real-valued functions which vanish at ∞, we see that $\mathcal{B}_0$ could also have been defined as the smallest σ-algebra with respect to which the functions in $C_0(X)$ are measurable.

Perhaps the most compelling feature of Baire measures is that *regularity comes for free on σ-bounded sets.* In order to discuss this result it will be convenient to introduce some notation. $\mathcal{K}_0$ will denote the family of all compact G_δs, and $\mathcal{O}_0$ will denote the family of all open *Baire* sets. A Baire measure $\mu : \mathcal{B}_0 \to [0, +\infty]$ is called *inner regular* on a family $\mathcal{F}$ of Baire sets if for every $F \in \mathcal{F}$ we have

$$\mu(F) = \sup\{\mu(K) : K \subseteq F, K \in \mathcal{K}_0\}.$$

Similarly, μ is *outer regular* on $\mathcal{F}$ if for every $F \in \mathcal{F}$ we have

$$\mu(F) = \inf\{\mu(O) : O \supseteq F, O \in \mathcal{O}_0\}.$$

4.2 Separation properties of $\mathcal{K}_0$ and $\mathcal{O}_0$. There are enough open Baire sets to form a base for the topology on X; more generally, given any compact set K and an open set U containing K, there exist sets $K_0 \in \mathcal{K}_0$ and $U_0 \in \mathcal{O}_0$ such that

$$K \subseteq K_0 \subseteq U_0 \subseteq U,$$

see [1, Theorem D, §50]. Indeed, by replacing U with a smaller open set having compact closure, if necessary, we see that one can even choose U_0 to be a *bounded* open Baire set. That is about all one can say about open Baire sets in general. On the other hand, the only compact Baire sets are the obvious ones, namely the compact G_δs. The proof of the latter is not so easy, and we shall not require the result the sequel. The reader is referred to [1, Theorem A, §51] for a proof.

Proposition 4.3. *Let μ be a Baire measure which is finite on compact G_δs. Then μ is both inner regular and outer regular on σ-bounded Baire sets.*

proof. We will show that for every σ-bounded Baire set E one has both properties

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, K \in \mathcal{K}_0\}, \tag{4.3.A}$$

$$\mu(E) = \inf\{\mu(U) : U \supseteq E, U \in \mathcal{O}_0\}. \tag{4.3.B}$$

To this end, we claim that for every Baire set E and every $K \in \mathcal{K}_0$, the intersection $K \cap E$ satisfies both 4.3.A and 4.3.B. Indeed, let $\mathcal{A}$ denote the family of all Baire sets E for which this assertion is true. It suffices to show that $\mathcal{A}$ contains $\mathcal{K}_0$ and is a σ-algebra.

To see that $\mathcal{A}$ contains $\mathcal{K}_0$, choose $E \in \mathcal{K}_0$. The assertion 4.3.A is trivial because E itself belongs to $\mathcal{K}_0$. In order to prove 4.3.B, let $K \in \mathcal{K}_0$. Since $K \cap E$ is a compact G_δ we may find *bounded* open Baire sets U_n such that

$$K \cap E = U_1 \cap U_2 \cap \dots .$$

By replacing U_n with $U_1 \cap U_2 \cap \dots \cap U_n$ we can assume that $U_1 \supseteq U_2 \supseteq \dots$. For each n we have $\mu(U_n) < \infty$ because U_n is bounded, and thus 4.3.B follows from upper continuity of μ:

$$\mu(K \cap E) = \lim_{n\to\infty} \mu(U_n).$$

In order to show that $\mathcal{A}$ is a σ-algebra we have to show that $\mathcal{A}$ is closed under complementation and countable unions. We show first that $\mathcal{A}$ is closed under complementation. Choose $E \in \mathcal{A}$, fix $K \in \mathcal{K}_0$ and $\epsilon > 0$. Since $E \in \mathcal{A}$ there are sets $L \in \mathcal{K}_0$ and $U, V \in \mathcal{O}_0$ such that

$$\begin{aligned} L \subseteq K \cap E \subseteq U, \\ K \subseteq V \end{aligned}$$

and such that both $\mu(U \setminus L)$ and $\mu(V \setminus K)$ are smaller than ϵ. Using the remarks 4.2 we may assume that both U and V are bounded, by replacing them with smaller ones if necessary. Define sets A, B by

$$A = K \cap U^c, \quad B = V \cap L^c.$$

A belongs to $\mathcal{K}_0$, B belongs to $\mathcal{O}_0$, and we have

$$A \subseteq K \cap E \subseteq B.$$

Both A and B are bounded sets, and therefore have finite measure. Moreover, since

$$\mu(B) = \mu(V \setminus L) = \mu(V \setminus K) + \mu(K \setminus L)$$

and since

$$\mu(A) = \mu(K \setminus U) = \mu(K) - \mu(K \cap U) \geq \mu(K) - \mu(U)$$

we have

$$\mu(B) - \mu(A) \leq \mu(V \setminus K) + \mu(K \setminus L) - \mu(K) + \mu(U) = \mu(V \setminus K) + (\mu(U) - \mu(L)) \leq 2\epsilon.$$

Since ϵ is arbitrary, it follows that E^c belongs to $\mathcal{A}$.

We claim now that $\mathcal{A}$ is closed under countable unions. Choose $E_1, E_2, \dots \in \mathcal{A}$, fix $K \in \mathcal{K}_0$ and $\epsilon > 0$. Because 4.3.A and 4.3.B are valid for $K \cap E_n$ for every n, we may find $K_n \in \mathcal{K}_0$, and bounded sets $U_n \in \mathcal{O}_0$ such that

$$K_n \subseteq K \cap E_n \subseteq U_n$$

and for which

$$\mu(U_n) - \mu(K_n) \leq \epsilon/2^n.$$

Put $U = \cup_n U_n$, $L_n = K_1 \cup K_2 \cup \cdots \cup K_n$. Then $L_n \in \mathcal{K}_0, U \in \mathcal{O}_0$ and we have $L_n \subseteq K \cap E \subseteq U$. Moreover,

$$\mu(U) - \mu(\cup_n L_n) \leq \sum_{n=1}^{\infty} (\mu(U_n) - \mu(K_n)) \leq \epsilon.$$

Since the sets L_n increase to $\cup_n L_n$ as $n \to \infty$ and since $\cup_n L_n \subseteq K \cap E$ has finite measure, it follows that $\mu(L_n) \to \mu(\cup_n L_n)$. Hence the difference $\mu(U) - \mu(L_n)$ is smaller than 2ϵ when n is sufficiently large. Since ϵ is arbitrary, we conclude that $\cup_n E_n \in \mathcal{A}$.

Thus, $\mathcal{A}$ contains all Baire sets. Now let E be any σ-bounded Baire set, say

$$E \subseteq \cup_n K_n$$

where K_n is compact. Then by the remarks 4.2 we may assume that $K_n \in \mathcal{K}_0$ by slightly enlarging each K_n. Hence E is itself a countable union of sets

$$E = \cup_n (K_n \cap E)$$

each of which is, by what has already been proved, a Baire set of finite measure which is both inner and outer regular. It is easy to see that this implies E is both inner and outer regular. Indeed, choose $\epsilon > 0$. For each n we find $L_n \in \mathcal{K}_0$ and a bounded set $U_n \in \mathcal{O}_0$ such that

$$L_n \subseteq K_n \cap E \subset U_n$$

and for which

$$\mu(U_n) - \mu(K_n) \leq \epsilon/2^n.$$

Putting $L = \cup_n L_n$, and $U = \cup_n U_n$ we have

$$L \subseteq E \subseteq U$$

and by estimating as we have done above we also have

$$\mu(U \setminus L) \leq \epsilon.$$

If $\mu(E)$ is infinite then so is $\mu(U)$ and the preceding inequality implies $\mu(L) = +\infty$. By lower continuity of μ,

$$\lim_{n\to\infty} \mu(L_1 \cup L_2 \cup \cdots \cup L_n) = \mu(L) = +\infty.$$

Since $L_1 \cup L_2 \cup \cdots \cup L_n$ is a compact G_δ subset of E, this establishes inner and outer regularity at E in this case. If $\mu(E) < +\infty$, then the preceding inequality implies that both $\mu(L)$ and $\mu(U)$ are within ϵ of $\mu(E)$. Since

$$\mu(L) = \lim_{n\to\infty} \mu(L_1 \cup L_2 \cup \cdots \cup L_n)$$

and since $L_1 \cup L_2 \cup \cdots \cup L_n$ is a $\mathcal{K}_0$-subset of E for each n, we conclude that E is both inner and outer regular in this case as well. $\square$

Corollary. *Every finite Baire measure on a compact Hausdorff space is both inner regular and outer regular.*

Finally, it is significant that the correspondence between Radon measures (defined on $\mathcal{B}$) and Baire measures (defined on $\mathcal{B}_0$) is bijective. However, even here one must be careful in the formulation if X is not σ-compact. More precisely, if one restricts a Radon measure μ to $\mathcal{B}_0$ then one obtains a Baire measure which is finite on compact sets. However, the inner regularity of Radon measures on $\mathcal{B}$ does not immediately imply that their restrictions to $\mathcal{B}_0$ are inner regular Baire measures as defined in the paragraphs preceding Proposition 4.3. The problem is that a given Baire set may have many more compact subsets than it has compact G_δ subsets.

Nevertheless, Proposition 4.3 implies that the restriction of a Radon measure on $\mathcal{B}$ to the σ-ring $\mathcal{R}_0$ of σ-bounded Baire sets *is* inner regular. If X is σ-compact, then this restriction is already a regular Baire measure in both the inner and outer senses. But if X is not σ-compact then in order to obtain an inner regular Baire measure we must first restrict the Radon measure to the σ-ring $\mathcal{R}_0$ and then use the latter to define an inner regular Baire measure on the full Baire σ-algebra much as we did in the proof of Proposition 2.1. After these shenanigans one can say that every Radon measure "restricts" to an inner regular Baire measure in general. The following asserts that this map is a bijection.

Proposition 4.4. *Let μ be an inner regular Baire measure which is finite on compact Baire sets. Then μ extends uniquely to a Radon measure on $\mathcal{B}$.*

proof. For the existence of a Radon extension of μ we note that since functions in $C_c(X)$ are Baire measurable and μ-integerable, we may define a positive linear functional Λ on $C_c(X)$ by

$$\Lambda(f) = \int_X f\,d\mu.$$

By Theorem 3.3 there is a Radon measure ν such that

$$\int_X f\,d\nu = \Lambda(f) = \int_X f\,d\mu, \qquad f \in C_c(X).$$

In order to show that μ is the "restriction" of ν as described in the preceding discussion, let K be any compact G_δ. Notice that there is a sequence of functions $f_n \in C_c(X)$ such that

$$f_n(x) \downarrow \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } x \notin K. \end{cases}$$

Indeed, we can write $K = \cap U_n$ where each U_n is an open set having compact closure. Choosing a continuous function g_n taking values in $[0,1]$ such that $g_n = 1$ on K and $g_n = 0$ on the complement of U_n, we may take $f_n = g_1 \wedge g_2 \wedge \cdots \wedge g_n$. By the monotone convergence theorem we have

$$\nu(K) = \lim_n \int_X f_n\,d\nu = \lim_n \int_X f_n\,d\mu = \mu(K),$$

and the desired conclusion follows from the inner regularity of μ on $\mathcal{B}_0$.

For uniqueness, suppose that ν_1 and ν_2 are two Radon measures which extend μ in the sense described above. Notice that for every $f \in C_c(X)$ satisfying $f \geq 0$, the value of the integral

$$\int_X f \, d\nu_1$$

is entirely determined by the values of the function $F(t) = \nu_1(\{x \in X : f(x) \geq t\})$ for $t > 0$; i.e., by the values of $\nu_1 \restriction_{\mathcal{K}_0} = \mu \restriction_{\mathcal{K}_0}$. The same applies to ν_2, hence

$$\int_X f \, d\nu_1 = \int_X f \, d\mu = \int_X f \, d\nu_2.$$

Hence $\nu_1 = \nu_2$ by the uniqueness assertion of 3.3. $\square$

Corollary. *If X is compact, then every finite Baire measure extends uniquely to a measure on $\mathcal{B}$ which is both inner and outer regular.*

There are two possible reformulations of the Riesz-Markov theorem in terms of Baire measures; the proofs follow from the preceding discussion.

Theorem 4.5. *For every positive linear functional Λ defined on $C_c(X)$ there is a unique measure μ defined on the σ-ring of all σ-bounded Baire sets (resp. a unique inner regular Baire measure μ) such that*

$$\Lambda(f) = \int_X f \, d\mu, \qquad f \in C_c(X).$$

5. The dual of $C_0(X)$

We now discuss one of the useful consequences of the Riesz-Markov theorem. $C_0(X)$ will denote the space of all real-valued continuous functions f on X which vanish at ∞ in the sense that for every $\epsilon > 0$ the set $\{x \in X : |f(x)| \geq \epsilon\}$ is compact. $C_0(X)$ is a real algebra in that it is closed under the usual linear operations and pointwise multiplication. The norm

$$\|f\| = \sup_{x \in X} |f(x)|$$

makes $C_0(X)$ into a Banach space in which

$$\|fg\| \leq \|f\| \cdot \|g\|.$$

We will make essential use of the natural order $\leq$ on elements of $C_0(X)$, defined by $f \leq g \iff f(x) \leq g(x)$ for every $x \in X$. This ordering makes $C_0(X)$ into a lattice, and the lattice operations can be defined pointwise by

$$f \vee g\,(x) = max(f(x), g(x)),$$
$$f \wedge g\,(x) = min(f(x), g(x)), \qquad x \in X.$$

Finally, $\mathcal{P}$ will denote the cone of all positive functions,

$$\mathcal{P} = \{f \in C_0(X) : f \geq 0\}.$$

Notice that $\mathcal{P} \cap -\mathcal{P} = \{0\}$ and $\mathcal{P} - \mathcal{P} = C_0(X)$.

There is a natural ordering induced on the dual $C_0(X)^*$ by the ordering on $C_0(X)$, namely

$$\rho \leq \sigma \iff \rho(f) \leq \sigma(f), \qquad \text{for every } f \in \mathcal{P},$$

and a linear functional ρ is called *positive* if $\rho \geq 0$.

Proposition 5.1. *Every positive linear functional on $C_0(X)$ is bounded. Moreover, for every $\rho \in C_0(X)^*$ there is a smallest positive linear functional Λ such that $\rho \leq \Lambda$.*

Remarks. The second assertion means that if Λ' is another positive linear functional satisfying $\rho \leq \Lambda'$, then $\Lambda \leq \Lambda'$. After a simple argument (which we omit), this implies that the dual of $C_0(X)$ is itself a lattice with respect to the ordering defined above; and of course $\Lambda = \rho \vee 0$.

proof. To establish the first assertion let Λ be a positive linear functional. Since every element in the unit ball of $C_0(X)$ is a difference of positive functions in the unit ball of $C_0(X)$, to show that Λ is bounded it suffices to show that

$$\sup\{\Lambda(f) : f \in \mathcal{P}, \|f\| \leq 1\} < +\infty.$$

But if this supremum is infinite then we can find a sequence $f_n \in \mathcal{P}$ satisfying $\|f_n\| \leq 1$ and $\Lambda(f_n) > 2^n$. Letting $g \in \mathcal{P}$ be the function defined by the absolutely convergent series

$$g = \sum_{k=1}^{\infty} 2^{-k} f_k,$$

we have $g \geq \sum_{k=1}^{n} 2^{-k} f_k$ for every n, hence

$$\Lambda(g) \geq \Lambda(\sum_{k=1}^{n} 2^{-k} f_k) = \sum_{k=1}^{n} 2^{-k} \Lambda(f_k) > n.$$

The latter is absurd for large n.

To prove the second assertion choose $\rho \in C_0(X)^*$. For every $f \in \mathcal{P}$ we define a nonnegative number $\Lambda_0(f)$ by

$$\Lambda_0(f) = \sup\{\rho(u) : 0 \leq u \leq f\}.$$

Clearly $\Lambda_0(\alpha f) = \alpha \Lambda_0(f)$ for every nonnegative scalar α and every $f \in \mathcal{P}$. We claim that for all $f, g \in \mathcal{P}$,

$$\Lambda_0(f + g) = \Lambda_0(f) + \Lambda_0(g).$$

The inequality $\geq$ follows from the fact that if $0 \leq u \leq f$ and $0 \leq v \leq g$ then $0 \leq u + v \leq f + g$, hence

$$\Lambda_0(f + g) \geq \rho(u + v) = \rho(u) + \rho(v).$$

$\geq$ follows after taking the sup over u and v. For the opposite inequality, fix $f, g \in \mathcal{P}$ and choose u satisfying $0 \leq u \leq f + g$. Now for each $x \in X$, $f \wedge u\,(x)$ is either $f(x)$ or $u(x)$, and in either case $u(x) \leq (f \wedge u)\,(x) + g(x)$. It follows that

$$0 \leq u - f \wedge u \leq g.$$

Therefore

$$\rho(u) - \rho(f \wedge u) = \rho(u - f \wedge u) \leq \Lambda_0(g),$$

and hence

$$\rho(u) \leq \rho(f \wedge u) + \Lambda_0(g) \leq \Lambda_0(f) + \Lambda_0(g).$$

The inequality $\leq$ now follows after taking the supremum over u.

Finally, we claim that Λ_0 extends uniquely to a linear functional Λ defined on all of $C_0(X)$. Given that, the remaining assertions of 5.1 follow; for if Λ' is another positive linear functional satisfying $\rho \leq \Lambda'$, then for every $f \in \mathcal{P}$ and every $0 \leq u \leq f$ we have

$$\rho(u) \leq \Lambda'(u) \leq \Lambda'(f),$$

and $\Lambda(f) \leq \Lambda'(f)$ follows from the preceding inequality after taking the sup over u. In order to extend Λ_0, choose an arbitrary $f \in C_0(X)$, write $f = f_1 - f_2$ in any way as the difference of two elements $f_k \in \mathcal{P}$, and define $\Lambda(f) = \Lambda_0(f_1) - \Lambda_0(f_2)$. The only question is whether or not this is well-defined; but that is clear from the fact that if $f_k, g_k \in \mathcal{P}$ and $f_1 - f_2 = g_1 - g_2$ then

$$\Lambda_0(f_1) + \Lambda_0(g_2) = \Lambda_0(f_1 + g_2) = \Lambda_0(g_1 + f_2) = \Lambda_0(g_1) + \Lambda_0(f_2),$$

hence $\Lambda_0(f_1) - \Lambda_0(f_2) = \Lambda_0(g_1) - \Lambda_0(g_2)$. The linearity of the extended Λ follows from the restricted linearity of the original Λ_0 on the positive cone. Uniqueness of the extension is obvious from the fact that $\mathcal{P} - \mathcal{P} = C_0(X)$ $\square$

Remarks. Let Λ be a positive linear functional on $C_0(X)$, and let $\|\Lambda\|$ be its norm. By the Riesz-Markov theorem there is a positive Radon measure μ such that

$$\Lambda(f) = \int_X f\, d\mu, \qquad f \in C_c(X).$$

Notice that if $f \in C_c(X)$ satisfies $0 \leq f \leq 1$, then

$$\int_X f\, d\mu \leq \Lambda(f) \leq \|\Lambda\|.$$

From this, we may deduce that for every compact set $K \subset X$ we have $\mu(K) \leq \|\Lambda\|$, and finally by inner regularity

$$\mu(X) \leq \|\Lambda\| < +\infty.$$

In particular, μ is a finite measure, and hence is both outer and inner regular (Proposition 1.2). It is easy to see that in fact $\mu(X) = \|\Lambda\|$.

More generally, we may deduce the following description of the dual of $C_0(X)$; this result (together with myriad other results which bear it no relation whatsoever) is (are) often called the Riesz Representation Theorem.

Theorem 5.2. *For every bounded linear functional ρ on $C_0(X)$ there is a finite signed Borel measure μ such that*

$$\rho(f) = \int_X f\, d\mu, \qquad f \in C_0(X).$$

proof. Note first that ρ can be decomposed into a difference $\Lambda_1 - \Lambda_2$ of positive linear functionals. Indeed, using 5.1 we may define $\Lambda_1 = \rho \vee 0$, and it is clear from the statement of 5.1 that the linear functional $\Lambda_2 = \Lambda_1 - \rho$ is positive. Thus the existence of μ follows from the Riesz-Markov theorem and 5.1 above. $\square$

Remarks. If one stipulates that the total variation measure $|\mu|$ of μ should be a Radon measure, then μ is unique, and moreover $|\mu|$ is outer regular as well as inner regular because it is finite. We omit the argument [2, Chapter 13, §5].

If we agree to define a signed Radon measure as a *finite* signed measure μ whose variation $|\mu| = \mu_+ + \mu_-$ is inner regular (and therefore also outer regular), then the vector space $M(X)$ of all signed Radon measures becomes a Banach space with norm

$$\| \mu \| = \sup \sum_k |\mu(E_k)|$$

the supremum being extended over all finite families $E_1, E_2, \dots, E_n$ of mutually disjoint Borel sets. Such a measure gives rise to a linear functional $\rho \in C_0(X)^*$ as in 5.2, and it is not hard to show that

$$\| \rho \| = \| \mu \|,$$

[2, Chapter 13, §5]. Thus the dual of $C_0(X)$ is naturally isometrically isomorphic to the Banach space $M(X)$ in such a way that the order structure on the dual of $C_0(X)$ corresponds to the natural ordering of signed measures.

References

1. Halmos, P. R., *Measure Theory*, Van Nostrand, Princeton, 1950.
2. Royden, H. L., *Real Analysis, third edition*, Macmillan, New York, 1988.