A COMPUTABLE $\aleph_0$-CATEGORICAL STRUCTURE WHOSE THEORY COMPUTES TRUE ARITHMETIC

BAKHADYR KHOUSSAINOV AND ANTONIO MONTALBÁN

1. Introduction

The goal of this paper is to construct a computable $\aleph_0$-categorical structure whose first order theory is computably equivalent to the true first order theory of arithmetic. Recall that a structure is *computable* if its atomic open diagram, that is the set of all atomic statements and their negations true in the structure, is a computable set. Computability of an infinite structure $\mathcal{A} = (A; P_0^{n_0}, P_1^{n_1}, \dots)$ is equivalent to saying that the domain A is either finite or ω and that there exists an algorithm that given an $i \in \omega$ and elements $x_1, \dots, x_{n_i}$ of the domain decides whether $P_i^{n_i}(x_1, \dots, x_{n_i})$ is true. If a structure $\mathcal{B}$ is isomorphic to a computable structure $\mathcal{A}$ then $\mathcal{A}$ is called a *computable presentation* of $\mathcal{B}$. We often identify computable and computably presentable structures. If there exists an algorithm that decides the full diagram of a structure $\mathcal{A}$ then $\mathcal{A}$ is called a *decidable structure*. Clearly, decidable structures are computable but the opposite is not always true. Each computable structure is countable. Therefore, in this paper we restrict ourselves to countable structures.

One of the major themes in computable model theory investigates computable models of theories. Let T be a deductively closed consistent theory. If T is decidable then the Henkin's construction can be carried out effectively for T. Therefore, a complete theory T has a decidable model if and only if T is decidable. For complete decidable theories T the class of all decidable models of T has been well studied starting in the 70s. See for example the results by Goncharov [GN73] [Gon78], Millar [Mil78] [Mil81], Morley [Mor76], Harrington [Har74], and Peretyatkin [Per78]. These results investigate decidability of specific models of T such as prime models, saturated models, and homogeneous models. Roughly, prime models are the smallest models since they can be embedded into all models of T, and saturated models are the largest models since all (countable) models of T can be embedded into saturated models. Prime and saturated models are unique up to isomorphism, and homogeneous models are characterized by

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the types they realize. Goncharov, Millar, and Morely found characterizations for these models to be decidable. For instance, the prime model of T is decidable if and only if the set of all principle types of T is uniformly computable [Har74] [GN73]. Similarly, the saturated model of T is decidable if and only if the set of all types of T is uniformly computable [Mor76]. If T is undecidable then one would like to study the class of computable models of T. One simple observation is that if a complete theory T has a computable model then $\mathbf{0}^{(\omega)}$ the ω-jump of the computable degree computes T. This bound is sharp given by the model of arithmetic $(\omega; 0, S, +, \times)$. However, it is perhaps quite an ambitious goal to hope for results of general character that say something reasonable and deep about computable models of T. Therefore, one would like to study computable models of specific theories T.

Ershov proves that all computably enumerable extensions of the theory of trees have computable models [Ers73]. Lerman and Schmerl prove that all Δ^0_2-extensions of the theory of linear orders have computable models [LS79]. Khisamiev in [Khi98] studies computable models of the theory of Abelian groups. A series of results investigate computable models of $\aleph_1$-categorical theories [GHL+03] [KLLS07] [KNS97] [Kud80]. For example, all models of a trivial strongly minimal theory with a computable model are decidable in $\mathbf{0}''$ [GHL+03]. The current paper contributes to this line of research by considering computable models of $\aleph_0$-categorical theories. Below we give a brief background to known results about computable models of $\aleph_0$-categorical theories.

A complete theory T is $\aleph_0$-*categorical* if all countable models of T are isomorphic to each other. A structure $\mathcal{A}$ is $\aleph_0$-categorical if its theory is $\aleph_0$-categorical. It is well-known that T is $\aleph_0$-categorical if and only if for each n the number of complete n-types of T is finite (e.g. see [Hod93]). If T is $\aleph_0$-categorical then T is decidable if and only if all of its models (and hence exactly one model of T) are decidable. Schmerl in [Sch78] proves that for every computably enumerable degree X there exists a decidable $\aleph_0$-categorical theory T such that the type function of T is Turing equivalent to X. In [LS79] Lerman and Schmerl show that if T is an arithmetical $\aleph_0$-categorical theory such that the set of all $\exists_{n+2}$-sentences of T is a Σ^0_{n+1}-set for each $n < \omega$, then T has a computable model. Knight extends this result in [Kni94] to include non-arithmetical $\aleph_0$-categorical theories. These results, however, do not provide examples of computable $\aleph_0$-categorical structures of high arithmetical complexity. In [GK04] Khoussainov and Goncharov, for every $n \geq 0$ build $\aleph_0$-categorical computable structures whose theories are equivalent to $\mathbf{0}^{(n)}$, the n-jump of the computable degree. It has been a long standing open question whether there exists a computable $\aleph_0$-categorical structure whose first order theory is not arithmetical. In this paper we solve this problem by proving the following theorem:

Theorem 1.1. *There exists a computable $\aleph_0$-categorical structure whose first order theory is 1-equivalent to true first order arithmetic $Th(\omega; 0, S, +, \times)$.*

The rest of this paper is devoted to proving this theorem.

2. General Idea

We start by roughly describing the idea of the proof. Suppose we want to code one bit of Σ_n information, say φ. We will define two n-graphs $G^{\Sigma,n}$ and $G^{\Pi,n}$ which are $\aleph_0$-categorical and not elementary equivalent. By an *n-graph* we mean a structure (V, E) where E is an n-ary relation on V such that, for all tuples $(x_1, \ldots, x_n)$, if $(x_1, \ldots, x_n) \in E$ then all $x_1, \ldots, x_n$ are pairwise distinct. Furthermore, for an n-graph $G = (V, E)$ we often write $G(\bar{x})$ instead of $E(\bar{x})$. Later, we will define a computable procedure that, given a Σ_n-sentence φ, produces a computable n-graph G^φ such that

$$G^\varphi \cong \begin{cases} G^{\Sigma,n} & \text{if } \varphi \\ G^{\Pi,n} & \text{if } \neg\varphi. \end{cases}$$

We define the n-graphs $G^{\Sigma,n}$ and $G^{\Pi,n}$ inductively. For $n = 1, 2, 3$ these graphs are defined as follows. The 1-graph $G^{\Pi,1}$ is a unary relation that holds of every element, and $G^{\Sigma,1}$ is a unary relation that holds on an infinite and co-infinite set of elements. For example, $G^{\Sigma,1}$ can be defined by flipping a coin randomly. The 2-graph $G^{\Pi,2}$ is the usual random directed graph. In this random graph for each pair (a_1, a_2) we flip a coin to decide whether $G^{\Pi,2}(a_1, a_2)$ holds. The directed graph $G^{\Sigma,2}$ has two types of elements. The first type of elements are connected (via the edge of the graph) to all other elements of the graph. The second type of elements are connected to an infinite co-infinite set of elements in a random way. The same idea is applied in defining the 3-graphs $G^{\Pi,3}$ and $G^{\Sigma,3}$. In $G^{\Pi,3}$, for every element b we have that the graph obtained by $G_b(a_1, a_2) = G^{\Pi,3}(b, a_1, a_2)$ is isomorphic to $G^{\Sigma,2}$. Moreover, these 2-graphs G_b for the different b's are, in a certain sense, randomly independent. In the 3-graph $G^{\Sigma,3}$, there is an infinite set of elements b such that G_b is isomorphic to $G^{\Sigma,2}$ and there is an infinite set of elements b such that G_b is isomorphic to $G^{\Pi,2}$. Precise definitions of n-graphs $G^{\Pi,n}$ and $G^{\Sigma,n}$ for $n > 3$ are given in Section 4.

In order to ensure that these graphs are $\aleph_0$-categorical, we will define them inside of a random structure that we know is $\aleph_0$-categorical. To be able to decode the bit of information φ we will have that the sentence

$$\psi_n \equiv (\exists x_1)\neg(\exists x_2 \neq x_1)\neg\ldots\neg(\exists x_n \neq x_1, \ldots, x_{n-1})\neg G(x_1, \ldots., x_n).$$

holds in $G^{\Sigma,n}$ but not in $G^{\Pi,n}$. Decoding information will work in a nice way. Let our sentence φ be $\exists x_1 \neg\exists x_2 \neg\ldots\neg\exists x_n \neg R(x_1, \ldots., x_n)$, a Σ^0_n-sentence of arithmetic, written in a certain standard form that will be explained later. Let q be a computable 'random' projection from n-tuples to n-tuples as we will also define later, and let G^φ be the computable n-graph defined by $G^\varphi(\bar{x}) = R(q(\bar{x}))$. The surprising fact is that the isomorphism type of the

n-graph (ω, G^{φ}) does not depend on what φ is, but only on whether φ holds. Moreover, G^{φ} is isomorphic to either $G^{\Sigma,n}$ or $G^{\Pi,n}$ depending on whether φ holds. Moreover, the connection between φ and ψ_n will be such that φ is true in the arithmetic if and only if ψ_n is true in G^{φ}.

Suppose now we want to code another bit of Σ_n information. We will now consider the graphs $G^{\Sigma,n}(x_1, ..., x_n)$ or $G^{\Pi,n}(x_1, ..., x_n)$ again, but now, we will think of x_1 as a member of ω^{n+1}. The definitions of these new graphs will be random enough, that all the m-tuples of elements for $m \leq n$ will have the same m-type, as they will be part of $(n+1)$-tuples satisfying all the possible $(n+1)$-types. Furthermore, these new graphs will be randomly independent from the n-graphs defined previously, and hence will not add any new m-type for $m \leq n$. This will allow as to define infinitely many such graphs keeping the number of m-types finite, and hence preserving $\aleph_0$-categoricity.

3. Random string maps

We want to work with finite strings all whose entries are different. So we need to develop a bit of notation to work with these objects. Recall that by an *n-graph* we mean a structure (V, E) where E is an n-ary relation on V such that, for all tuples $(x_1, \ldots, x_n)$, if $(x_1, \ldots, x_n) \in E$ then all $x_1, \ldots, x_n$ are pairwise distinct.

We will use the following notation. We let $V^{\langle n \rangle}$ be the set of n-tuples from V all whose entries are different. So, an n-graph is nothing more than a subset $r \subseteq V^{\langle n \rangle}$. We also set

$$V^{\leq\langle n \rangle} = \bigcup_{i=1}^{n} V^{\langle i \rangle} \quad \text{and} \quad V^{<\langle \omega \rangle} = \bigcup_{i=1}^{\omega} V^{\langle i \rangle}.$$

We call any pair of the form (V, p) where $p \colon V^{<\langle \omega \rangle} \to \omega$ a *string ω-map.* Also, we call pairs (V, r) where $r \colon V^{<\langle \omega \rangle} \to \{0, 1\}$ *string* 2*-map.*

We identify each string 2-map (V, r) with the relational structure with infinitely many predicates $(V; P_1, P_2, \ldots)$, where $P_n(x_1, \ldots, x_n)$ if and only if $r(x_1, \ldots, x_n) = 1$. Similarly, we identify each string ω-map (V, p) with the following structure $(V; P^i_n)_{i,n\in\omega}$, where $P^i_n(x_1, \ldots, x_n)$ if and only if $p(x_1, \ldots, x_n) = i$.

Now we want to consider random string maps and random string sets. They are just the Fraïssé limits of the class of finite sting maps and of the class of finite string sets. Recall that a structure is *ultra-homogeneous* if every two tuples which satisfy the same quantifier free types are automorphic.

Theorem 3.1. *Let α be either 2 or ω. Let (V, r) be a string α-map. The following properties are equivalent:*

(1) *(V, r) is ultra-homogeneous and every finite string α-maps is isomorphic to a substructure of (V, r).*

(2) *For each finite set* $V_0 \subseteq V$ *and function* $r_0\colon (V_0 \cup x)^{<\langle\omega\rangle} \to \alpha$ *that extends* $r \restriction V_0^{<\langle\omega\rangle}$ *there exists* $a \in V \setminus V_0$ *such that for every* σ *in the set* $(V_0 \cup \{x\})^{<\langle\omega\rangle}$ *we have*

$$r_0(\sigma) = r(\sigma_{x\mapsto a}),$$

where $r(\sigma_{x\mapsto a})$ *is obtained by replacing* a *with* x *in the domain of* r*.*

Furthermore, there is a structure unique up to isomorphism satisfying any of these properties. For $\alpha = 2$*, this structure is* $\aleph_0$*-categorical.*

Proof. The proof of this theorem is standard (e.g. see [Hod93]). □

We single out the structures that are specified in the theorem above in the following definition.

Definition 3.2. We call any structure that satisfies any of the conditions of the theorem the *random string* α*-map.*

The next lemma says that each random string α-map is a computable structure unique up to computable isomorphism.

Lemma 3.3. *Let* α *be either 2 or* ω*. There exists a computable random* α*-map. Moreover, any two computable random* α*-maps are isomorphic via a computable map.*

Proof. For the first part, one builds the α-map r by stages, finitely much at a time, satisfying the requirements for (2) of the theorem above stage by stage. The second part is a typical back and forth argument that can be carried out effectively. □

We will see now how to use random string ω-maps to transform subset of $\omega^{<\omega}$ into random string 2-maps. We need a couple definitions.

Definition 3.4. For a sting ω-map p, we set $\bar{p}\colon V^{<\langle\omega\rangle} \to \omega^{<\omega}$ as follows:

$$\bar{p}(a_1a_2...a_k) = (p(a_1), p(a_1a_2), \dots, p(a_1a_2...a_k)).$$

We may abuse notation and write $\bar{p}(a_1a_2...a_k)$ as $p(a_1)p(a_1a_2)\dots p(a_1a_2...a_k)$.

By Lemma 3.3, if (V, p) is a random ω-string map then we can assume that (V, r) is a computable structure, and hence we identify V with ω. With this identification, the following observation is easy check.

Observation 3.5. If (V, p) is a random string ω-map then $\bar{p}$ satisfies the following properties:

(1) For every σ, $|\bar{p}(\sigma)| = |\sigma|$;
(2) If $\sigma \subseteq \tau$, then $\bar{p}(\sigma) \subseteq \bar{p}(\tau)$;
(3) Given $\sigma_0, \sigma_1, \tau_1$ such that $n = |\sigma_0| + |\tau_1| = |\sigma_0| + 1 + |\sigma_1|$, and σ_0 and σ_1 do not share any entry, there exist infinitely many $a \in V$ such that $\bar{p}(\sigma_0 a \sigma_1) = \bar{p}(\sigma_0)\tau_1$.

(4) Given $\{(\sigma_0^i, \sigma_1^i, \tau_1^i) : i = 1, \dots, s\}$ such that σ_0^i and σ_1^i do not share any entry and $n = |\sigma_0^i| + |\tau_1^i| = |\sigma_0^i| + 1 + |\sigma_1^i|$ for all $i = 1, \dots, s$, there exist infinitely many $a \in V$ such that for each $i = 1, ..., s$ we have $\bar{p}(\sigma_0^i a \sigma_1^i) = \bar{p}(\sigma_0^i)\tau_1^i$.

Definition 3.6. A map $q : V^{<\omega} \to \{0, 1\}$ is *diverse* if for every $\sigma \in V^{<\omega}$ there exist k_0 and k_1 such that $q(\sigma k_0) = 0$ and $q(\sigma k_1) = 1$.

Diverse maps are not necessarily string 2-maps. However, for a diverse map q and string ω-map p, the composition $q \circ \bar{p}$ is a string 2-map.

Lemma 3.7. *If (V, p) is a random string ω-map and q is a diverse map, then $q \circ \bar{p}$ is a random string 2-map.*

Proof. Let $r = q \circ \bar{p}$. We show that Condition (2) of Theorem 3.1 is satisfied. Let $V_0 \subset \omega$ be a finite set and let $r_0 \colon (V_0 \cup x)^{<\langle\omega\rangle} \to \{0, 1\}$ be a function extending $r \restriction V_0^{<\langle\omega\rangle}$. We need to show that there exists an element $a \in V \setminus V_0$ such that for every $\sigma \in (V_0 \cup x)^{<\langle\omega\rangle}$ we have

$$r_0(\sigma) = r(\sigma_{x \mapsto a}).$$

For each $\sigma \in (V_0 \cup \{x\})^{<\langle\omega\rangle}$, write σ as $\sigma_0 x \sigma_1$. Since q is diverse, for the given σ there exists τ_σ of length $|\sigma_1| + 1$ such that $q(p(\sigma_0)\tau_\sigma) = r_0(\sigma)$. Note that the set $(V_0 \cup \{x\})^{<\langle\omega\rangle}$ is finite. Now, from the last part of the observation above, we have that there exist infinitely many a such that $p(\sigma_{x \mapsto a}) = p(\sigma_0)\tau_\sigma$. Hence, Condition (2) of Theorem 3.1 is satisfied. □

4. The coding structures

In this section we turn our interest to defining the n-graphs $G^{\Sigma,n}$ and $G^{\Pi,n}$ as suggested in Section 2. Our definition will proceed by induction using the random string 2-map.

Definition 4.1. Let (ω, r) be the random string 2-map. For each $m \leq n$ with $1 \leq m$, and each $\bar{b} \in \omega^{n-m}$ we define two m-graphs $G_{\bar{b}}^{\Sigma,m}$ and $G_{\bar{b}}^{\Pi,m}$ inductively as follows. When $m = 1$, $|\bar{b}| = n - 1$, and $|a| = 1$, we let

$$G_{\bar{b}}^{\Sigma,1}(a) = r(\bar{b}a) \quad \text{and} \quad G_{\bar{b}}^{\Pi,1}(a) = 1.$$

Let

$$\begin{aligned} G_{\bar{b}}^{\Sigma,m}(a_1, ..., a_m) &= \begin{cases} G_{\bar{b}a_1}^{\Sigma,m-1}(a_2, ..., a_m) & \text{if } r(\bar{b}a_1) = 1 \\ G_{\bar{b}a_1}^{\Pi,m-1}(a_2, ..., a_m) & \text{if } r(\bar{b}a_1) = 0 \end{cases} \\ G_{\bar{b}}^{\Pi,m}(a_1, ..., a_m) &= G_{\bar{b}a_1}^{\Sigma,m-1}(a_2, ..., a_m). \end{aligned}$$

The isomorphism types of the structures $G_{\bar{b}}^{\Sigma,n}$ and $G_{\bar{b}}^{\Pi,n}$ do not depend on the parameter $\bar{b}$. In particular, they are all isomorphic to $G_{\langle\rangle}^{\Sigma,n}$ and $G_{\langle\rangle}^{\Pi,n}$, respectively, where $\langle\rangle$ is the empty tuple. Also, since there is only one random string 2-map up to isomorphism, the structures $G_{\bar{b}}^{\Sigma,n}$ and $G_{\bar{b}}^{\Pi,n}$ up to isomorphism do not depend on the presentation of r. However, the particular presentations of $G^{\Sigma,n}_{\bar b}$ and $G^{\Pi,n}_{\bar b}$ do depend on the particular presentation of r. Note that the n-graphs $G^{\Sigma,n}$ and $G^{\Pi,n}$ obtained for the cases when $n = 1, 2, 3$ are exactly as in Section 2. We now prove the following theorem.

Theorem 4.2. *The n-graphs $G^{\Sigma,n}$ and $G^{\Pi,n}$ have the following properties:*

(1) *The n-graphs $G^{\Sigma,n}$ and $G^{\Pi,n}$ are $\aleph_0$-categorical.*
(2) *There is a $\exists_n$ formula ψ_n in the language of n-graphs which is true in $G^{\Sigma,n}$ but false in $G^{\Pi,n}$.*
(3) *There is a uniform computable procedure that given a Σ^0_n sentence φ in the language of arithmetic, builds an n graph G^φ such that*

$$G^\varphi \cong \begin{cases} G^{\Sigma,n} & \text{if } \varphi \text{ holds} \\ G^{\Pi,n} & \text{if } \neg\varphi \text{ holds.} \end{cases}$$

Proof. For Part 1 note that both $G^{\Sigma,n}$ and $G^{\Pi,n}$ are definable in the structure (ω, r) which is $\aleph_0$-categorical. This implies $G^{\Sigma,n}$ and $G^{\Pi,n}$ are also $\aleph_0$-categorical because the number of k-types in each of these structures is at most the number of k-types in (ω, r).

For Part 2, the sentence distinguishing $G^{\Sigma,n}$ and $G^{\Pi,n}$ is the following:

$$\psi_n \equiv (\exists x_1)\neg(\exists x_2 \neq x_1)\neg ... \neg(\exists x_n \neq x_1, ..., x_{n-1})\neg G(x_1, ..., x_n).$$

When $n = 1$ the sentence says that there is an element outside of the unary relation G. This is satisfied by $G^{\Sigma,1}$ and falsified by $G^{\Pi,1}$. When $n = 2$, the statements says there is an element that is connected to all other elements of the structure. This is satisfied by $G^{\Sigma,2}$ and falsified by $G^{\Pi,2}$. The rest is proved by induction on n. Suppose G is isomorphic to either $G^{\Sigma,n}$ or $G^{\Pi,n}$. Then we have that $G \cong G^{\Sigma,n}$ if and only if there exists x such that the graph G_x defined by $G_x(\bar a) = G(x, \bar a)$ is isomorphic to $G^{\Pi,n-1}$.

The last part of the theorem can be proved in several ways. One way of proving this would be to show that $G^{\Sigma,n}$ is n-back-and-forth below $G^{\Pi,n}$ and that these structures are n-friendly, and then use Ash and Knight's theorem [AK00] (see Thm 18.6). These would require some combinatorial work and notation needed to apply Ash and Knight's theorem. Instead, we give a direct construction of G^φ which is interesting in its own right.

Consider a Σ^0_n formula φ. Write φ as $\exists x_1 \neg\exists x_2 \neg ... \neg\exists x_n \neg R^\varphi(x_1, ..., x_n)$.

Definition 4.3. Let $\varphi_i(x_1, ..., x_{n-i})$ be $\exists x_{n-i+1}\neg ... \neg\exists x_n \neg R^\varphi(x_1, ..., x_n)$. We say that φ is in *semi-diverse form* if for each i and each $\bar a$ of length $n - i - 1$ there exists some b such that $\varphi_i(\bar a, b)$ holds.

Thus, for the Σ^0_n formula φ, the definition above states that $\varphi_i(x_1, ..., x_{n-i})$ is a Σ^0_i sub-formula of φ obtained by removing the first $(n - i)$ quantifiers, and hence it has $(n - i)$ free variables. Furthermore, we have that $\varphi \equiv \varphi_n$, $\varphi_{i+1} \equiv \exists x_{n-i}\neg\varphi_i$ and $\varphi_0 \equiv R(x_1, \ldots, x_n)$.

Observation 4.4. Every Σ_n formula φ is equivalent to one in semi-diverse form. This can be done by steps starting with $i = n-1$ and going down to $i = 0$: at each step one modifies the formula $\varphi_i(\bar{b}a)$ for some a without changing the value of $\varphi_{i+1}(\bar{b})$. It is only necessary to make a change when $\forall a \neg\varphi_i(\bar{b}a)$, but in this case we have $\varphi_{i+1}(\bar{b}) \equiv \exists x_{n-i} \neg\varphi_i(\bar{b}x_{n-i})$ even if we change $\varphi_i(\bar{b}a)$ for one value of a.

An intuition for why we need formulas in semi-diverse form is the following. In the graphs $G^{\Sigma,m}$, there are two types of elements, the ones for which the rest of the graph is $G^{\Pi,(m-1)}$, and the ones for which the rest of the graph is $G^{\Sigma,(m-1)}$. Therefore we would like similar things to happen with the formulas φ_i. Namely, if this existential formula is true, then we want it to have some witnesses, but at the same time we also want to have some elements which are not witnesses. This intuition is made precise in the reasoning below that constitutes the proof of Part (3) of the theorem.

Definition 4.5. Let p be a random string ω-map. Given a Σ_n formula φ, written in semi-diverse form as $\exists x_1 \neg \exists x_2 \neg ... \neg \exists x_n \neg R^{\varphi}(x_1, ..., x_n)$, we define

$$G^{\varphi} = R^{\varphi} \circ \bar{p},$$

where $\bar{p}$ is defined in Definition 3.4.

Clearly G^{φ} is a computable n-graph and the definition is uniform on φ.

It is not hard to prove that $G^{\varphi} \models \psi_n$ if and only if φ is a true sentence of the arithmetic. To prove this one uses an induction on i to show that the statement $\varphi_i(\bar{p}(\bar{b}))$ is true if and only if

$$G^{\varphi} \models (\exists x_{n-i+1} \neq b_1, ..., b_{n-i}) \neg ... \neg (\exists x_n \neq b_1, ..., b_{n-i}, x_{n-i+1}, ..., x_{n-1}) \neg G(\bar{b}, x_{n-i+1}, ..., x_n).$$

Indeed, when $i = 0$ then the statement is simply the definition of G^{φ}. For the inductive case one uses the fact that p is random and hence it is onto on every coordinate. What is left to prove is that G^{φ} satisfies Part (3) of the theorem.

Let $Q^{\varphi} \colon \omega^{<\omega} \to \{0, 1\}$ be defined as follows. For a non-empty tuple $\bar{b}a \in \omega^{\leq n}$ with $|\bar{b}a| = i$, let

$$Q^{\varphi}(\bar{b}a) = \begin{cases} \varphi_i(\bar{b}a) & \text{whenever } a \neq 0 \text{ or } \exists x \neg\varphi_i(\bar{b}x), \\ 0 & \text{if } a = 0 \text{ and } \forall x \varphi_i(\bar{b}x). \end{cases}$$

For $\bar{b}a \in \omega^{>n}$, define $Q^{\varphi}(\bar{b}a)$ in any way so that Q^{φ} is diverse. Also, note that this makes Q^{φ} diverse because φ is chosen to be in a semi-diverse form. Now we apply Lemma 3.7, and have the following statement:

Lemma 4.6. *The mapping $Q^{\varphi} \circ p$ is a random string 2-map.*

Let $r = Q \circ p$ and consider $G^{\Sigma,m}_{\bar{b}}$ and $G^{\Pi,m}_{\bar{b}}$ as in Definition 4.1.

The next lemma establishes the connection between $G_{\bar b}^{\Sigma,m}$, $G_{\bar b}^{\Pi,m}$, and G^φ. For the lemma, we need another bit of notation. For every $\bar b$ of length $n-i$ and $\bar a$ of length i, we set $G^\varphi_{\bar b}(\bar a) = G^\varphi(\bar b\bar a)$.

Lemma 4.7. *For every $\bar b$ of length $n-i$ and $\bar a$ of length i, we have:*

$$G^\varphi_{\bar b}(\bar a) = \begin{cases} G_{\bar b}^{\Sigma,n-i}(\bar a) & \text{if } \varphi_i(\bar p(\bar b)) \\ G_{\bar b}^{\Pi,n-i}(\bar a) & \text{if } \neg\varphi_i(\bar p(\bar b)). \end{cases}$$

So, in particular when $i=0$, Part (3) of the Theorem is satisfied.

The proof is by reverse induction on i. Suppose $i = n-1$. If $\varphi_i(\bar p(\bar b)) \equiv \exists x_n \neg R(\bar p(\bar b)x_n)$ holds, then $G^\varphi_{\bar b}(a) = r(\bar b a)$ because $Q^\varphi(\bar b a) = R(\bar b a)$. Therefore, $G^\varphi_{\bar b} = G^{\Sigma,1}_{\bar b}$. Otherwise, if $\neg\varphi_i(\bar p(\bar b)) \equiv \forall x_n R(\bar p(\bar b)x_n)$ holds, then $G_{\bar b}$ is the whole universe and hence it is isomorphic to $G^{\Pi,1}$. For the induction step we proceed as follows. If $\varphi_{i-1}(\bar p(\bar b)) \equiv \exists x_{n-i+1}\neg\varphi_i(\bar p(\bar b)x_{n-i+1})$ holds, then for every a, $Q^\varphi(\bar b a) = 1$ if and only if $\varphi_i(\bar b a)$, and hence

$$\begin{aligned} G^\varphi_{\bar b}(a\bar a) &= G^\varphi_{\bar b a}(\bar a) \\ &= \begin{cases} G_{\bar b a}^{\Sigma,n-i}(\bar a) & \text{if } \varphi_i(\bar p(\bar b a)) \\ G_{\bar b a}^{\Pi,n-i}(\bar a) & \text{if } \neg\varphi_i(\bar p(\bar b a)) \end{cases} && \text{by induction hypothesis} \\ &= \begin{cases} G_{\bar b a}^{\Sigma,n-i}(\bar a) & \text{if } r(\bar b a) = 1 \\ G_{\bar b a}^{\Pi,n-i}(\bar a) & \text{if } r(\bar b a) = 0 \end{cases} && \text{because } Q^\varphi(\bar b a) = \varphi_i(\bar b a) \\ &= G_{\bar b}^{\Sigma,n-i+1}(a\bar a) && \text{by definition of } G_{\bar b}^{\Sigma,n-i+1}. \end{aligned}$$

When $\varphi_{i-1}(\bar p(\bar b)) \equiv \exists x_{n-i+1}\neg\varphi_i(\bar p(\bar b)x_{n-i+1})$ does not hold, we have that for every a, $\varphi_i(\bar p(\bar b a))$ holds, and hence

$$\begin{aligned} G_{\bar b}(a\bar a) &= G_{\bar b a}(\bar a) \\ &= \begin{cases} G_{\bar b a}^{\Sigma,n-i}(\bar a) & \text{if } \varphi_i(\bar p(\bar b a)) \\ G_{\bar b a}^{\Pi,n-i}(\bar a) & \text{if } \neg\varphi_i(\bar p(\bar b a)) \end{cases} && \text{by induction hypothesis} \\ &= G_{\bar b a}^{\Sigma,n-i}(\bar a) && \text{because } \forall a \varphi_i(\bar p(\bar b a)) \\ &= G_{\bar b}^{\Pi,n-i+1}(a\bar a) && \text{by definition of } G_{\bar b}^{\Pi,n-i+1}. \end{aligned}$$

This concludes the proof of the lemma, and hence of the theorem. □

5. Coding many bits

Now we want to encode infinitely many bits of information into our structures, each bit being a Σ_n-sentence of the arithmetic for various n. So we will use infinitely many graphs. Since we do not want the different graphs to have any interaction between each other we will use a variation of the graphs defined in the previous section.

Definition 5.1. Let (ω, r) be a random string 2-map and $l, n, m \in \omega$ with $1 \le m \le n$. For each $\bar b \in \omega^{l+n-m}$ we define two m-graphs $G_{\bar b}^{\Sigma,l,m}$ and $G_{\bar b}^{\Pi,l,m}$

inductively as follows. When $m = 1$, $|\bar{b}| = l + n - 1$, and $|\bar{a}| = 1$, set:

$$G^{\Sigma,1}_{\bar{b}}(a) = r(\bar{b}a) \quad \text{and} \quad G^{\Pi,1}_{\bar{b}}(a) = 1.$$

When $1 < m < n$, $|\bar{b}| = l + n - m$, and $|\bar{a}| = m$, we let

$$\begin{aligned} G^{\Sigma,l,m}_{\bar{b}}(a_1, ..., a_m) &= \begin{cases} G^{\Sigma,l,m-1}_{\bar{b}a_1}(a_2, ..., a_m) & \text{if } r(\bar{b}a_1) = 1 \\ G^{\Pi,l,m-1}_{\bar{b}a_1}(a_2, ..., a_m) & \text{if } r(\bar{b}a_1) = 0 \end{cases} \\ G^{\Pi,l,m}_{\bar{b}}(a_1, ..., a_m) &= G^{\Sigma,m-1}_{\bar{b}a_1}(a_2, ..., a_m). \end{aligned}$$

Finally, we define $(l + n)$-graphs:

$$\begin{aligned} G^{\Sigma,l,n}(a_1, ..., a_{l+n}) &= \begin{cases} G^{\Sigma,l,n-1}_{a_1...a_{l+1}}(a_{l+2}, ..., a_{l+n}) & \text{if } r(a_1...a_{l+1}) = 1 \\ G^{\Pi,l,n-1}_{a_1...a_{l+1}}(a_{l+2}, ..., a_{l+n}) & \text{if } r(a_1...a_{l+1}) = 0 \end{cases} \\ G^{\Pi,l,n}(a_1, ..., a_{l+n}) &= G^{\Sigma,n-1}_{a_1...a_{l+1}}(a_{l+2}, ..., a_{l+n}). \end{aligned}$$

Note that the definition of $G^{\Sigma,l,n}(a_1, ..., a_{l+n})$ is essentially the same as the one for $G^{\Sigma,n}(a_1, ..., a_n)$ if we treat the first $l + 1$ coordinates as a single one. In particular, the structure $G^{\Sigma,0,n}(a_1, ..., a_n)$ is the same as $G^{\Sigma,n}(a_1, ..., a_n)$. We now outline the proof of the following theorem that simply extends Theorem 4.2.

Theorem 5.2. *The $(l+n)$-graphs $G^{\Sigma,l,n}(a_1, ..., a_{l+n})$ and $G^{\Pi,l,n}(a_1, ..., a_{l+n})$ have the following properties:*

(1) *The structures $(\omega, G^{\Sigma,l,n})$ and $(\omega, G^{\Pi,l,n})$ are $\aleph_0$-categorical.*
(2) *There is a $\exists_n$ formula $\psi_{l,n}$ in the language of $(l + n)$-graphs which is true in $(\omega, G^{\Sigma,l,n})$ but false in $(\omega, G^{\Pi,l,n})$. These formulas are:*

$$\begin{aligned} \psi_{l,n} \equiv\ & (\exists x_1, ..., x_{l+1} \text{ all different}) \\ & \neg(\exists x_{l+2} \neq x_1, .., x_{l+1})\neg...\neg(\exists x_{l+n} \neq x_1, ..., x_{n+l-1})\neg G(x_1, ..., x_{n+l}). \end{aligned}$$

(3) *There is a uniform computable procedure that given a Σ^0_n sentence φ in the language of arithmetic, builds an n graph $G^{\varphi,l}$ such that*

$$G^{\varphi,l} \cong \begin{cases} G^{\Sigma,l,n} & \text{if } \varphi \text{ holds} \\ G^{\Pi,l,n} & \text{if } \neg\varphi \text{ holds.} \end{cases}$$

Proof. The first two parts of the theorem are proved almost in exactly the same way as the first two parts of Theorem 4.2 in the previous section.

For Part (3) we need to define $G^{\varphi,l}$ with a slight modification of G^{φ}. Suppose φ is written in semi-diverse form as $\exists x_1 \neg\exists x_2 \neg...\neg\exists x_n \neg R^{\varphi}(x_1, ..., x_n)$. Let

$$R^{\varphi,l}(x_1, ..., x_{l+n}) = R^{\varphi}(\langle x_1, ..., x_{l+1}\rangle, x_{l+2}, ..., x_{l+n})$$

where $\langle \cdot, ..., \cdot\rangle$ is a computable bijection $\omega^{l+1} \to \omega$. We now define:

$$G^{\varphi,l} = R^{\varphi,l} \circ \bar{p}.$$

To show that $G^{\varphi,l}$ is the desired structure that satisfy Part (3) of the theorem we proceed as follows. First, we consider the mapping Q^{φ} as in the previous section. Second, we modify Q^{φ} in the following way. For $\bar{c} \in \omega^{<\omega}$ with $l < |\bar{c}| \leq l + n$, write $\bar{c}$ as $\bar{b}\bar{a}$ where $\bar{b} \in \omega^{l+1}$, and $\bar{a} \in \omega^{<n}$ and define $Q^{\varphi,l}(\bar{b}\bar{a}) = Q^{\varphi}(\langle \bar{b} \rangle \bar{a})$. For $\bar{c} \in \omega^{<\omega}$ with either $l \geq |\bar{c}|$ or $l + n > |\bar{c}|$ define $Q^{\varphi,l}$ in any way that makes it diverse.

As in the previous section the map $Q \circ \bar{p}$ is a random string 2-map. An analogous version of Lemma 4.7 is now proved in a similar matter. □

6. Putting the n-Graphs together

This is the last step of the proof of our main theorem. The main idea is to put the n-graphs built in the previous sections into one computable structure which is defined using the random 2-map.

Lett $S \subseteq \omega$ be a set which is one-to-one equivalent with $0^{(\omega)}$. Suppose that we have a list of sentences of the arithmetic $\varphi_1, \varphi_2, ...$, where each φ_i is Σ_i-sentence, such that for all $i \geq 1$ we have $i \in S$ if and only if φ_i holds.

Definition 6.1. Let (ω, r) be a random string 2-map. Define the following structure

$$\mathcal{A}_S = (\omega, H_1, H_2 ...),$$

where for each i, H_i is the $(1 + 2 + ... + i)$-ary relation

$$H_i = \begin{cases} G^{\Sigma, l_i, i} & \text{if } i \in S \\ G^{\Pi, l_i, i} & \text{if } i \notin S. \end{cases}$$

where $l_i = 1 + 2 + ... + (i - 1)$.

Lemma 6.2. *The structure $\mathcal{A}_S$ satisfies the following properties:*

(1) *The structure is $\aleph_0$-categorical.*
(2) *The theory of the structure is one-to-one reducible to S.*
(3) *The structure is computable.*

The first part of the lemma is obvious since $\mathcal{A}_S$ is being defined from the structure (ω, r) which is $\aleph_0$-categorical by Theorem 3.1.

The second part follows from the use of the formulas $\psi_{l,n}$ defined in the previous section (Theorem 5.2). Indeed, one can see that S is one-to-one reducible to the first order theory of the structure $\mathcal{A}_S$ since $i \in S$ if and only if $(\omega, H_i) \models \psi_{l_i, i}$.

For the last part, one notices that given i, l the structures $G^{\varphi_i, l}$ can be constructed effectively. Therefore the structure

$$(\omega, G^{\varphi_1, 0}, G^{\varphi_2, 1}, G^{\varphi_3, 3}, G^{\varphi_4, 6}, ...)$$

must be computable. This structure is isomorphic to $\mathcal{A}_S$. More explicitly, the structure $\mathcal{A}_S$ can be constructed as follows. Define $Q \colon \omega^{<\omega} \to \{0, 1\}$ by letting $Q(\bar{c}) = Q^{\varphi_i, l_i}(\bar{c})$ where i is such that $l_i < |\bar{c}| \leq l_i + i$. The mapping Q is diverse. Hence, the mapping $r = Q \circ \bar{p}$ is a random string 2-map. Consider the graphs $G^{\Sigma, l_i, i}$ and $G^{\Pi, l_i, i}$ using this r. The structure $\mathcal{A}_S$ is then $(\omega, G^{\varphi_1, 0}, G^{\varphi_2, 1}, G^{\varphi_3, 3}, G^{\varphi_4, 6}, ...)$.

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Department of Computer Science, Auckland University, Auckland, New Zealand

E-mail address: bmk@cs.auckland.ac.nz

Department of Mathematics, University of Chicago, 5734 S. University ave., Chicago, IL 60637, USA

E-mail address: antonio@math.uchicago.edu

URL: www.math.uchicago.edu/∼antonio