June 7, 2007

THE ISOMORPHISM PROBLEM FOR TORSION-FREE ABELIAN GROUPS IS ANALYTIC COMPLETE.

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Abstract. We prove that the isomorphism problem for torsion-free Abelian groups is as complicated as any isomorphism problem could be in terms of the analytical hierarchy, namely Σ^1_1 complete.

1. Introduction

This paper is concerned with the classification problem for countable torsion-free Abelian groups. The question we ask is, given two countable torsion-free Abelian groups, how hard is it to tell if they are isomorphic or not. We answer it from the viewpoint of Computability Theory, by showing that the isomorphism problem is Σ^1_1 complete. In other words, telling whether two countable torsion-free Abelian groups are isomorphic is as hard as it could be in the analytical hierarchy. We look at the complexity of the set of pairs of countable torsion-free Abelian groups which are isomorphic in two natural ways. One is to view this set as a class of reals (i.e. a set of infinite binary sequences coding the groups); and the set of natural numbers which are codes for pairs of computable torsion-free Abelian groups which are isomorphic (i.e. the isomorphism problem for recursively presentable torsion-free Abelian groups with solvable word problems).

Let us explain our result to the reader that is not familiar with these logic terms. (For background on complexity classes see Subsection 1.1 below.) To check whether two countable torsion-free Abelian groups G_1 and G_2 are isomorphic, the first idea one could have is to go through all the functions $f\colon G_1 \to G_2$ and see if any of these functions is actually an isomorphism. Given such a function f, checking whether it is an isomorphism or not, even though it cannot be done computably, is relatively simple; it is Π^0_2. However, going through all the continuum many possible functions $f\colon G_1 \to G_2$ is considerably harder. This is what makes the isomorphism problem Σ^1_1.

2000 *Mathematics Subject Classification.* 03D80 ,20F10.

Key words and phrases. torsion-free abelian groups, isomorphism problem, classification, complexity.

The first author's research was partially supported by The Marsden Found of New Zealand. The second author was partially supported by NSF Grant DMS-0600824 and by the the Marsden Found of New Zealand. Thanks to Professor C. F. Miller III, for advice on finitely presented groups and, in partciular, the Baumslag, Dyer and Miller material on the integral homology of finitely presented groups.

In some cases, one can find simpler ways to check whether two structures are isomorphic. This happens for example with torsion-free Abelian groups of finite rank n, (or equivalently, subgroups of $\mathbb{Q}^n$), where the isomorphism problem is Σ^0_3, much simpler than Σ^1_1. The reason is that to check isomorphism one has to find an n-tuple of elements in each group, say $\{g^1_1, ..., g^1_n\} \subseteq G_1$ and $\{g^2_1, ..., g^2_n\} \subseteq G_2$ satisfying the following two conditions: $\{g^1_1, ..., g^1_n\}$ generates G_1 using addition and division by integers and $\{g^2_1, ..., g^2_n\}$ generates G_2; and the function that maps one tuple to the other, namely $g^1_i \mapsto g^2_i$, can be extended to an isomorphism of the groups. Checking these two condition is again relatively simple (Π^0_2). Searching over all the possible n-tuples is not as hard as searching over all the functions $f\colon G_1 \to G_2$, because there are only countably many pairs of n-tuples and we can easily enumerate them.

Another case where it is easier to check for isomorphism is when when one of the two groups is fixed and easy to describe. For example, to check whether a torsion-free Abelian group G is isomorphic to $\mathbb{Q}^\infty$ (the group of sequences of rational numbers which are eventually 0) all we need to do is verify that every element of G is divisible and G has infinite rank. These are Π^0_2 and Π^0_3 questions respectively. Actually, it is not difficult to prove completeness here. We include a proof in Section 4.

We show that for the case of countable torsion-free Abelian groups we will not be able to avoid doing such a search though a whole set of functions with infinite countable domain and countable range. Moreover, we show that any other problem which requires such a search over a whole set of functions, can be reduced to the isomorphism problem for torsion-free Abelian groups.

We remark that a similar approach was taken by Slaman and Woodin [SW98] who used computational methods to show that partial orderings with dense extensions cannot have an reasonable characterization, as again the computable partial orderings with dense extensions formed a Σ^1_1 complete class.

Also, it is know that the isomorphism problem for p-groups is Σ^1_1 complete, as proved by Friedman and Stanley [FS89]. Therefore, isomorphism problem for the whole class of countable Abelian is already known to be Σ^1_1 complete. This has no implications about the class of torsion-free Abelian groups.

In the last few years, there has been a lot of work done on the classification problem for countable torsion-free Abelian groups from the view point of Descriptive Set Theory and some from Computability Theory. Methods from these areas allow us to attack questions like whether one classification is more involved than another, and whether there is any reasonable set of invariants available for classification. In Descriptive Set Theory, it was Friedman and Stanley [FS89] who started analyzing the complexity of the isomorphism problem between structures. In Computability Theory, Goncharov and Knight [GK02] and Calvert and Knight [CK06] studied possible ways of classifying structures and proving that structures are not classifiable.

For the case of torsion-free Abelian groups of finite rank, Hjorth, Kechris, Thomas and others attacked the question using Borel relations by showing that there is no Borel map which would transfer invariants from the rank $n+1$ to the rank n case [Tho]. This shows that a set of invariant for groups of rank n would necessarily get more and more complicated as n increases. From the view point of complexity classes, as we mentioned before, the isomorphism problem for torsion-free Abelian groups of rank n is Σ_3^0. Calvert [Cal05] showed that is actually Σ_3^0 complete. (He actually show that the set of pairs of indices of computable isomorphic torsion-free Abelian groups of rank n is a Σ_3^0 m-complete set of natural numbers.) For the general case of countable torsion-free Abelian groups of any rank, Greg Hjorth proved that isomorphism problem is not *Borel*, showing that it is indeed a complicated problem. Calvert [Cal05] modified Hjorth's proof and proved that the set of pairs of indices of computable isomorphic torsion-free Abelian groups is not a hyperarithmetic set of natural numbers. Our main results extend these.

Theorem 1.1. *The set of pairs of reals which correspond to isomorphic countable torsion-free abelian groups is Σ_1^1 complete.*

Theorem 1.2. *The set of pairs of indices of isomorphic computable torsion-free Abelian groups is an m-complete Σ_1^1 set of natural numbers.*

We prove these theorems using another well known Σ_1^1 problem, namely the problem of deciding whether a tree has an infinite path or not. We do it by defining a computable operator $G(\cdot)$ from trees to torsion-free Abelian groups which is well-defined on isomorphism types and such that trees with infinite paths are map to different groups than trees without infinite paths. The way we guarantee this last property is by showing that, for a specify group G_0 that we construct, we have that a tree T has an infinite path if and only if G_0 embeds in $G(T)$. As a corollary we get that the class of groups which contain a copy of G_0 is Σ_1^1-complete. The construction of this operator uses the idea of *eplag group* developed by Hjorth in [Hjo].

We actually prove a slightly stronger result than Theorem 1.2. We build a single computable torsion-free Abelian group such that the set of indices of computable groups which are isomorphic to it is Σ_1^1 complete. This implies, for example, that its Scott rank is either ω_1^{CK} or $\omega_1^{CK}+1$. What this says is that this group is very hard to describe, as opposed to, for instance, $\mathbb{Q}^\infty$, which is relatively simple to describe.

Theorem 1.2 is more natural to computability theorist than 1.1 because it talks about the complexity of a set of natural numbers rather than a set of reals. The restriction to computable groups is very natural. A group is *computable* if its domain and group operation are computable. The *index* of a computable group is the natural number that corresponds the pair of programs computing its domain and group operation, in some numbering of the pairs of programs. In Combinatorial Group Theory, these are the groups which can be presented with an effective set of generators and relations where the word problem is solvable. (Actually, we would only need to have an effective set of generators and relations, since Khisamiev [Khi86] showed that any such torsion-free Abelian group is isomorphic to one with a solvable word problem.) These groups arise very naturally classically. We call a group that is presented by an effective set of generators and and effective set of relations a *c.e. presented group*. For instance, as observed by Baumslag, Dyer and Miller [BDM83], the c.e. presented groups presented are exactly the groups that appear in integral homology sequences of finitely presented groups. Moreover, given any computable sequence $A_1, A_2, \ldots,$ of c.e. presented torsion-free Abelian groups, groups with the first two finitely generated, there exists a finitely presented group G whose integral homology is the given sequence. They also obtain this result when the groups $A_1, A_2, \ldots,$ are all computably presented. Observing that the construction in [BDM83] of G from the sequence $A_1, A_2, \ldots,$ is effective, at least in the case when all A_i are 0 except for one, we get the following corollaries of Theorem 1.2.

Corollary 1.3. *Deciding whether two finitely presented groups have the same homology sequence is Σ^1_1 m-complete.*

Corollary 1.4. *Deciding of two finitely presented groups G and K have $H_j(G) \cong H_j(K)$, for $j \leq 3$ is Σ^1_1 m-complete.*

These two corollaries also follow from Friedman and Stanley [FS89] result that the isomorphism problem for computable p-groups is also Σ^1_1 m-complete.

There are many other results in the literature saying that properties about finitely presented groups cannot be decided computably, or are Σ^0_1 m-complete, as for example the isomorphism problem, or even Π^0_2 m-complete, as for example being torsion-free (Lempp [Lem97]). But no other decision problem about finitely presented groups is known to be as high up as Σ^1_1 m-complete. (See [Mil92] for a survey on decision problems for finitely presented groups.)

A question that reminds open is whether the class of torsion-free Abelian groups is *Borel complete*. That is, if for any class of structures K there is a Borel operator from K to the class of torsion-free Abelian groups which is well-defined and one-to-one on isomorphism types. The notion of Borel Completeness was introduced by Friedman and Stanley [FS89]. In [FS89] they proved that if a class of structures is Borel complete classes, then its isomorphism problem is Σ^1_1 complete. They also show that the reversal of this statement is not true by showing that the class of p-groups is Σ^1_1 complete but not Borel complete. In that paper they leave the Borel completeness of the torsion-free Abelian groups as a main open question. They conjectured the positive answer.

1.1. **Background on Complexity Hierarchies.** Typical problems in, for instance, combinatorial group theory are *arithmetical* in that they can be extressed in relatively simple terms. They are usually of either Σ^0_n or Π^0_n form, with n relatively small. Here a set $A \subseteq \mathbb{N}$ is Σ^0_2, for instance, if there exists a computable relation R such that for all x, $x \in A$ iff $\exists y \forall z R(x, y, z)$ (where the quantification concerns individual numbers), and A is Π^0_2 iff the complement of A is Σ^0_2. For instance, deciding if a computable Abelian group is divisible is easily seen to be Π^0_2. The "n" in Σ^0_n (Π^0_n) refers to the number of alternations of quantifiers in the definition where Σ^0_n means n alternations beginning with an existential quantifier, and Π^0_n beginning with a universal quantifier. A set $A \subseteq \mathbb{N}$ is called arithmetical iff it is Σ^0_n or Π^0_n for some n.

Subsets of the set of natural numbers, or infinite binary sequences, are usually referred as *reals*. We use $2^{\mathbb{N}}$ to denote *Cantor Space*, the set of all reals. As for the subset of $\mathbb{N}$ we say that a set $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is Σ^0_2, for instance, if there exists a computable relation R such that for all $X \in 2^{\mathbb{N}}$, $X \in \mathcal{A}$ iff $\exists y \forall z R(X, y, z)$ (here the computable relation R is allowed to access X as an oracle). When we think of a countable group $G = (D, +_G)$, we will assume that it domain D is a subset of $\mathbb{N}$ and hence that $+_G \subseteq \mathbb{N}^3$. Then, via some effective bijection between $\mathbb{N}$ and $\mathbb{N} \cup \mathbb{N}^3$, we think of G as a single subset of $\mathbb{N}$, and hence as a real.

Beyond the arithmetical sets lie the *analytic* sets. To define an analytic set, we are also allowed to quantify over *functions*. A set A is called Σ^1_1 (analytic) iff there is a computable relation R such that for all x, $x \in A$ iff $\exists f \forall n R(x, f, n)$ where the quantification for f concerns functions from $\mathbb{N}$ to $\mathbb{N}$. Analogously, we say that $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is Σ^1_1 iff there is a computable relation R such that for all $X \in 2^{\mathbb{N}}$, $X \in \mathcal{A}$ iff $\exists f \forall n R(X, f, n)$.

Given a set complexity class Γ, as for example Π^0_3 or Σ^1_1, we say that a set $A \subseteq \mathbb{N}$ is Γ *m-complete* if for every Σ^1_1 set $B \subseteq \mathbb{N}$, there is a computable function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every $x \in \mathbb{N}$, $x \in B \iff f(x) \in A$. We say that a set $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is Γ *complete* if for every Σ^1_1 set $\mathcal{B} \subseteq 2^{\mathbb{N}}$, there is a computable operator $F\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that for every $X \in 2^{\mathbb{N}}$, $X \in \mathcal{B} \iff F(X) \in \mathcal{A}$.

2. From trees to groups

A *tree* is a downward subset of $\mathbb{N}^{<\mathbb{N}}$, the set of finite strings of natural numbers. Two trees T_0 and T_1 are *isomorphic* if there is a bijection $f\colon T_0 \to T_1$ which preserves inclusion of strings. A tree is *well-founded* if it has no infinite path. It is known that set of pairs of isomorphic trees is Σ^1_1 complete (see for instance [GK02]) and that the set of non-well-founded trees is Σ^1_1-complete (Kleene).

Theorem 2.1. *There is a computable operator G, that assigns to each tree T a torsion-free group $G(T)$, in a way that*

(1) *if $T_0 \cong T_1$, then $G(T_0) \cong G(T_1)$,*
(2) *if T_0 is well-founded and T_1 is not, then $G(T_0) \not\cong G(T_1)$.*

We start by defining the operator G. Let T be a tree. Let $\mathbb{Q}^T$ the group whose elements are formal sums

$$\sum_{\sigma\in V} q_\sigma \sigma,$$

where V is a finite subset of T, $q_\sigma \in \mathbb{Q}$ and addition is computed componentwise. Note that if T is infinite, then $\mathbb{Q}^T$ is isomorphic to $\mathbb{Q}^\infty$. $G(T)$ will be a subgroup of $\mathbb{Q}^T$. We think of T as a subset of $\mathbb{Q}^T$. Let $\mathbb{P} = \{p_0, p_1, ...\}$ be the set of prime numbers, listed in increasing order. $G(T)$ is defined so that $\sigma \in T$ can be divided by all the powers of $p_{2|\sigma|}$, and if $|\sigma| > 0$, then $\sigma^- + \sigma$ can be divided by all the powers $p_{2|\sigma|-1}$, where σ^- is σ with its last element removed (i.e. $\sigma^- = \sigma \restriction |\sigma| - 1$). In other words, $G(T)$ is the subgroup of $\mathbb{Q}^T$ generated under addition by

$$\{\frac{1}{p^k_{2|\sigma|}}\sigma : \sigma \in T, k \in \mathbb{N}\} \cup \{\frac{1}{p^k_{2|\sigma|-1}}(\sigma^- + \sigma) : \sigma \in T, |\sigma| > 0, k \in \mathbb{N}\}.$$

For the reader familiar with Hjorth [Hjo], we note that $G(T)$ is the *group eplag* corresponding to the *prime labeled graph* (V, E, f), where $V = T$, $E = \{(\sigma^-, \sigma) : \sigma \in T\}$, $f(\sigma) = p_{2|\sigma|}$, and $f((\sigma^-, \sigma)) = p_{2|\sigma|-1}$.

Note that the isomorphism type of $G(T)$ only depends on the isomorphism type of the tree T. This gives part (1) of Theorem 2.1. The second part follows immediately from the following lemma.

Lemma 2.2. *A tree T is non-well-founded if and only if in the group $G(T)$ there exists an infinite sequence $g_0, g_1, ...$ of elements such that for each i, g_i divisible by all the powers of p_{2i} and $g_i + g_{i+1}$ is divisible by all the powers of p_{2i+1}.*

Before we prove this lemma, we need to prove some basic properties of $G(T)$. Properties similar to these are proved in [Hjo] about the group eplags.

We will use the following well-known fact from number theory. Given a finite set of prime numbers P, we use $\mathbb{Q}_P$ to denote the set of rational numbers whose denominators are products of powers of primes in P. Note that $\mathbb{Q}_\emptyset = \mathbb{Z}$. The facts we will use are that if P and R are sets of prime numbers then

$$\mathbb{Q}_P \cap \mathbb{Q}_R = \mathbb{Q}_{P\cap R} \quad \text{and} \quad \mathbb{Q}_P + \mathbb{Q}_R = \mathbb{Q}_{P\cup R}.$$

Lemma 2.3. *Let $h = \sum_{\sigma\in V} r_\sigma \sigma \in G(T)$ where $V \subseteq T$ and each $r_\sigma \neq 0$. If h is divisible by all the powers of p_{2n}, then $|\sigma| = n$ for every $\sigma \in V$.*

Proof. Multiply h by some integer and divide it by some power of p_{2n}, and obtain $g = \sum_{\sigma\in V} q_\sigma \sigma \in G(T)$ so that all the coefficients q_σ are of the form

$\frac{m_\sigma}{p_{2n}^{i_\sigma}}$ for $m_\sigma \in \mathbb{Z}$, $i_\sigma \in \mathbb{Z}^+$, and $p_{2n} \not| m_\sigma$. In other words, all the coefficients of $g \in G(T)$ are in $\mathbb{Q}_{p_{2n}} \setminus \mathbb{Z}$. By the definition of $G(T)$, every element of $G(T)$ can be written as follows:

$$g = \sum_{\tau \in W} a_\tau \tau + \sum_{(\tau^-,\tau) \in U} b_\tau(\tau^- + \tau),$$

where $W \subseteq T$, $U \subseteq \{(\tau^-, \tau) : \tau \in T \setminus \{\emptyset\}\}$, $a_\tau \in \mathbb{Q}_{p_{2|\tau|}}$, and $b_\tau \in \mathbb{Q}_{p_{2|\tau|-1}}$. Consider now $\sigma \in V$; we want to show that $|\sigma| = n$. We have that q_σ is equal to the coefficient of σ in the sum above. This coefficient is

$$a_\sigma + \left(\sum_{(\sigma,\tau) \in U} b_\tau \right) + b_\sigma,$$

where a_σ and b_σ might be 0. On the one hand we have that $q_\sigma \in \mathbb{Q}_{p_{2n}} \setminus \mathbb{Z}$. On the other hand, the coefficient above belongs to $\mathbb{Q}_{p_{2|\sigma|-1}, p_{2|\sigma|}, p_{2|\sigma|+1}}$. If $p_{2n} \neq p_{2|\sigma|}$, then $(\mathbb{Q}_{p_{2n}} \setminus \mathbb{Z}) \cap \mathbb{Q}_{p_{2|\sigma|-1}, p_{2|\sigma|}, p_{2|\sigma|+1}} = \emptyset$. Therefore $p_{2n} = p_{2|\sigma|}$ and $|\sigma| = n$ as wanted. □

Lemma 2.4. *Let $h = \sum_{\sigma \in V} r_\sigma \sigma \in G(T)$ where $V \subseteq T$ and each $r_\sigma \neq 0$. If h is divisible by all the powers p_{2n+1}, then, for every $\sigma \in V$ with $|\sigma| = n$, there exists $\tau \in V$ with $\sigma = \tau^-$*

Proof. As in the proof of the previous lemma, multiplying h by the right scalar, we obtain $g = \sum_{\sigma \in V} q_\sigma \sigma \in G(T)$ all whose coefficients are in $\mathbb{Q}_{p_{2n+1}} \setminus \mathbb{Z}$. Again, since $g \in G(T)$, we get that

$$g = \sum_{\tau \in W} a_\tau \tau + \sum_{(\tau^-,\tau) \in U} b_\tau(\tau^- + \tau),$$

where $W \subseteq T$, $U \subset \{(\tau^-, \tau) : \tau \in T \setminus \{\emptyset\}\}$, $a_\tau \in \mathbb{Q}_{p_{2|\tau|}}$, and $b_\tau \in \mathbb{Q}_{p_{2|\tau|-1}}$. Consider now $\sigma \in V$ with $|\sigma| = n$. We have that q_σ is equal to the coefficient of σ in the sum above. This coefficient is

$$a_\sigma + \left(\sum_{(\sigma,\tau) \in U} b_\tau \right) + b_\sigma,$$

where a_σ and b_σ might be 0. So, we have that $q_\sigma \in \mathbb{Q}_{p_{2n+1}} \setminus \mathbb{Z}$ and that the coefficient above belongs to $\mathbb{Q}_{p_{2n-1}, p_{2n}, p_{2n+1}}$. Therefore, the middle term, $\sum_{(\sigma,\tau) \in U} b_\tau$ has to be in $\mathbb{Q}_{p_{2n+1}} \setminus \mathbb{Z}$: Because otherwise the coefficient above would belong to $\mathbb{Q}_{p_{2n-1}, p_{2n}}$, which has empty intersection with $\mathbb{Q}_{p_{2n+1}} \setminus \mathbb{Z}$. So, there exists some $\tau \in T$ with $(\sigma, \tau) \in U$ and $b_\tau \in \mathbb{Q}_{p_{2n+1}} \setminus \mathbb{Z}$. Pick one such τ. Note that $\sigma = \tau^-$. We claim that $\tau \in V$. Let us look at the coefficient of τ in g (which we want to show is not 0):

$$a_\tau + \sum_{(\tau,\delta) \in U} b_\delta + b_\tau.$$

The first two terms in this sum are in $\mathbb{Q}_{p_{2n+2},p_{2n+3}}$, and the third one in $\mathbb{Q}_{p_{2n+1}} \setminus \mathbb{Z}$. Therefore this coefficient is not 0, and $\tau \in V$. $\square$

Proof of Lemma 2.2. If T is not well-founded and X is an infinite path through T, then $\{g_i = X \restriction i : i \in \mathbb{N}\} \subseteq T \subseteq G(T)$ is a sequence in $G(T)$ as wanted.

Suppose now that $\{g_i : i \in \mathbb{N}\}$ is a sequence as in Lemma 2.2. Since g_i is divisible by all the powers of p_{2i}, by Lemma 2.3, we get that $g_i = \sum_{\sigma \in V_i} q_\sigma \sigma$, where V_i is a finite subset of $T \cap \mathbb{N}^i$, and $q_\sigma \neq 0$. Since $g_i + g_{i+1} = \sum_{\sigma \in V_i \cap V_{i+1}} q_\sigma \sigma$ is divisible by all the powers of p_{2i+1}, then, by Lemma 2.4 we get that for every $\sigma \in V_i$, there exists $\tau \in V_{i+1}$ extending σ. Therefore, by induction we can choose a sequence $\sigma_i \in V_i$, for $i \in \mathbb{N}$, such that $\sigma_i \subset \sigma_{i+1}$. Hence T is not well-founded. $\square$

Let $T_0 = \{0^n : n \in \mathbb{N}\}$ where 0^n is the string with n many zeros $\langle 0, 0, ..., 0\rangle$. Let $G_0 = G(T_0)$. From the proof above we can get the following corollary.

Corollary 2.5. *A tree T has an infinite path in and only if G_0 embeds in $G(T)$.*

Now we have proved all we needed about the operator $G(\cdot)$.

3. Trees and Σ^1_1-completeness

The following lemma about trees is essentially known. For completeness, and since we have not been able to find it in this form in the literature we sketch a proof of it. We prove it only after showing how it implies our main theorems.

Lemma 3.1. *There are computable operators S and R which map trees to trees and satisfy the following properties:*

1. *$R(T)$ is well-founded if and only if T is well-founded.*
2. *$S(T)$ is never well-founded and if $\omega_1^{T_0} = \omega_1^{T_1}$, then $S(T_0) \cong S(T_1)$.*
3. *If $R(T)$ is not well-founded, then $R(T) \cong S(T)$.*

Here, ω_1^X denotes the first ordinal that is not computable in X.

Proof of Theorem 1.1. Let $T(X)$ be a computable operator that assigns a tree $T(X)$ to each real X, so that the set $\mathcal{X}$ of reals for which $T(X)$ is non-well-founded is Σ^1_1 complete. (The existence of such an operator T is a well known result of Kleene.) We claim that $X \in \mathcal{X}$ if and only $G(R(T(X)))$ is isomorphic to $G(S(T(X)))$. If $X \notin \mathcal{X}$, then $T(X)$ is well-founded and hence so is $R(T(X))$. But $S(T(X))$ is never well-founded. So we have that $G(R(T(X)))$ is not isomorphic to $G(S(T(X)))$. Suppose now that $X \in \mathcal{X}$. So, $T(X)$ is not well-founded, and hence $R(T(X)) \cong S(T(X))$. So $G(R(T(X)))$ is isomorphic to $G(S(T(X)))$.

We have prove that the computable operator

$$X \mapsto \langle G(R(T(X)))), G(S(T(X)))\rangle$$

is a reduction of the Σ^1_1 complete set of reals $\mathcal{X}$, to the set of pairs of isomorphic torsion-free Abelian groups. □

Proof of Theorem 1.2. Let $\{T_n : n \in \mathbb{N}\}$ be a computable sequence of computable trees such that the set $\mathcal{X}$ of n such that T_n is not well-founded is a Σ^1_1 m-complete set of natural numbers. By the lemmas above, $n \in \mathcal{X}$ if and only $G(R(T_n))$ is isomorphic to $G(S(\emptyset))$. □

Observation 3.2. In the proof above, note that $G(S(\emptyset))$ is a computable group such that the set of indices of computable groups which are isomorphic to it is Σ^1_1 complete.

Sketch of the proof of Lemma 3.1. Given a linear ordering $\mathcal{L}$, we let $DS(\mathcal{L})$ be the tree of finite descending sequences of $\mathcal{L}$. Clearly the isomorphism type of $DS(\mathcal{L})$ depends only on the isomorphism type of $\mathcal{L}$, and $DS(\mathcal{L})$ is well-founded if and only if $\mathcal{L}$ is well-ordered.

Harrison [Har68] proved that there is a computable linear ordering $\mathcal{H}$ of order type $\omega_1^{CK}(1+\mathbb{Q})$. Relativizing this proof, we get that for every Y there is an Y-computable linear ordering $\mathcal{H}^Y$ of order type $\omega_1^Y(1+\mathbb{Q})$. A construction of $\mathcal{H}^Y$ that is uniform on the oracle Y can be obtain by relativizing the construction of $\mathcal{H}$ given in, for example, [Sac90, Lemmas III.2.1 and III.2.2]. ($\mathcal{H}^Y$ is build as the Kleene-Brower ordering of a non-well-founded Y-computable tree which has no Y-hyperarithmetical paths. Such a tree is build essentially by removing the hyperarithmetic paths of a tree with continuum many paths, using the fact that the set of Y-hyperarithmetic sets is $\Pi^1_1(Y)$.) We define $S(T) = DS(\mathcal{H}^T)$.

To define $R(T)$, we use the fact that there is a computable operation L which maps trees to linear orderings in a way that if T is well-founded, then $L(T)$ is well-founded, and if T is not well-founded, then $L(T)$ is isomorphic to the relativized Harrison linear ordering $\mathcal{H}^T = \omega_1^T(1+\mathbb{Q})$. Such an operator L is constructed in for example [CDH, Lemma 5.2], or [GK02, Theorem 4.4(d)]. It is not hard to see that the constructions in those papers can be relativized to any oracle. We then define $R(T) = DS(L(T))$. □

Using only the operation R and Corollary 2.5, we get the following corollary.

Corollary 3.3. *The class of torsion-free Abelian groups G such that G_0 embeds in G is Σ^1_1 complete.*

4. Identifying $\mathbb{Q}^\infty$.

Theorem 4.1. *The problem for deciding if a computable torsion-free Abelian groups is isomorphic to $\mathbb{Q}^\infty$ is Π^0_3 m-complete.*

Recall that $\mathbb{Q}^\infty$ is the the group of infinite sequences of rational numbers which are eventually 0, and where the group operation is addition computed coordinatewise.

Sketch of the proof. We have already observed that it is in the class Π^0_3. For each c.e. set C, we will uniformly build a computable free Abelian group G_C, such that G_C is isomorphic to $\mathbb{Q}^\infty$ if and only if C is coinfinite. Since the set of indices for coinfinite c.e. sets is Π^0_3 complete (see, for instance, Soare [Soa87]), this gives the desired result.

We consider $\mathbb{Q}^\infty$ as a vector space over $\mathbb{Q}$, with canonical basis $\{e_i : i \in \omega\}$, where e_i is the vector whose i-th coordinate is 1, and is zero elsewhere. We define a uniform procedure which, for each c.e. set C, defines a subspace $V \leq \mathbb{Q}^\infty$. We ensure that V is a computable space (i.e. as a set) and will ask that $G_C = \mathbb{Q}^\infty/V$ is finite dimensional iff C is cofinite.

We assume $0 \notin C$. We will make sure that

(1) $e_0 \notin V$,
(2) for each $i \geq 1$, if $i \in C$, then e_0 and e_1 are linearly dependent over V,
(3) if F is disjoint form C, then $\{e_i : i \in F\}$ is linearly independent over V.

It is not hard to see that these conditions imply that the dimension of $G_C = \mathbb{Q}^\infty/V$ is equal to the size of the complement of C. Therefore $G_C \cong \mathbb{Q}^\infty$ if and only if C is coinfinite.

$V_s \subseteq$ will denote that part of a basis of V generated by stage s. We use V_s^* to denote the subspace of $\mathbb{Q}^\infty$ generated by V_s, and $V = \bigcup_s V_s^*$. To make V a computable set, we will ask that, at each stage s, $V_s^* \cap \{0, \dots, s\} = V \cap \{0, \dots, s\}$.

At stage $s+1$ of the construction, suppose that c is enumerated into C. Then find λ so that $(V_s \cup (e_0 + \lambda e_c))^* \cap \{0, ..., s\} = V_s^* \cap \{0, \dots, s\}$. Such a λ exists because of the following reason. Since no point of the form $(e_0 + \lambda e_c)$ has been added to V_s^* yet, if $\lambda_1 \neq \lambda_2$ then $(V_s \cup (e_0 + \lambda_1 e_c))^* \cap (V_s \cup (e_0 + \lambda_2 e_c))^* = V_s^*$. Therefore, since there are infinitely many λ to choose from, there has to exists one such that $(V_s \cup (e_0 + \lambda e_c))^*$ is disjoint from $\{0, ..., s\} \setminus V_s^*$. Let $V_{s+1} = V_s \cup (e_0 + \lambda e_c)$. This guarantees part (2).

To verify (3), note that if F is finite and disjoint from C, then no sum of the form $\sum_{i \in F} q_i e_i$ is ever added to V_s^* unless that sum is 0. The reason is that if $i \notin C$, $i \neq 0$, then no term containing e_i is ever added to V_s. □

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