October 14, 2007

ON THE PI-ONE-ONE SEPARATION PRINCIPLE

ANTONIO MONTALBÁN

Abstract. We study the proof theoretic strength of the Π^1_1-separation axiom scheme. We show that Π^1_1-separation lies strictly in between the Δ^1_1-comprehension and Σ^1_1-choice axiom schemes over RCA_0.

1. Introduction

In this paper we analyze the strength of a particular subsystem of second order arithmetic. The topic is closely related to the program of Reverse Mathematics, which is the area of logic that studies the question of which subsystems of second order arithmetic are sufficient and necessary to prove theorems in classical mathematics. ([Sim99] is the standard reference for the subject.) A lot of work has been done on this program in the last couple of decades.

We study the following natural axiom scheme of second order arithmetic that we call Π^1_1-separation:

$$\Pi^1_1\text{-}\mathsf{SEP}: \nexists n \in \mathbb{N}\,(\varphi(n)\ \&\ \psi(n)) \ \Rightarrow\ \exists Z \subseteq \mathbb{N}\,\forall n \in \mathbb{N}\,(\varphi(n) \Rightarrow n \in Z\ \&\ \psi(n) \Rightarrow n \notin Z),$$

where φ and ψ are Π^1_1 formulas. In general, when φ and ψ are in some class of formulas Γ, we get the Γ-SEP axiom scheme. We use Π^1_1-SEP_0 to denote the system RCA_0+Π^1_1-SEP. We call the set Z in the statement of Π^1_1-SEP, a *separator* of φ and ψ.

Many separation statements have been studied in the context of reverse mathematics, computability theory and descriptive set theory. For example, in the context of computability theory, we know that a separator for Π^0_1 formulas can always be found to be computable. For Σ^1_0 formulas, we need PA degrees to compute separators, i.e. degrees which can compute completions of Peano Arithmetic. Σ^1_1 formulas can always be separated by hyperarithmetic sets, but this is not the case for Π^1_1 formulas. However, disjoint Π^1_1 formulas can always be separated by a hyperarithmetically low set. All these results have their counter part in Reverse Mathematics. See Table **??** below.

However, no equivalence of this sort has been found for Π^1_1-SEP_0. It will follow from our results that Π^1_1-SEP_0 is not equivalent to any other well-known subsystem of second order arithmetic. It is not hard to show that Π^1_1-SEP is in between two well-known systems of second order arithmetic, namely Δ^1_1-comprehension and Σ^1_1-choice:

$$\Delta^1_1\text{-}\mathsf{CA}: \quad \forall n \in \mathbb{N}\ (\varphi(n) \Leftrightarrow \neg\psi(n)) \Rightarrow \exists Z \subseteq \mathbb{N}\ \forall n \in \mathbb{N}\ (n \in Z \Leftrightarrow \varphi(n)),$$

$$\Sigma^1_1\text{-}\mathsf{AC}: \quad \forall n \in \mathbb{N}\ \exists X \subseteq \mathbb{N}(\varphi(n, X)) \Rightarrow \exists Z \subseteq \mathbb{N}^2\ \forall n \in \mathbb{N}\ (\varphi(n, Z^{[n]})),$$

The author would like to thank Kazuyuki Tanaka for pointing out that Π^1_1-SEP_0 is a statement of hyperarithmetic analysis and motivating the author to work on this question. This research was partially supported by NSF grant DMS-0600824.

Γ	Γ-SEP_0	reference in [Sim99]
Π^0_1	RCA_0	Exercise IV.4.8
Σ^0_1	WKL_0	Lemma IV.4.4
Π^0_n, $n \geq 2$	ACA_0	
Σ^1_1	ATR_0	Theorem V.5.1
Π^1_2	Δ^1_2-CA_0	Exercise VII.6.13
Σ^1_2	Π^1_2-CA_0	Exercise VII.6.14

TABLE 1. The second column lists the systems equivalent to Γ-SEP over RCA_0. To prove the last two rows, use the Π^1_1-uniformization principle [Sim99, Lemma VI.2.1].

where φ and ψ are Σ^1_1 formulas, and $Z^{[n]} = \{m : \langle n, m\rangle \in Z\}$. Together with RCA_0, these axioms schemes form the systems Δ^1_1-CA_0 and Σ^1_1-AC_0. The latter of these implications appears in [Sim99, Exercise V.5.7]. For the former implication, note that Δ^1_1-CA is a particular case of Π^1_1-SEP.

The main result of this paper is that Π^1_1-SEP_0 lies strictly in between Δ^1_1-CA_0 and Σ^1_1-AC_0. We will show that there is an ω-model of Π^1_1-SEP_0 where Δ^1_1-CA does not hold, and an ω-model of Δ^1_1-CA_0 where Π^1_1-SEP does not hold.

Theories of hyperarithmetic analysis. Consider HYP, the ω-model of second order arithmetic, whose second order objects are the hyperarithmetic sets. Let φ and ψ be Π^1_1 formulas. By the hyperarithmetic quantifier theorem of Spector [Spe60] and Gandy [Gan60] (see [Sim99, Theorem VII.3.2]), we have that the sets $\{n : \varphi^{HYP}(n)\}$ and $\{n : \psi^{HYP}(n)\}$ are Σ^1_1. Therefore, if these sets are disjoint, there exists a hyperarithmetic set Z which separates them. It follows that

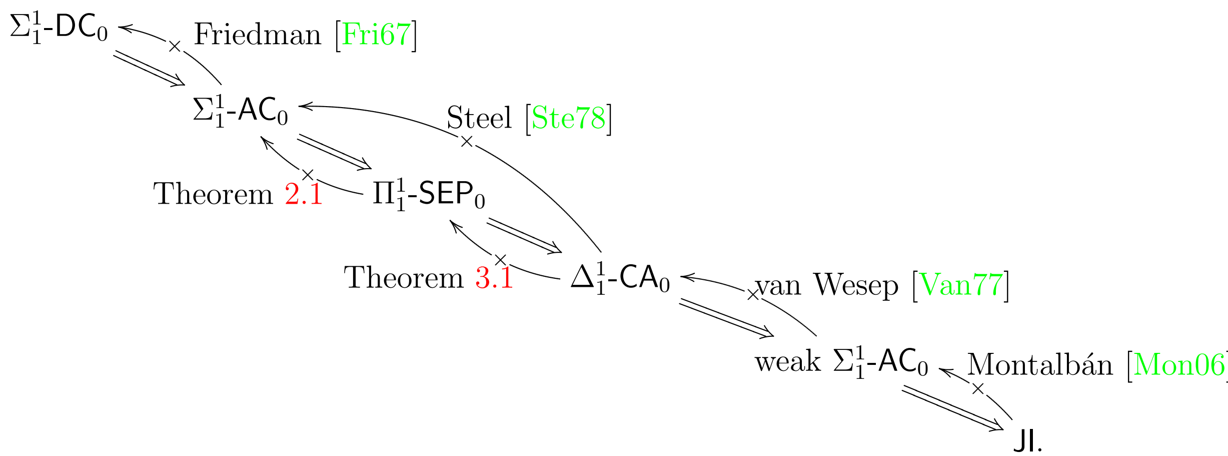
$$HYP \models \Pi^1_1\text{-}\mathsf{SEP}_0.$$

Moreover, Π^1_1-SEP_0 is what we call a theory of hyperarithmetic analysis.

Definition 1.1. A set of sentences of second order arithmetic S is a *theory of hyperarithmetic analysis* if for every $Y \subseteq \omega$, $HYP(Y)$ is the least ω-model of S containing Y.

Examples of known theories of hyperarithmetic analysis are the following schemes: Σ^1_1-dependent choice (Σ^1_1-DC_0), Σ^1_1-choice (Σ^1_1-AC_0), Δ^1_1-comprehension (Δ^1_1-CA_0), and weak-Σ^1_1-choice (weak-Σ^1_1-AC_0). The unrelativized versions of these results were proved by Harrison [Har68], Kreisel [Kre62], [Kle59] and [Sim99, Theorem VIII.4.16]. (See [Sim99, Section VII.6] for definitions of these statements.) The easier way to see that Π^1_1-SEP_0 is a theory of hyperarithmetic analysis is using the fact that it is in between Σ^1_1-AC_0 and Δ^1_1-CA_0. As listed above, these four statements go from strongest to weakest, they all imply ACA_0, and, except for Σ^1_1-DC_0, they are implied by ATR_0 (see [Sim99, VIII.3 and VIII.4]). Moreover, the implications Σ^1_1-$\mathsf{DC}_0 \Rightarrow \Sigma^1_1$-$\mathsf{AC}_0$, Σ^1_1-$\mathsf{AC}_0 \Rightarrow \Delta^1_1$-$\mathsf{CA}_0$, and Δ^1_1-$\mathsf{CA}_0 \Rightarrow$weak-$\Sigma^1_1$-$\mathsf{AC}_0$ cannot be reversed as proved by Friedman [Fri67], Steel [Ste78] and van Wesep [Van77], respectively.

In [Mon06], the author provides the first example of a theorem in classical mathematics that is a statement of hyperarithmetic analysis. This is a theorem that has to do with indecomposable linear orderings and was proved by Jullien in [Jul69]. It is still unknown whether this statement about linear ordering is equivalent to Δ^1_1-CA_0 or not. In [Mon06], the author also introduces five other statements of hyperarithmetic analysis, four of which are about finitely terminating games (clopen games) and the other one is about Turing jump iterations. The latter one, that we call JI, was very useful to prove that certain ω-models are closed under hyperarithmetic reducibility. JI says that for every set X and every ordinal α, if we have that $\forall\beta < \alpha(X^{(\beta)}$ exists), then $X^{(\alpha)}$ also exists, where $X^{(\beta)}$ is the βth iteration of the Turing jump of X.



As opposed to the "big five systems", namely RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-ACA_0, theories of hyperarithmetic analysis seem to have a very unstable behavior: Small modifications of theories of hyperarithmetic analysis give, in most of the cases, inequivalent theories.

The proofs. Our main tool is Steel's forcing with tagged trees [Ste78]. Steel used this forcing to prove that Σ^1_1-AC_0 is strictly stronger than Δ^1_1-CA_0. Then, van Wesep [Van77] used it to prove that Δ^1_1-CA_0 is strictly stronger than weak Σ^1_1-AC_0, and the author [Mon06] used it to prove that weak Σ^1_1-AC_0 is strictly stronger than JI.

To prove that Σ^1_1-AC is strictly stronger than Π^1_1-SEP, we will use the same model M_∞ that Steel constructed in [Ste78, Section 5], which satisfies Δ^1_1-CA but not Σ^1_1-AC. All we do is to modify the proof that $M_\infty \models \Delta^1_1$-$\mathsf{CA}$, to get that $M_\infty \models \Pi^1_1$-SEP. A modification of this model will be used to prove that Π^1_1-SEP is strictly stronger than Δ^1_1-CA.

For completeness we include the whole definition of the forcing notion and of these models. Familiarity with [Ste78] is not assumed. Our presentation is different than Steel's [Ste78], although some parts of our presentation are exactly the same. He works with models of set theory and takes the intersection with $\mathcal{P}(\omega)$ when he wants to consider models of second order arithmetic. We directly build ω-models of second order arithmetic, which is probably more natural for the reverse mathematics audience.

Notation. A *tree* is a downward closed subset of $\omega^{<\omega}$, the set of finite strings of natural numbers. Given a tree T, to each node $s \in T$ we assign its *well founded rank* $|s|_T$ which is either an ordinal or ∞ and satisfies $|s|_T = \sup\{|t|_T + 1 : t \in T, s \subsetneq t\}$. So $|s|_T = \infty$ exactly when s is not in the well-founded part of T. We will use $F \subset_f \omega$ to say that F is a finite subset of ω.

When we write formulas of second order arithmetic, we will use lower case roman characters x, y, z, n, m, d for variables that represent numbers and upper case roman characters X, Y, Z, D for sets of numbers. An *ω-model* of second order arithmetic is one whose first order part is the standard model of the natural numbers, namely ω.

Our forcing language will consist of computable infinitary formulas of a certain kind. A *computable infinitary Σ^0_0 or Π^0_0 formula* is just an open formula. Given a computable ordinal α, a *computable infinitary Σ^0_α formula* is given by a (possibly infinite) disjunction of a c.e. set of formulas $\exists\bar{x}\psi_i(\bar{x})$, where ψ_i is computable infinitary Π^0_β for some $\beta < \alpha$. A *computable infinitary Π^0_α formula* is given by a (possibly infinite) conjunction of a c.e. set of formulas $\forall\bar{x}\psi_i(\bar{x})$, where ψ_i is computable infinitary Σ^0_β for some $\beta < \alpha$. The class of formulas $\Sigma^0_\alpha(Y_1, ..., Y_k)$ is defined analogously but allowing Y_1,...,Y_k to appear as parameters. We say that $X \subseteq \mathbb{N}$ is a $\Sigma^0_\alpha(Y)$ set if it can be defined by a $\Sigma^0_\alpha(Y)$ formula. See [AK00, Chapter 7] for more background on computable infinitary formulas.

2. Π^1_1-SEP$_0$ does not imply Σ^1_1-AC$_0$

This section is dedicated to prove the following theorem.

Theorem 2.1. *There is an ω-model of Π^1_1-SEP_0 which is not a model of Σ^1_1-AC_0. Therefore, Π^1_1-SEP_0 does not imply Σ^1_1-AC_0.*

In [Ste78, Section 5], Steel defined an ω-model $M_\infty \cap P(\omega)$ and proved it is a model of Δ^1_1-CA_0, but not of Σ^1_1-AC_0. We will prove that it is also a model of Π^1_1-SEP_0. We will change Steel's notation and use M_∞ to refer to what Steel called $M_\infty \cap P(\omega)$. The definition of M_∞ is given below. Our presentation is slightly different than Steel's, as it is done in the context of ω-models of second order arithmetic, rather than models of set theory. But the construction is essentially the same as Steel's, except for Subsection 2.7 which contains the new lemmas necessary to show that $M_\infty \models \Pi^1_1$-SEP_0.

2.1. **The model M_∞.** To define M_∞ we define a generic object

$$G = \langle T^G, \{\alpha^G_i : i \in \omega\}, h^G\rangle,$$

where T^G is a tree on ω; $\{\alpha^G_i : i \in \omega\}$ is a set of paths though T^G; and $h^G : T^G \to \omega^{CK}_1 \cup \{\infty\}$ is the well founded rank function for T^G. In other words, $h^G(s) = |s|_{T^G}$. We use the function h^G to ensure that the tree T^G looks well-founded in $\mathcal{M}_\infty$, and to prove properties of the forcing notion.

For each $F \subset_f \omega$, we let M_F be the class of sets which are definable by a computable infinitary formula that may use T^G and α^G_i for $i \in F$ as parameters. In other words,

$$M_F = \{X \subseteq \omega : \exists\mu < \omega^{CK}_1 \ (X \in \Sigma^0_\mu(T^G, \alpha^G_i : i \in F))\},$$

Or equivalently,

$$M_F = \mathcal{P}(\omega) \cap L_{\omega^{CK}_1}(\{T^G\} \cup \{\alpha^G_i : i \in F\}),$$

where $L_{\omega^{CK}_1}(\{T^G\} \cup \{\alpha^G_i : i \in F\})$ is the set of Gödel-constructible sets up to level ω^{CK}_1, starting from $\{T^G\} \cup \{\alpha^G_i : i \in F\}$. (See [Ste78, page 57] for a definition of $L_\mu(A)$.) We will show in Lemma 2.9 that M_F is closed under hyperarithmetic reduction. Finally, let

$$M_\infty = \bigcup_{F \subset_f \omega} M_F.$$

We will show that for each $F \subset_f \omega$, $\{\alpha_i^G : i \in F\}$ is the set of paths though T^G which belong to M_F [Ste78, Lemma 7]. That is, $M_\infty \cap [T^G] = \{\alpha_i^G : i \in \omega\}$. It will then follow that the sequence $\langle \alpha_i^G : i \in \omega\rangle$ does not belong to M_∞. This is the fact used to show that $M_\infty \not\models \Sigma_1^1\text{-}\mathsf{AC}_0$.

For notational convenience, given $\mu < \omega_1^{CK}$, let us also define

$$M_{\mu,F} = \{X \subseteq \omega : \exists \nu < \mu\ (X \text{ is } \Sigma_\nu^0(T^G, \alpha_i^G : i \in F))\} \quad \& \quad M_{\mu,\infty} = \bigcup_{F \subset_f \omega} M_{\mu,F}.$$

Let us also name the elements of M_∞. Let $H_{F,1} = T^G \oplus \bigoplus_{i\in F} \alpha_i^G$. Given $\mu < \omega_1^{CK}$, and $e \in \omega$, let

$$H_{F,\mu} = \bigoplus_{\nu<\mu, e\in\omega} S_{\nu,F,e} \quad \text{and} \quad S_{\mu,F,e} = W_e^{H_{F,\mu}}$$

where W_e^Y is the eth c.e. set relative to an oracle Y.

It is not hard to see that the set $S_{\mu,F,e}$ is $\Sigma_\mu^0(T^G, \alpha_i^G : i \in F)$ and that $M_{\mu,F} = \{S_{\nu,F,e} : e \in \omega, \nu < \mu\}$.

2.2. **The forcing notion.** The forcing conditions are finite approximations to $\langle T^G, \{\alpha_i^G : i \in \omega\}, h^G\rangle$ in a standard way. The set of forcing conditions P, defined in [Ste78, page 68], consists of the triplets $p = \langle T^p, f^p, h^p\rangle$ where

(1) T^p is a finite tree;
(2) f^p is a nonempty finite function, $\operatorname{dom} f^p \subset_f \omega$, and *ran* $f^p \subseteq T^p$;
(3) $h^p : T^p \to \omega_1^{CK} \cup \{\infty\}$ so that
  (a) $\forall s, t \in T^p (s \subsetneq t \Rightarrow h^p(s) > h^p(t))$,
  (b) $\forall s \in T^p (\exists i (s \subseteq f^p(i)) \Rightarrow h^p(s) = \infty)$, and
  (c) $h^p(\emptyset) = \infty$.

By fiat, $\infty > \infty$ and $\infty > \omega_1^{CK}$.

For $p, q \in P$ define $p \le q$ iff

(4) $T^q \subseteq T^p$;
(5) (a) $\operatorname{dom} f^q \subseteq \operatorname{dom} f^p$;
  (b) $\forall i \in \operatorname{dom} f^q (f^q(i) \subseteq f^p(i))$;
  (c) $\forall i \in \operatorname{dom} f^q (\nexists s \in T^q (f^q(i) \subset s \subseteq f^p(i)))$;
  (d) $\forall i \in \operatorname{dom} f^p \smallsetminus \operatorname{dom} f^q \forall s \subseteq f^p(i) (s \neq \emptyset \Rightarrow s \notin T^q)$
(6) $h^q = h^p \restriction T^q$.

Let $\mathbf{P} = (P, \le)$ and G be a sufficiently $\mathbf{P}$-generic filter (generic enough to force all the Σ over $\mathcal{L}_F$ formulas of the forcing language defined below). Define $T^G = \bigcup_{p\in G} T^p$, $\alpha_i^G = \bigcup_{p\in G} f^p(i)$ for $i \in \omega$, and $h^G = \bigcup_{p\in G} h^p$.

Conditions (5c) and (5d) above are a bit subtle, and are the only ones that could be unexpected. They are necessary to prove Lemma 2.5. Part (5c) implies that if $p \in G$ and $f^p(i)$ is defined, then $f^p(i)$ is the largest initial segment of α_i^G which is in T^p. Part (5d), that is not mentioned in [Ste78] but needs to be added, says that if $p \in G$ and $f^p(i)$ is not defined, then no initial segment of α_i^G is in T^p, except for the empty string. The essential reason to include this clause, that will become clearer later, is that if $p \in G$ and $s \in T^p$, $s \neq \emptyset$, then whether or not there exists a path through T^G in M_∞ extending s depends only on whether $\exists i \in \operatorname{dom} f^p (s \subseteq f^p(i))$ or not, and is independent of the values of h^p. Conditions (5b), (5c) and (5d) can be stated as $\{(i,s) \in \omega \times \omega^{<\omega} : \emptyset \subset s \subseteq f^q(i)\} = \{(i,s) \in \omega \times \omega^{<\omega} : \emptyset \subset s \subseteq f^p(i)\} \cap \omega \times T^q$.

2.3. **The forcing language.** The forcing language is a modification of $\mathcal{L}_\infty$ defined in [Ste78, page 58] into a language adequate for second order arithmetic. The language $\mathcal{L}_\infty$ is appropriate to describe M_∞, and, for each $F \subset_f \omega$, the language $\mathcal{L}_F$ is appropriate to describe M_F.

The symbols of $\mathcal{L}_F$ are: $\in$, $=$, $+$, $\times$ $\leq$; constants for natural numbers $\mathbf{0}$, $\mathbf{1}$,...; number variables n, m, x, y, z, unranked set variables $X_H, Y_H, \ldots$ for $H \subseteq F$, ranked set variables $X_H^\lambda, X_H^\lambda, \ldots$ for $H \subseteq F$ and $\lambda < \omega_1^{CK}$; connectives $\wedge$, $\neg$; quantifier $\forall$ for both number and set variables. A class C^F of constants intended to name the elements of M_F is defined by recursion:

- $C_0^F = \{\mathbf{T}, \boldsymbol{\alpha}_i, i \in F\}$;
- $C_{\nu+1}^F = C_\nu^F \cup \{\mathbf{S}_{\nu,F,e} : e \in \omega\} \cup \{\mathbf{H}_{\nu,F}\}$;
- $C_\lambda^F = \bigcup_{\mu<\lambda} C_\mu^F$ for λ limit;
- $C^F = \bigcup_{\mu<\omega_1^{CK}} C_\mu^F$.

Each constant in C^F is also a symbol of $\mathcal{L}_F$. Let $C = \bigcup_{F \subset_f \omega} C^F$.

The language $\mathcal{L}_\infty$, in addition to the symbols of $\bigcup_{F \subset_f \omega} \mathcal{L}_F$, has ranked set variables $X^\nu, Y^\nu, \ldots$ for $\nu < \omega_1^{CK}$ and unranked set variables $X, Y, \ldots$. We call a variable F-restricted iff it is subscripted G for some $G \subseteq F$; a formula of $\mathcal{L}_\infty$ is F-restricted iff all its bounded variables are F-restricted.

The semantics of the various $\mathcal{L}_F$ and of $\mathcal{L}_\infty$ is straightforward. Simply remember that $\mathbf{T}$ denotes T, $\boldsymbol{\alpha}_i$ denotes α_i^G, $\mathbf{S}_{\nu,F,e}$ denotes $S_{\nu,F,e}$; X_F^ν ranges over $M_{\nu,F}$, X^ν ranges over $M_{\nu,\infty}$, X_F ranges over M_F and X ranges over M_∞.

A formula of $\mathcal{L}_\infty$ is *ranked* if all its bounded variables are ranked. If ψ is a formula of $\mathcal{L}_\infty$, $o(\psi)$ is the least upper bound of $\{\nu : \nu$ is the superscript of a quantified variable in $\psi\} \cup \{\nu + 1 :$ some constant of the form $\mathbf{S}_{\nu,F,e}$ or $\mathbf{H}_{\nu,F}$ occurs in $\psi\}$. If $c \in C$, let $o(c) = o(\emptyset \in c)$. If $\psi \in \mathcal{L}_\infty$, let

$$\mathrm{rk}(\psi) = \omega_1^{CK} \cdot u(\psi) + \omega^2 \cdot o(\psi) + \omega \cdot r(\psi) + n(\psi),$$

where $u(\psi)$ is the number of unranked quantifiers in ψ, $r(\psi)$ is the number of ranked quantifiers, and $n(\psi)$ is the number of connectives.

2.4. **The forcing relation.** The definition of the forcing relation is also standard. It can be proved by transfinite induction on the rank of formulas that $p \Vdash \psi$ if and only if whenever G is a sufficiently generic filter, $p \in G$ and $\mathcal{M}_\infty$ is the model defined from G, we have that $\mathcal{M}_\infty \models \psi$. This property is what motivated the definition of $p \Vdash \sigma \in T$ below.

Definition 2.2. The forcing relation for formulas of $\mathcal{L}^\infty$ is defined as usual:

(1) $p \Vdash \psi$ if ψ holds when ψ is a quantifier free formula of arithmetic;
(2) $p \Vdash \sigma \in \mathbf{T}$ if either $|\sigma| < 2$, $\sigma \in T^p$, or $\sigma^- \in T^p$ and $h^p(\sigma^-) \geq 1$;
(3) $p \Vdash \langle \mathbf{n}, \mathbf{m} \rangle \in \boldsymbol{\alpha}_i$ if $i \in \mathrm{dom}\, f^p$ and $f^p(i)(n) = m$;
(4) $p \Vdash \mathbf{n} \in \mathbf{S}_{\nu,F,e}$ if $p \Vdash \exists s R(\mathbf{H}_{\nu,F}; e, s, \mathbf{n})$ where R is a Δ_0^0 formula which codes a universal Turing machine;
(5) $p \Vdash \langle \mathbf{e}, \mathbf{n}, \mu \rangle \in \mathbf{H}_{\nu,F}$ if $\mu < \nu$ and $p \Vdash \mathbf{n} \in \mathbf{S}_{\mu,F,e}$;

(6) $p \Vdash \forall x \psi(x)$ if for all $n \in \omega$, $p \Vdash \psi(\mathbf{n})$;
(7) $p \Vdash \forall X_F^\lambda \psi(X_F^\lambda)$ if for all $\nu < \lambda, e \in \omega$, $p \Vdash \psi(\mathbf{S}_{\nu,F,e})$;
(8) $p \Vdash \forall X_F \psi(X_F^\lambda)$ if for all $\nu < \omega_1^{CK}, e \in \omega$, $p \Vdash \psi(\mathbf{S}_{\nu,F,e})$;
(9) $p \Vdash \forall X^\lambda \psi(X_F^\lambda)$ if for all $\nu < \lambda, e \in \omega, F \subset_f \omega$, $p \Vdash \psi(\mathbf{S}_{\nu,F,e})$;
(10) $p \Vdash \forall X \psi(X_F^\lambda)$ if for all $\nu < \omega_1^{CK}, e \in \omega, F \subset_f \omega$, $p \Vdash \psi(\mathbf{S}_{\nu,F,e})$;
(11) $p \Vdash \bigwedge_{i\in\omega} \psi_i$ if for every i, $p \Vdash \psi_i$;
(12) $p \Vdash \neg\psi$ if for every $q \leq p$, $q \nVdash \psi$.

2.5. **Retaggings.** The next definition is a key concept when forcing with tagged trees.

Definition 2.3. Let $p, p^* \in \mathbf{P}$, $F \subset_f \omega$ and $\mu \in \omega_1^{CK}$. We say that p^* is an *μ-F-absolute retagging* of p, and we write $Ret(\mu, F, p, p^*)$, if

(1) $T^p = T^{p^*}$, and $f^p \restriction F = f^{p^*} \restriction F$;
(2) for all $\sigma \in T^p$, if $h^p(\sigma) < \mu$, then $h^{p^*}(\sigma) = h^p(\sigma)$; and
(3) if $h^p(\sigma) \geq \mu$, then $h^{p^*}(\sigma) \geq \mu$.

Note that for fixed F and μ, $Ret(\mu, F; \cdot, \cdot)$ is an equivalence relation on P. Intuitively, two conditions are retaggings of each other if they are indistinguishable to somebody living in $M_{\mu,F}$. This idea is reflected in the following lemma.

Lemma 2.4. [Ste78, Lemma 6] *Let ψ be a formula ranked in $\mathcal{L}^F$ and let $p, p^* \in \mathbf{P}$ then*

$$Ret(\omega \cdot \mathrm{rk}(\psi), F, p, p^*) \quad \Rightarrow \quad (p \Vdash \psi \Leftrightarrow p^* \Vdash \psi). \tag{2.1}$$

To prove this lemma, the following lemma is key.

Lemma 2.5. [Ste78, Sublemma 4] *Let p^* is an $\omega \cdot \beta$-F-*absolute retagging* of p and let $\gamma < \beta$. Let $q \leq p$. Then, there exists $q^* \leq p^*$ such that $Ret(\omega \cdot \gamma, F, q, q^*)$.*

$$\begin{tikzcd}
p \arrow[r, dotted, dash, "Ret(\omega\beta\ldots)"] \arrow[d, dash] & p^* \arrow[d, dash] \\
q \arrow[r, dotted, dash, "Ret(\omega\gamma\ldots)"] & q^*
\end{tikzcd}$$

Proof. Let us start proving the following claim. Suppose $Ret(\delta+1, F, p, p^*)$, $q \leq p$ and $T^q \smallsetminus T^p$ has only one element σ. Then there exists $q^* \leq p^*$ such that $Ret(\delta, F, q, q^*)$. The lemma will then follow by induction on the size of $T^q \smallsetminus T^p$.

Let $T^{q^*} = T^q = T^{p^*} \cup \{\sigma\}$ and $f^{q^*} = f^q \restriction F$. For $\tau \in T^{p^*}$, let $h^{q^*}(\tau) = h^{p^*}(\tau)$. Let $\tau = \sigma^-$. There are two cases. The first case is $h^q(\sigma) \leq \delta$: define $h^{q^*}(\sigma) = h^q(\sigma)$. The second case is $h_j^q(\sigma) > \delta$. In this case, if $\sigma \subseteq f^q(i)$ for some $i \in \mathrm{dom}\, f^q$, then let $h^{q^*}(s) = \infty$, otherwise it is safe to let $h^{q^*}(s) = \delta$. It is not hard to check that q^* is as wanted. Notice that to prove that q^* satisfies Condition (3a) in the definition of $\mathbf{P}$, we have to use Condition (5c) and (5d) of the definition of $q \leq p$. $\square$

Proof of Lemma 2.4. The proof is by transfinite induction on β, the rank of ψ. All the cases are trivial except for $\psi = \neg\varphi$. Suppose that $p^* \Vdash \neg\varphi$; we want to show that $p \Vdash \neg\varphi$. Consider $q \leq p$; we need to show that $q \nVdash \varphi$. Let $\gamma < \beta$ be the rank of φ. By the previous lemma there is a $q^* \leq p^*$ which is a $\omega \cdot \gamma$-F-absolute retagging of q. Then, since $p^* \Vdash \neg\varphi$, we have that $q^* \nVdash \varphi$, and hence $q \nVdash \varphi$. $\square$

Let $P_\beta = \{p \in P : ran\ h^p \subseteq \beta \cup \{\infty\}\}$. It follows from Lemma 2.4 that if we have $\psi = \neg\varphi$ of rank β and $p \in P_{\omega\cdot\beta}$, Condition 12 in the definition of forcing can be replaced by

(12′) $p \Vdash \neg\psi$ if for every $q \in P_{\omega\cdot\beta}$, $q \leq p$ we have that $q \nVdash \psi$.

The following corollary can then be proved by transfinite induction on β.

Corollary 2.6. [Ste78, Lemma 8] *For a formula ψ of rank β, $0^{(\beta)}$ can decide whether $p \Vdash \psi$ uniformly in ψ, p and β. (Actually, less than $0^{(\beta)}$ is required.)*

Lemma 2.7. [Ste78, Lemma 7] $M_F \cap [T^G] = \{\alpha_i^G : i \in F\}$ *for each* $F \subset_f \omega$.

Proof. Suppose, toward a contradiction, that $S = S_{\nu,F,e} \in \mathcal{M}_F$ is a path through T^G which is different from α_i^G for $i \in F$. There exists $\sigma \subset S$ such that $\sigma \not\subset \alpha_i^G$ for every $i \in F$. Let $p \in G$ be such that $\sigma \in T^p$ and

$$p \Vdash \mathbf{S} \in [\mathbf{T}]\ \&\ \sigma \subseteq \mathbf{S}\ \&\ \forall i \in F(\sigma \not\subseteq \boldsymbol{\alpha}_i)$$

Let β be greater than the ω times the rank of the formula forced above and big enough so that $p \in P_\beta$. Since S is a path in T^G and $\sigma \subseteq S$, we have that $h^G(\sigma) = \infty$ and hence $h^p(\sigma) = \infty$. Now, we define p^* such that $Ret(\beta, F, p, p^*)$ and $h^{p^*}(\sigma) \in \omega_1^{CK}$. To define p^*, all we have to do is to change the values of $h^p(\tau)$ for $\tau \supseteq \sigma$ to ordinals in ω_1^{CK} which are above β. Since $\forall i \in F(\sigma \not\subseteq \alpha_i^G)$, there is no difficulty defining p^* as desired. But then, by Lemma 2.4, we have that

$$p^* \Vdash \mathbf{S} \in [\mathbf{T}]\ \&\ \sigma \subseteq \mathbf{S}\ \&\ \forall i \in F(\sigma \not\subseteq \boldsymbol{\alpha}_i).$$

But, on the other hand, since $h^{p^*}(\sigma) \in \omega_1^{CK}$, σ is in the well-founded part of T^{G^*} for any generic filter G^* extending p^*. □

Corollary 2.8. [Ste78, Corollary of Lemma 7] $\mathcal{M}_\infty \not\models \Sigma_1^1$-*AC.*

Proof. Let ψ be the following Σ_1^1 formula:

$$\psi(n, X) \quad \equiv \quad \exists X = \langle X_1, ..., X_n\rangle\ \forall i \leq n\ (X_i \in [T^G]) \quad \&\ \forall i \neq j (X_i \neq X_j).$$

Since $\forall i(\alpha_i^G \in [T^G])$, $M_\infty \models \forall n \exists X\ \psi(n, X)$. If we had $\mathcal{M}_\infty \models \Sigma_1^1$-AC, then there would be set $Z \in M_\infty$ such that for every n, $Z^{[n]} \in [T^G]$ and for $n \neq m$, $Z^{[n]} \neq Z^{[m]}$. However, this set Z would have to be in M_F for some finite F. But this will imply that $[T^G]$ has infinitely many paths in M_F contradicting the previous lemma. □

We say that a formula is Σ-over-$\mathcal{L}_F$ iff it is built up from ranked, F-restricted formulas using $\wedge$, $\forall n$, and $\exists X$. For any formula ψ and $\mu < \omega_1^{CK}$, ψ^μ is the result of replacing "X" by "X^μ" for each unranked (i.e. unsuperscripted) variable X. We observe that if ψ is Σ-over-$\mathcal{L}_F$, $\mu < \omega_1^{CK}$ and $\mu > o(\mathbf{d})$ for any constant $\mathbf{d}$ in ψ, then $\psi^\mu \Rightarrow \psi$.

Lemma 2.9. [Ste78, Lemma 9]

(1) *Let $p \Vdash \psi$ where $\psi \in \mathcal{L}_\infty$, ψ is Σ-over-$\mathcal{L}_F$, and $F \subset_f \omega$. Then $\exists \mu < \omega_1^{CK} \forall \rho(\mu \leq \rho < \omega_1^{CK} \Rightarrow p \Vdash \psi^\rho)$.*

(2) *$M_F \models \Sigma_1^1$-AC_0, and hence M_F is hyperarithmetically closed. Moreover $M_F = HYP(T \oplus \bigoplus_{i\in F} \alpha_i^G)$.*

Proof. Note that by persistence upwards of Σ formulas, we only need to show that for some $\mu < \omega_1^{CK}$, $p \Vdash \psi^\mu$. We use induction on the way ψ is build from ranked, F-restricted formulas. If $\psi \equiv \forall n \varphi(n)$, then, by the induction hypothesis, for each $n \in \omega$, exists μ_n such that $p \Vdash \varphi^{\mu_n}(\mathbf{n})$. Since whether $p \Vdash \varphi^{\mu_n}(\mathbf{n})$ can be decided hyperarithmetically, there is a $\mu < \omega_1^{CK}$ which bounds all the μ_n. For this μ, $p \Vdash \psi^\mu$. The case $\psi \equiv \varphi_0 \wedge \varphi_1$ is similar, but easier.

Suppose now that $\psi \equiv \exists X \varphi(X)$. That $p \Vdash \psi$ means that for every $q \leq p$ exists $r \leq q$ and $\mathbf{S} \in C$ such that $r \Vdash \varphi(\mathbf{S})$, and then by the induction hypothesis $r \Vdash \varphi^\rho(\mathbf{S})$ for some $\rho < \omega_1^{CK}$. Again, since whether $r \Vdash \varphi^\rho(\mathbf{S})$ can be decided hyperarithmetically, for each $\beta < \omega_1^{CK}$ there exists $\gamma < \omega_1^{CK}$ such that for every $q \in P_\beta$, there exists $r \in P_\gamma$, $\mathbf{S} \in C_\gamma$ and $\gamma_q < \gamma$ such that $r < q$, $\omega \cdot \mathrm{rk}(\varphi^{\gamma_q}(\mathbf{S})) < \gamma$ and

$$r \Vdash \varphi^{\gamma_q}(\mathbf{S}).$$

Since we can find γ from β hyperarithmetically, there exists a limit ordinal μ such that for every $\beta < \mu$ we can find such a γ also $< \mu$. We claim that $p \Vdash \psi^\mu$. To prove this claim, consider $q < p$. Let q^* be obtained by changing the value of $h^q(s)$ to ∞ for every s with $h^q(s) \geq \mu$. So, $Ret(\mu, \mathrm{dom}\, f^q, q, q^*)$ and since μ is a limit ordinal, we get that $q \in P_\beta$ for some $\beta < \mu$. Then, there exists γ, $r^* \in P_\mu$, $\mathbf{S} \in C_\mu$ and $\gamma_q < \mu$ such that $r^* < q^*$, $\omega \cdot \mathrm{rk}(\varphi^{\gamma_q}(\mathbf{S})) < \gamma$ and $r^* \Vdash \varphi^{\gamma_q}(\mathbf{S})$. By Lemma 2.5, there exists $r \leq q$ such that $Ret(\gamma, F, r, r^*)$, and then by Lemma 2.4, $r \Vdash \varphi^{\gamma_q}(\mathbf{S})$, and in particular $r \Vdash \varphi^\mu(\mathbf{S})$.

For the second part, suppose that $M_F \models \forall n \exists X \varphi(n, X)$, where φ is Σ^1_1. Let $p \in G$ be such that $p \Vdash \forall n \exists X \varphi(n, X)$. By the first part, there exists $\mu < \omega_1^{CK}$ such that $p \Vdash \forall n \exists X^\mu \varphi^\mu(n, X)$. So, for each $n \in \omega$, there exists $S_{F,\mu,e_n} \in M_{\mu,F}$ such that $M_F \models \varphi^\mu(\mathbf{n}, \mathbf{S}_{F,\mu,e_n})$. The function $n \mapsto e_n$ is computable in $H_{F,\mu+\omega}$, and hence, the set $\bigoplus_{e\in\omega} S_{F,\mu,e_n}$ belongs to M_F. So, $M_F \Vdash \exists Z \forall n (\varphi(n, Z^{[n]}))$. □

2.6. **Automorphisms of P.** Another useful tool will be the automorphisms of $\mathbf{P}$. Let $\pi\colon \omega \to \omega$ be a permutation of ω. Then, π induces an automorphism $\hat{\pi}$ of $\mathbf{P}$ as follows: $T^{\hat{\pi}(p)} = T^p$, $h^{\hat{\pi}(p)} = h^p$ and $f^{\hat{\pi}(p)}(\pi(i)) = f^p(i)$. Given $\psi \in \mathcal{L}_\infty$, let $\pi\psi$ be the formula obtained from ψ by replacing $\boldsymbol{\alpha}_i$ by $\boldsymbol{\alpha}_{\pi(i)}$ for each i.

Lemma 2.10. [Ste78, Lemma 10] *Let π be a permutation of ω, $p \in P$, and $\psi \in \mathcal{L}_\infty$. Then $p \Vdash \psi \Leftrightarrow \hat{\pi}(p) \Vdash \pi\psi$.*

Proof. Use induction on the rank of ψ. □

Observation 2.11. The main usage of these automorphisms is the following one. Let F be a finite subset of ω and let $K \subset \omega$ be finite and disjoint from F. Suppose that $p \in P$ has $\mathrm{dom}\, f^p \subseteq F$, and that ψ has constants in $\mathcal{L}_F$. Now, suppose that there exists a condition $r \in P$ such that $r \leq p$ and $r \Vdash \psi$. Then, using an automorphism if necessary, we can replace r by a condition $\hat{\pi}(r)$ such that $\hat{\pi}(r) \leq p$, $\hat{\pi}(r) \Vdash \psi$ and $\mathrm{dom}\, f_{\hat{\pi}(r)}$ is disjoint from K. In other words, replacing r by $\hat{\pi}(r)$ if necessary, we could have started assuming that $\mathrm{dom}\, r \cap K = \emptyset$.

Lemma 2.12. [Ste78, Lemma 11] *Let $F \subset_f \omega$ and ψ be a Σ-over-$\mathcal{L}_F$ sentence. Suppose $\sigma = \mathrm{rk}(\psi^\mu)$, where $\mu < \omega_1^{CK}$. Then,*

$$Ret(\omega\sigma + \omega^2, F, p, p^*) \;\&\; \mathrm{dom}\, f^p \subseteq F \quad\Rightarrow\quad (p \Vdash \psi^\mu \;\Rightarrow\; p^* \Vdash \psi^\mu).$$

Note that ψ^μ is not in $\mathcal{L}_F$ because is might have quantifiers of the form $\exists X^\mu$.

Proof. By induction on the number k of steps needed to build ψ from ranked F-restricted formulas, we show that the lemma holds with "$\omega\sigma+\omega 2k$" replacing "$\omega\sigma+\omega^2$". The case $k=0$ follows from Lemma 2.4. All the cases are easy to prove except for when ψ is of the form $\exists X\varphi(X)$. We need to prove that $\forall q^*\leq p^*\ \exists r^*\leq q^*\exists\mathbf{S}\in C_\mu\ (r^*\Vdash\varphi^\mu(\mathbf{S}))$.

Let $q^*\leq p^*$ be given. Using Lemma 2.5, get $q\leq p$ with $Ret(\omega\sigma+\omega(2k+1),F,q,q^*)$. We can get such a q with $\operatorname{dom} f^q\subseteq F$. Since $p\Vdash\psi^\mu$, there exists $r\leq q$ and $\mathbf{S}\in C_\mu$ such that $r\Vdash\varphi^\mu(\mathbf{S})$. Choose H so that $\mathbf{S}\in C_\mu^{F\cup H}$, $\operatorname{dom} f^r\subseteq F\cup H$ and $F\cap H=\emptyset$. Using an automorphism of $\mathbf{P}$ if necessary, we could choose r, $\mathbf{S}$ and H so that $H\cap\operatorname{dom} f^{q^*}=\emptyset$. We now build $r^*\leq q^*$ so that $Ret(\omega\sigma+\omega 2k,F,r,r^*)$. Then, we would have that $r^*\Vdash\varphi^\mu(\mathbf{S})$ by the induction hypothesis. Define r^* as follows.

(1) $T^{r^*}=T^r$;
(2) (a) $f^{r^*}(i)=f^{q^*}(i)$ for $i\in\operatorname{dom} f^{q^*}\setminus F$;
  (b) $f^{r^*}(i)=f^r(i)$ for $i\in F\cup H$;
(3) (a) $h^{r^*}(s)=h^{q^*}(s)$ for $s\in T^{q^*}$;
  (b) $h^{r^*}(s)=h^r(s)$ for s with $h^r(s)<\omega\sigma+\omega 2k$;
  (c) $h^{r^*}(s)=\infty$ if $\exists i(s\subseteq f^{r^*}(i))$;
  (d) $h^{r^*}(s)=\omega\sigma+\omega 2k+|s|_Q$ otherwise, where $Q=\{t\in T^{r^*}: t$ not covered by the previous cases$\}$.

Now, we just need to check that r^* is as desired. This is a straightforward checking of the conditions of the definitions of $\mathbf{P}$ and of Ret. □

2.7. **Two new lemmas.** So far, all the lemmas we have proved are proved in [Ste78], the only difference being that our setting is oriented to second order arithmetic. The following two lemmas are new. The first one is a simple extension of Lemma 2.12 ([Ste78, Lemma 11]). This proof of the second one is a modification of [Ste78, Lemma 12].

Lemma 2.13. *Let $F\subset_f\omega$ and ψ be a Σ-over-$\mathcal{L}_F$ sentence. Suppose $\sigma=\operatorname{rk}(\psi^\mu)$, where $\mu<\omega_1^{CK}$. Then,*

$$Ret(\omega\sigma+\omega^2+\omega,F,q,q^*)\ \&\ \operatorname{dom} f^q\subseteq F\quad\Rightarrow\quad(q^*\Vdash\neg\psi^\mu\ \Rightarrow\ q\Vdash\neg\psi^\mu).$$

Proof. Suppose, toward a contradiction, that there is an $r\leq q$ such that $r\Vdash\psi^\mu$. Note that if no such an r exists, then $q\Vdash\neg\psi^\mu$. Using an automorphism of $\mathbf{P}$ if necessary, we can choose r such that $\operatorname{dom} f_r\cap\operatorname{dom} f^{q^*}\subseteq F$. So, we actually have $Ret(\omega\sigma+\omega^2+\omega,\operatorname{dom} f_r,q,q^*)$. By [Ste78, Sublemma 4], there exists $r^*\leq q^*$ so that $Ret(\omega\sigma+\omega^2,\operatorname{dom} f_r,r,r^*)$. Then, by Lemma 2.12, we have that $r^*\Vdash\psi^\mu$, contradicting $q^*\Vdash\neg\psi^\mu$. □

Lemma 2.14. $M_\infty\models\Pi^1_1$-*SEP*.

Proof. Let $\varphi(n)$, $\psi(n)$ be Σ-over-$\mathcal{L}_F$ with only n free, $F\subset_f\omega$, and $M_\infty\models\forall n\ (\psi(n)\vee\varphi(n))$. We need to build $D\in M_\infty$ such that

$$M_\infty\models\forall n(\neg\varphi(n)\Rightarrow n\in D\ \&\ \neg\psi(n)\Rightarrow n\notin D).$$

Let $p\in G$ be such that $p\Vdash\forall n(\psi(n)\vee\varphi(n))$. Enlarge F if necessary so that $\operatorname{dom} f^p\subseteq F$. By Lemma 2.9, there exists $\mu<\omega_1^{CK}$ with

$$p\Vdash\forall n(\psi^\mu(n)\vee\varphi^\mu(n))$$

and $\mu > o(\mathbf{d})$ for any constant $\mathbf{d}$ occurring in ψ or φ.

Define the notion of being ν-good exactly as in [Ste78, Lemma 12]: For $g\colon T' \to \omega_1^{CK} \cup \{\infty\}$ where $T' \subseteq_f T^G$, we say that g is *ν-good* iff $\forall s \in T'((h^G(s) < \nu \Rightarrow g(s) = h^G(s))\ \&\ (h^G(s) \geq \nu \Rightarrow g(s) \geq \nu))$. Note that deciding whether g is ν-good is hyperarithmetic in g, T^G and ν, since it requires at most $\nu + \omega$ many Turing jumps of T^G. Fix ν such that $p \in P_\nu$ and $\mathrm{rk}(\varphi^\mu(\mathbf{n}) \vee \psi^\mu(\mathbf{n})) < \nu$ for all $n \in \omega$.

We define a set D which we will show is a separator for $\neg\psi$ and $\neg\varphi$ in $\mathcal{M}_\infty$ as follows: Let $d \in D$ if and only if there exists $q \in P_{\omega\nu+\omega^2+\omega 2}$, $q \leq p$ such that

(1) $q \Vdash \neg\varphi^\mu(\mathbf{d})$
(2) $T^q \subset T^G$
(3) h^q is $\omega\nu + \omega^2 + \omega 2$-good
(4) $\forall i \in F$ ($f^q(i)$ is the longest initial segment of α_i^G on T^q)

Note that D is hyperarithmetic in $T \oplus \bigoplus_{i\in F} \alpha_i^G$. Then, since $M_F = HYP(T \oplus \bigoplus_{i\in F} \alpha_i^G)$, we have that $D \in M_F \subseteq M_\infty$. Now we need to show that if $\neg\varphi(d)$ then $d \in D$, and that if $\neg\psi(d)$ then $d \notin D$.

Assume first that $d \in \omega$ and $\neg\varphi(d)$ holds; we claim that $d \in D$. Since φ is Σ-over-$\mathcal{L}_F$, $\neg\varphi^\mu(d)$ also holds. Let $q^* \in G$, $q^* \leq p$ and $q^* \Vdash \neg\varphi^\mu(\mathbf{d})$. Note that q^* satisfies all the conditions (1)-(4) but q^* might not be in $P_{\omega\nu+\omega^2+\omega 2}$, so we need to modify it a bit. Define q by: $T^q = T^{q^*}$, $f^q = f^{q^*}$, $h^q(s) = \infty$ if $h^{q^*}(s) \geq \omega\nu + \omega^2 + \omega 2$, and $h^q(s) = h^{q^*}(s)$ otherwise. Is not hard to see that q satisfies conditions (2)-(4) above. Note that $Ret(\omega\nu + \omega^2 + \omega 2, \mathrm{dom}\, f^q, q, q^*)$. Using Lemma 2.13, we get that q also satisfies (1) too. So, q witnesses that $d \in D$.

Assume now that $\neg\psi(d)$; we claim the $d \notin D$. Let $r \leq p$, $r \in G$ be such that $r \Vdash \neg\psi^\mu(\mathbf{d})$. Suppose, toward a contradiction, that $d \in D$ and q witnesses it. Using an automorphism of $\mathbf{P}$ if necessary, we can choose q so that $\mathrm{dom}\, f^q \subseteq F \cup H$ where $H \cap F = \emptyset$ and $H \cap \mathrm{dom}\, f_r = \emptyset$. The next step is to define $q^* \leq q$, $r^* \leq r$ and $s^* \leq p$ such that $Ret(\omega\nu+\omega^2+\omega, F, s^*, q^*)$, $Ret(\omega\nu+\omega^2+\omega, F, s^*, r^*)$, and $\mathrm{dom}\, f^{s^*} \subseteq F$. Then, since $q^* \Vdash \neg\varphi^\mu(\mathbf{d})$ and $r^* \Vdash \neg\psi^\mu(\mathbf{d})$, by Lemma 2.13, we have that $s^* \Vdash \neg\varphi^\mu(\mathbf{d}) \wedge \neg\psi^\mu(\mathbf{d})$. But since $s^* \leq p$, this contradicts $p \Vdash \forall n(\psi(n)^\mu \vee \varphi^\mu(n))$.

$$\begin{tikzcd}
& p \arrow[dl, dash] \arrow[dr, dash] \arrow[dd, dash] & \\
q \arrow[d, dash] & & r \arrow[d, dash] \\
q^* \arrow[r, dotted, dash, "Ret"] & s^* \arrow[r, dotted, dash, "Ret"] & r^*
\end{tikzcd}$$

All that is left now is to define q^*, r^* and s^* as wanted. The definition of q^* and r^* are exactly as in [Ste78, Lemma 12], but using $\omega\nu + \omega^2 + \omega$ instead of $\omega\nu + \omega^2$. The one of s^* slightly differs from the one of s in [Ste78, Lemma 12].

Define q^* by

(1) $T^{q^*} = T^r \cup T^q$;
(2) (a) $f^{q^*} = f^q(i)$ for $i \in H$;
    (b) $f^{q^*}(i) = \alpha_i^G \restriction n$ where n is the largest so that $\alpha_i^G \restriction n \in T^{q^*}$, for $i \in F$;

(3) (a) $h^{q^*}(t) = h^q(t)$ for $t \in T^q$;
(b) $h^{q^*}(t) = h^r(t) = h(t)$ if $h^r(t) < \omega\nu + \omega^2 + \omega$;
(c) $h^{q^*}(t) = \infty$ if $\exists i(s \subseteq f^{q^*}(i))$;
(d) $h^{q^*}(t) = \omega\nu + \omega^2 + \omega + |s|_Q$ otherwise, where $Q = \{t \in T^{q^*} : t$ not covered by (a), (b) or (c)$\}$.

One may verify that $q^* \in P$, that h^{q^*} is $\omega\nu+\omega^2+\omega$-good, and that $q^* \leq q$. Now define r^* by:

(1) $T^{r^*} = T^{q^*} = T^r \cup T^q$;
(2) (a) $f^{r^*}(i) = f^r(i)$ for $i \in \mathrm{dom}\, f^r \setminus F$;
(b) $f^{q^*}(i) = \alpha_i^G \restriction n$ where n is the largest so that $(\alpha_i^G \restriction n) \in T^{r^*}$, for $i \in F$;
(3) $h^{r^*} = h \restriction T^{r^*}$.

One may verify that $r^* \in P$, $r^* \leq r$ and h^{r^*} is $\omega\nu + \omega^2 + \omega$-good. Finally, define s by:

(1) $T^{s^*} = T^{q^*} = T^{r^*}$;
(2) $f^{s^*} = f^{q^*} \restriction F = f^{r^*} \restriction F$;
(3) (a) $h^{s^*}(t) = h^{q^*}(t) = h^{r^*}(t) = h(t)$ for if $h(t) < \omega\nu + \omega^2 + \omega$;
(b) $h^{s^*}(t) = \infty$ if $h(t) \geq \omega\nu + \omega^2 + \omega$.

Verifying that q^*, r^* and s^* are as claimed is straightforward, though it requires some checking. □

3. Δ^1_1-CA$_0$ does not imply Π^1_1-SEP$_0$

This section is dedicated to prove the following theorem.

Theorem 3.1. *There is an ω-model of Δ^1_1-CA_0 which is not a model of Π^1_1-SEP_0. Therefore, Δ^1_1-CA_0 does not imply Π^1_1-SEP_0.*

The model we build, $\tilde{M}_\infty$, is a modification of M_∞ defined in the previous section. Again, we define a generic object $G = \langle T^G, \{\alpha_i^G : i \in \omega\}, h^G\rangle$, and build $\tilde{M}_\infty$ from it exactly as in the previous section.

$$\tilde{M}_F = \{X \subseteq \omega : \exists \mu < \omega_1^{CK}\ (X \text{ is } \Sigma^0_\mu(T, \alpha_i^G : i \in F))\},$$

and

$$\tilde{M}_\infty = \bigcup_{F \subset_f \omega} \tilde{M}_F.$$

We define $\tilde{M}_{\nu,F}$, $\tilde{M}_{\nu,\infty}$, $S_{\nu,F,e}$ in the same way we did in Subsection 2.1.

We will use the same forcing language, though we will change the forcing notion a bit. This time, we want T^G to satisfy the following property. For every n, either $h^G(\langle 2n\rangle) = \infty$, or $h^G(\langle 2n+1\rangle) = \infty$ or both, and if $h^G(\langle m\rangle) = \infty$, then there are infinitely many α_i^G that start with m. The idea is that the formulas that say "there is no path in T^G starting with $2n$" and "there is no path in T^G starting with $2n+1$" are disjoint Π^1_1 formulas in $\tilde{M}_\infty$ and we will show that there is no separator for them in $\tilde{M}_\infty$. Thus $\tilde{M}_\infty \not\models \Pi^1_1$-$\mathsf{SEP}$. We will then have to prove that $\tilde{M}_\infty$ still satisfies Δ^1_1-CA_0.

The forcing notion is defined as follows. The new set of conditions $\tilde{P}$ consists of the conditions $p \in P$ (defined in Subsection 2.2) which satisfy the following extra conditions

($\tilde{1}$) For every n, if either $\langle 2n\rangle \in T^p$ or $\langle 2n+1\rangle \in T^p$, then both $\langle 2n\rangle$ and $\langle 2n+1\rangle$ are in T^p.

($\tilde{2}$) For every n, if $\langle 2n\rangle \in T^p$ and $\langle 2n+1\rangle \in T^p$, then either $h^p(\langle 2n\rangle) = \infty$, or $h^p(\langle 2n+1\rangle) = \infty$ or both.

($\tilde{3}$) For $i \in \operatorname{dom} f^p$, $f^p(i) \neq \emptyset$.

For $p, q \in \tilde{P}$ we define $p \leq q$ exactly as we did in Subsection 2.2 except that we substitute Condition (5d) by

($\tilde{5d}$) $\forall i \in \operatorname{dom} f^p \setminus \operatorname{dom} f^q \ (\forall s \subseteq f^p(i)(|s| \geq 2 \Rightarrow s \notin T^q))$.

This new condition says that if $p \in G$ and $f^p(i)$ is not defined, then no initial segment of α_i^G is in T^p, except maybe for the empty string or $\langle m\rangle$, where $m = \alpha_i^G(0)$.

Let $\tilde{\mathbf{P}} = (\tilde{P}, \leq)$ and G be a sufficiently $\tilde{\mathbf{P}}$-generic filter.

The forcing relation is defined exactly as in Subsection 2.4. The notion of retaggings is also the same as the one in the previous section. However, most of the lemmas about retaggings proved in the previous section do not hold for this new forcing notion, but the following slight modifications do, where we assume $F \subseteq \operatorname{dom} f^p$.

Lemma 3.2. *Let p^* is an $(\omega \cdot \beta)$-F-absolute retagging of p and let $\mu < \beta$. Let $q \leq p$. Assume that $F \subseteq \operatorname{dom} f^p$. Then, there exists $q^* \leq p^*$ such that $Ret(\omega \cdot \mu, F, q, q^*)$.*

The reason why Lemma 2.5 does not hold, is that we do not have condition (5d) anymore, so we cannot guarantee that q^*, build in the proof of Lemma 2.5, is in $\tilde{P}$ because it might not satisfy (3a) for $i \in F \cap \operatorname{dom} f^q \setminus \operatorname{dom} f^p$. But in the lemma above, there is no $i \in F \setminus \operatorname{dom} f^p$, so we do not have this problem. It is not hard to verify that the same construction of Lemma 2.5 goes though here.

The proof of the following lemmas are the same as the proofs of the corresponding lemmas in the previous sections but assuming that $F \subseteq \operatorname{dom} f^p$.

Lemma 3.3. [Ste78, Lemma 6] *Let ψ be a ranked formula in $\mathcal{L}^F$ and let $p, p^* \in \mathbf{P}$, be such that $F \subseteq \operatorname{dom} f^p, \operatorname{dom} f^{p^*}$, then*

$$Ret(\omega \cdot \operatorname{rk}(\psi), F, p, p^*) \quad \Rightarrow \quad (p \Vdash \psi \Leftrightarrow p^* \Vdash \psi). \tag{3.1}$$

Corollary 3.4. *For a formula ψ of rank β, with constants in C^F, and p with $F \subseteq \operatorname{dom} f^p$, $0^{(\beta)}$ can decide whether $p \Vdash \psi$ uniformly in ψ, p, F and β.*

Lemma 3.5. *$\tilde{M}_F \cap [T^G] = \{\alpha_i^G : i \in F\}$ for each $F \subset_f \omega$.*

Lemma 3.6. *(1) Let $p \Vdash \psi$ where $\psi \in \mathcal{L}_\infty$, ψ is Σ-over-$\mathcal{L}_F$, and $F \subset_f \omega$. Assume that $F \subseteq \operatorname{dom} f^p$. Then $\exists \mu < \omega_1^{CK} \forall \rho (\mu \leq \rho < \omega_1^{CK} \Rightarrow p \Vdash \psi^\rho)$.*

(2) $\tilde{M}_F \models \Sigma_1^1$-$\mathsf{AC}_0$, and hence $\tilde{M}_F$ is hyperarithmetically closed.

Automorphisms of $\tilde{\mathbf{P}}$ work the same way as automorphisms of $\mathbf{P}$.

Lemma 3.7. [Ste78, Lemma 10] *Let π be a permutation of ω, $p \in \tilde{P}$, and $\psi \in \mathcal{L}_\infty$. Then $p \Vdash \psi \Leftrightarrow \hat{\pi}(p) \Vdash \pi\psi$.*

3.1. Not a Model of Π_1^1-SEP_0.

Lemma 3.8. *$\tilde{M}_\infty \not\models \Pi_1^1$-$\mathsf{SEP}$.*

Proof. Let $\psi_0(n)$ be the Π_1^1 formula that says "there is no path in T^G starting with $2n$" and let $\psi_1(n)$ say "there is no path in T^G starting with $2n+1$". The first observation is that for $j = 0, 1$, $\tilde{M}_\infty \models \psi_j(\mathbf{n})$ if and only if $h^G(\langle 2n+j\rangle) \neq \infty$: Clearly if $h^G(\langle 2n +$

$j\rangle) \neq \infty$ then there is no path in T^G starting with $2n+j$. On the other hand, if $h^G(\langle 2n+j\rangle) = \infty$, then by genericity, there will be some $p \in G$ and $i \in \operatorname{dom} f^p$ such that $f^p(i)(0) = 2n+j$. Then, since for every n, $h^G(\langle 2n\rangle) = \infty$, or $h^G(\langle 2n+1\rangle) = \infty$ or both, we have that

$$\tilde{M}_\infty \models \nexists n\ (\psi_0(n)\ \&\ \psi_1(n)).$$

In terms of the forcing relation, we get that $p \Vdash \psi_j(n)$ if and only if $h^p(\langle 2n+j\rangle) \neq \infty$.

Suppose toward a contradiction that there exists $S \in \tilde{M}_\infty$ which is a separator of ψ_0 and ψ_1. Let F be such that $S \in \tilde{M}_F$. There exists $p \in G$ such that

$$p \Vdash \forall n\ (\psi_0(n) \Rightarrow n \in \mathbf{S}\ \&\ \psi_1(n) \Rightarrow n \notin \mathbf{S}),$$

and $F \subseteq \operatorname{dom} f^p$. Let m be such that $h^G(\langle 2m\rangle) = h^G(\langle 2m+1\rangle) = \infty$ and $\langle 2m\rangle \notin T^p$. Either $m \in S$ or not. Suppose without loss of generality that $m \in S$. Let $q \in G$, $q \leq p$ be such that

$$q \Vdash \mathbf{m} \in \mathbf{S}$$

and both $\langle 2m\rangle$ and $\langle 2m+1\rangle$ are in T^q. Let μ be ω times the rank of the formula forced above. Let q^* be such that $q^* \leq p$, $Ret(\mu, F, q, q^*)$ and $h^{q^*}(\langle 2m+1\rangle) \neq \infty$. Such a q^* can be easily obtained by changing the values of $h^q(\sigma)$ for σ with $\sigma(0) = 2m+1$ which are ∞ to ordinals greater than μ. But then we have that

$$q^* \Vdash \forall n\ (\psi_0(n) \Rightarrow n \in \mathbf{S}\ \&\ \psi_1(n) \Rightarrow n \notin \mathbf{S})\ \ \&\ \ \mathbf{m} \in \mathbf{S},$$

and at the same time $q^* \Vdash \psi_1(\mathbf{m})$, getting a contradiction. □

3.2. **New Retagging notion.** When we consider Σ-over-$\mathcal{L}_F$ formulas, a new notion of retagging will be necessary.

Definition 3.9. Let $p, p^* \in \tilde{\mathbf{P}}$, $F \subset_f \omega$ and $\mu \in \omega_1^{CK}$. We say $Ret_{\leq}(\beta, p, p^*)$, if $Ret(\mu, \operatorname{dom} f^p; p, p^*)$ and for every m with $\langle m\rangle \in T^p$ we have that if $h^p(\langle m\rangle) = \infty$, then $h^{p^*}(\langle m\rangle) = \infty$.

Now, for fixed μ, $Ret_{\leq}(\mu, \cdot, \cdot)$ is not an equivalence relation anymore, but a quasi-ordering (or pre-ordering). The key property of this new retagging notion is the following lemma.

Lemma 3.10. [Ste78, Lemma 11] *Let $F \subset_f \omega$ and ψ be a Σ-over-$\mathcal{L}_F$ sentence. Suppose $\sigma = \operatorname{rk}(\psi^\mu)$, where $\mu < \omega_1^{CK}$. Then,*

$$Ret_{\leq}(\omega\sigma + \omega^2, p, p^*)\ \&\ F \subseteq \operatorname{dom} f^p \quad \Rightarrow \quad (p \Vdash \psi^\mu\ \Rightarrow\ p^* \Vdash \psi^\mu).$$

Before proving the lemma, we need the following version of Lemma 2.5

Lemma 3.11. [Ste78, Sublemma 4] *If $Ret_{\leq}(\omega \cdot \beta, p^*, p)$, $q \leq p$ and $\gamma < \beta$, then, there exists $q^* \leq p^*$ such that $Ret_{\leq}(\omega \cdot \gamma, q^*, q)$.*

Proof. The proof is the same as the proof of Lemma 2.5, where $F = \operatorname{dom} f^q$. Note that $\operatorname{dom} f^{q^*} = \operatorname{dom} f^q = F$. One just has to observe that q^* constructed in the proof of Lemma 2.5 satisfies $Ret_{\leq}(\delta, q^*, q)$. □

Proof of Lemma 3.10. The proof is essentially the same as the one of Lemma 2.12. The case $k = 0$ can still be proved using Lemma 2.4 since we have that $Ret(\omega\sigma, F, p, p^*)$. Again, the case where ψ is $\exists X \varphi(X)$ is the interesting one. The definition of q, that was done using Lemma 2.5, is now done using Lemma 3.11. So we have $Ret_{\leq}(\omega\sigma + \omega(2k+1), q, q^*)$. The rest of the proof is exactly the same except for a small change in the definition of r^* needed to get $Ret_{\leq}(\omega\sigma + \omega 2k, p^*, p)$. The change is that before condition (3d) we need to add the following condition
(3c'): $h^{r^*}(s) = \infty$ if $|s| = 1$ and s not covered by the previous cases.
The compatibility of (3a) and (3c) in the definition of r^* is not immediate anymore. Now, as opposed to in the proof of Lemma 2.12, there could be some $s \in T^{q^*}$, $s \neq \emptyset$ and an $i \in \operatorname{dom} f^r \setminus F$ such that $s \subseteq f^{r^*}(i) = f^r(i)$. Here is where we have to use $Ret_{\leq}(\omega\sigma + \omega(2k+1), q, q^*)$. Because, for such an s, we have that $h^q(s) = h^r(s) = \infty$, and therefore $h^{q^*}(s) = \infty$ too. □

Lemma 3.12. $\tilde{M}_\infty \models \Delta^1_1\text{-}\mathsf{CA}_0$.

Proof. This proof is a slight modification of [Ste78, Lemma 12]. It suffices to show the following.

> $(*)$ Let $\varphi(n)$, $\psi(n)$ be Σ-over-$\mathcal{L}_F$ with only n free, $F \subset_f \omega$, and $\tilde{M}_\infty \models \forall n(\psi(n) \Leftrightarrow \neg\varphi(n))$.
> Then $\tilde{M}_\infty \models \exists D \forall n(\psi(n) \Leftrightarrow n \in D)$.

Assume the hypothesis of $(*)$. Let $p \in G$ be such that $p \Vdash \forall n(\psi(n) \Leftrightarrow \neg\varphi(n))$. Enlarge F if necessary so that $\operatorname{dom} f^p \subseteq F$. By Lemma 2.9 there exists $\mu < \omega_1^{CK}$ with $p \Vdash \forall n(\psi^\mu(n) \vee \varphi^\mu(n))$ and $\mu > o(\mathbf{S})$ for any constant $\mathbf{S}$ occurring in ψ or φ.

Define the notion of being ν-good exactly as in Lemma 2.14: For $g\colon T' \to \omega_1^{CK} \cup \{\infty\}$ where $T' \subseteq_f T^G$, we say that g is *ν-good* iff $\forall s \in T'((h^G(s) < \nu \Rightarrow g(s) = h^G(s))$ & $(h^G(s) \geq \nu \Rightarrow g(s) \geq \nu))$. Recall that deciding whether g is ν-good is hyperarithmetic in g, T^G and ν. Fix ν such that $ran\ h^p \subseteq \nu \cup \{\infty\}$ and $\operatorname{rk}(\varphi^\mu(\mathbf{n}) \vee \psi^\mu(\mathbf{n})) < \nu$ for all $n \in \omega$.

We define a set D in $\tilde{M}_\infty$ as follows: $d \in D$ if and only if there exists $q \in P_{\omega\nu+\omega^2}$, $q \leq p$ such that

(1) $q \Vdash \psi^\mu(\mathbf{d})$
(2) $T^q \subset T^G$
(3) h^q is $\omega\nu + \omega^2$-good
(4) $\forall i \in F$ ($f^q(i)$ is the longest initial segment of α_i^G on T^q)

Note that D is hyperarithmetic in $T \oplus \bigoplus_{i\in F} \alpha_i^G$. Then, since $\tilde{M}_F = HYP(T \oplus \bigoplus_{i\in F} \alpha_i^G)$, we have that $D \in \tilde{M}_F \subseteq \tilde{M}_\infty$. Now we show that $\neg\varphi(d)$ if and only if $d \in D$.

Assume first that $d \in \omega$ and $\neg\varphi(d)$ holds; we claim that $d \in D$. By Σ persistence, $\neg\varphi^\mu(d)$ also holds in $\tilde{M}_\infty$. Let $q^* \in G$, $q^* \leq p$ and $q^* \Vdash \neg\varphi^\mu(\mathbf{d})$, and hence $q^* \Vdash \psi^\mu(\mathbf{d})$. Note that q^* satisfies all the conditions (1)-(4) but q^* might not be in $P_{\omega\nu+\omega^2}$, so we need to modify it a bit. Define q by: $T^q = T^{q^*}$, $f^q = f^{q^*}$, $h^q(s) = \infty$ if $h^{q^*}(s) \geq \omega\nu + \omega^2$, and $h^q(s) = h^{q^*}(s)$ otherwise. It is not hard to see that q satisfies conditions (2)-(4) above. Note that $Ret_{\leq}(\omega\nu + \omega^2, q^*, q)$, so by Lemma 3.10, we get that q also satisfies (1). So, q witnesses that $d \in D$.

Assume now that $\varphi(d)$; we claim that $d \notin D$. Let $r \leq p$, $r \in G$ be such that $r \Vdash \neg\psi(d)$, and hence $r \Vdash \neg\psi^\mu(\mathbf{d})$ & $\varphi^\mu(\mathbf{d})$. Suppose, toward a contradiction, that

$d \in D$ and q witnesses it. So we have that $q \Vdash \psi^\mu(\mathbf{d})$. Using an automorphism of $\tilde{\mathbf{P}}$ if necessary, we can choose q so that dom $f^q \subseteq F \cup H$ where $H \cap F = \emptyset$ and $H \cap$dom $f^r = \emptyset$. The next step is to define $q^* \leq q$, $r^* \leq r$ and $s^* \leq p$ such that $Ret_{\leq}(\omega\nu + \omega^2, q^*, s^*)$ and $Ret_{\leq}(\omega\nu + \omega^2, r^*, s^*)$. Then, by Lemma 3.10, this would imply that $s^* \Vdash \varphi^\mu(\mathbf{d}) \wedge \psi^\mu(\mathbf{d})$. But since $s^* \leq p$, this contradicts $p \Vdash \forall n(\psi^\mu(n) \Leftrightarrow \varphi^\mu(n))$.

$$\begin{tikzcd}
& p \arrow[dl, dash] \arrow[dd, dash] \arrow[dr, dash] & \\
q \arrow[d, dash] & & r \arrow[d, dash] \\
q^* \arrow[r, dotted, dash, "Ret_{\leq}"] & s^* \arrow[r, dotted, dash, "Ret_{\geq}"] & r^*
\end{tikzcd}$$

Now, we just need to define q^*, r^* and s^* as wanted. The definitions of q^*, r^* and s^* are exactly as in [Ste78, Lemma 12]. So, the definition of q^* and r^* are exactly as in Lemma 2.14 using $\omega\nu + \omega^2$ instead of $\omega\nu + \omega^2 + \omega$.

Define q^* by

(1) $T^{q^*} = T^r \cup T^q$;
(2) (a) $f^{q^*} = f^q(i)$ for $i \in H$;
  (b) $f^{q^*}(i) = \alpha_i^G \restriction n$ where n is the largest so that $\alpha_i^G \restriction n \in T^{q^*}$, for $i \in F$;
(3) (a) $h^{q^*}(t) = h^q(t)$ for $t \in T^q$;
  (b) $h^{q^*}(t) = h^r(t) = h(t)$ if $h^r(t) < \omega\nu + \omega^2$;
  (c) $h^{q^*}(t) = \infty$ if $\exists i(s \subseteq f^{q^*}(i))$;
  (d) $h^{q^*}(t) = \omega\nu + \omega^2 + |s|^q$ otherwise, where $Q = \{t \in T^{q^*} : t$ not covered by (a), (b) or (c)$\}$.

One may verify that $q^* \in \tilde{P}$, that h^{q^*} is $\omega\nu + \omega^2$-good, and that $q^* \leq q$. Now define r^* by:

(1) $T^{r^*} = T^{q^*} = T^r \cup T^q$;
(2) (a) $f^{r^*}(i) = f^r(i)$ for $i \in$ dom $f^r \setminus F$;
  (b) $f^{q^*}(i) = \alpha_i^G \restriction n$ where n is the largest so that $\alpha_i^G \restriction n \in T^{r^*}$, for $i \in F$;
(3) $h^{r^*} = h \restriction T^{r^*}$.

One may verify that $r^* \in \tilde{P}$, $r^* \leq r$ and h^{r^*} is $\omega\nu + \omega^2$-good. Finally, define s by:

(1) $T^{s^*} = T^{q^*} = T^{r^*}$;
(2) $f^{s^*} = f^{q^*} \cup f^{r^*}$;
(3) (a) $h^{s^*}(t) = h^{q^*}(t) = h^{r^*}(t) = h(t)$ for if $h(t) < \omega\nu + \omega^2$;
  (b) $h^{s^*}(t) = \infty$ if $h(t) \geq \omega\nu + \omega^2$.

□

References

[AK00] C.J. Ash and J. Knight. *Computable Structures and the Hyperarithmetical Hierarchy*. Elsevier Science, 2000.

[Fri67] Harvey Friedman. *Subsystems of set theory and analysis*. PhD thesis, Massachusetts institute of Techonlogy, 1967.

[Gan60] R. O. Gandy. Proof of Mostowski's conjecture. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 8:571–575, 1960.

[Har68] J. Harrison. Recursive pseudo-well-orderings. *Transactions of the American Mathematical Society*, 131:526–543, 1968.

[Jul69] Pierre Jullien. *Contribution à l'étude des types d'ordre dispersés*. PhD thesis, Marseille, 1969.

[Kle59] S. C. Kleene. Quantification of number-theoretic functions. *Compositio Math.*, 14:23–40, 1959.

[Kre62] G. Kreisel. The axiom of choice and the class of hyperarithmetic functions. *Nederl. Akad. Wetensch. Proc. Ser. A 65 = Indag. Math.*, 24:307–319, 1962.

[Mon06] Antonio Montalb'an. Indecomposable linear orderings and hyperarithmetic analysis. *Journal of Mathematical Logic*, 6(1):89–120, 2006.

[Sim99] Stephen G. Simpson. *Subsystems of second order arithmetic*. Springer, 1999.

[Spe60] C. Spector. Hyperarithmetical quantifiers. *Fund. Math.*, 48:313–320, 1959/1960.

[Ste78] John R. Steel. Forcing with tagged trees. *Ann. Math. Logic*, 15(1):55–74, 1978.

[Van77] Robert Alan Van Wesep. *Subsystems of Second-order Arithmetic, and Descriptive Set Theory under the Axiom of Determinateness*. PhD thesis, University of California, Berkeley, 1977.

Department of Mathematics, University of Chicago, Chicago, IL, USA
E-mail address: antonio@math.uchicago.edu
URL: www.math.uchicago.edu/∼antonio