COUNTING THE BACK-AND-FORTH TYPES

ANTONIO MONTALBÁN

Abstract. Given a class of structures $\mathbb{K}$ and $n \in \omega$, we study the dichotomy between there being countably many n-back-and-forth equivalence classes and there being continuum many. In the latter case we show that, relative to some oracle, every set can be weakly coded in the $(n-1)$st jump of some structure in $\mathbb{K}$. In the former case we show that there is a countable set of infinitary Π_n relations that captures all of the Π_n information about the structures in $\mathbb{K}$. In most cases where there are countably many n-back-and-forth equivalence classes, there is a computable description of them. We will show how to use this computable description to get a complete set of computably infinitary Π_n formulas. This will allow us to completely characterize the relatively intrinsically Σ^0_{n+1} relations in the computable structures of $\mathbb{K}$, and to prove that no Turing degree can be coded by the $(n-1)$st jump of any structure in $\mathbb{K}$ unless that degree is already below $0^{(n-1)}$.

1. Introduction

This paper is part of the study of the interactions between the structural properties of a structure and the computational properties of its presentations. Given a class of structures $\mathbb{K}$ and $n \in \omega$, we study the interaction between three different types of properties of the nth Turing jump of the structures in $\mathbb{K}$.

(1) Relations that can be recognized by n jumps. We will work with the notion of a complete set of $\Pi^{\mathtt{c}}_n$ formulas, which is a set of formulas that capture all of the structural information about $\mathbb{K}$ that can be recognized by n jumps. (The superscript "$\mathtt{c}$" in $\Pi^{\mathtt{c}}_n$ stands for *computable infinitary*.) When there is such a set and the formulas are somewhat natural, we can find a relatively simple description of all the relations on a structure that are always c.e. in the nth jump of the structure. Another application of complete sets of $\Pi^{\mathtt{c}}_n$ formulas is the Jump Inversion Theorem for Structures (Theorem 1.3). We will study when is that such a set of formulas exists.

(2) Structures that cannot be distinguished by n jumps. Intuitively, two structures are n-back-and-forth equivalent if they are indistinguishable using just n Turing jumps. We will study the dichotomy between there being countably many n-back-and-forth equivalence classes and there being continuum many. In cases where there are countably many n-back-and-forth equivalence classes, we will get a classification of all the relatively intrinsically Σ^0_{n+1} relations as in the paragraph above, possibly relative to some oracle. In the continuum case, we will see that any set of numbers can be, in some way, coded in the $(n-1)$st jump of some structure in $\mathbb{K}$.

(3) Information coded in n jumps. The dichotomy here is that, relative to some fixed oracle, either no non-trivial information can be coded by the $(n-1)$st jump of any structure in $\mathbb{K}$, or otherwise, every infinite binary sequence can be so coded.

Let $\mathcal{A}$ be a structure and R a relation on it. A common way of measuring the computational, or arithmetical, complexity of the relation R is in terms of the following hierarchy. We say that

[0] Saved: Sept 18th, 2010. Corrected version
Compiled: October 1, 2010

This research was partially supported by NSF grant DMS-0901169 and the AMS centennial fellowship.

R is *relatively intrinsically* Σ^0_{n+1} if for every presentation $(\mathcal{B}, Q)$ of $(\mathcal{A}, R)$, we have that Q is computably enumerable in the nth Turing jump of $\mathcal{B}$. These relations are exactly the relations that we have available when we are working with a certain number of Turing jumps. This is why the question of what relations on a given structure are relatively intrinsically Σ^0_{n+1} is important and useful in the area of computable structure theory. A very satisfactory answer was given by Ash, Knight, Mennasse and Slaman [AKMS89], and independently Chisholm [Chi90]. They produced a characterization of these relations in syntactic terms.

Theorem 1.1 ([AKMS89, Chi90], see [AK00, Theorem 10.1])**.** *Let $\mathcal{A}$ be a computable structure, and let R be a relation in $\mathcal{A}$. The following are equivalent.*

- *R is relatively intrinsically Σ^0_{n+1}.*
- *R is definable in $\mathcal{A}$ by a computable infinitary $\Sigma^{\mathsf{c}}_{n+1}$ formula with finitely many parameters from $\mathcal{A}$.*

This theorem shows the importance of the computable infinitary Σ^{c}_n formulas, which are one of the main focuses of this paper. (For background information on infinitary languages, see Section 1.1 or [AK00, Chapters 6 and 7].)

Complete sets of Π_n formulas. For certain kinds of structures, one can find a much better characterization of the relatively intrinsically Σ^0_{n+1} relations than the one given in Theorem 1.1. For example, the relatively intrinsically Σ^0_2 relations on a computable linear ordering are exactly the $0'$-computable unions of relations defined by finitary existential formulas in the language with $\leq$ and successor (and finitely many parameters). In general, this type of characterization exists when there is a natural list of computably infinitary Π^{c}_n formulas $\{P_0, P_1, ...\}$ that captures all of the Π^{c}_n structural information about the structure. The following definition extends the one in [Mon09].

Definition 1.2. Let $\mathbb{K}$ be a class of $\mathcal{L}$-structures. Let $\{P_0, P_1, ...\}$ be a finite or infinite computable list of Π^{c}_n formulas. We say that $\{P_0, P_1, ...\}$ is a *complete set of* Π^{c}_n *formulas for* $\mathbb{K}$ if every $\Sigma^{\mathsf{c}}_{n+1}$ $\mathcal{L}$-formula is equivalent in $\mathbb{K}$ to a $\Sigma^{\mathsf{c},0^{(n)}}_1$ formula in the language $\mathcal{L} \cup \{P_0, P_1, ...\}$, and there is a computable procedure to find this equivalent formula. What this says is that every computable infinitary $\Sigma^{\mathsf{c}}_{n+1}$ formula can be written as a $0^{(n)}$-computable disjunction of finitary existential formulas that may use the predicates $P_0, P_1, \ldots$.

Note that to show that $\{P_0, P_1, ...\}$ is a complete set of Π^{c}_n formulas for $\mathbb{K}$, it suffices to show that every Π^{c}_n $\mathcal{L}$-formula is equivalent to a $\Sigma^{\mathsf{c},0^{(n)}}_1$ $\mathcal{L} \cup \{P_0, P_1, ...\}$-formula (in a uniform way). We will see examples of complete sets of Π^{c}_n formulas in Section 4. For instance, for the class of linear orderings and $n = 1$, the successor relation, together with relations that recognize the first and last elements, form a complete set of Π^{c}_1 formulas. Therefore, to understand the relations on a linear ordering recognized by one Turing jump, we only need to understand the successor relation, and have parameters for the first and last elements.

The main application of having a complete set of Π^{c}_n formulas is the following theorem.

Theorem 1.3 (Jump Inversion Theorem)**.** [Mon09] *Let $\{P_0, P_1, ...\}$ be a complete set of Π^{c}_n formulas for $\mathbb{K}$, let $\mathcal{A}$ be a structure in $\mathbb{K}$, and let $Y \geq_T 0^{(n)}$. Then if $(\mathcal{A}, P^{\mathcal{A}}_0, P^{\mathcal{A}}_1, ...)$ has a copy computable in Y, there exists X with $X^{(n)} \equiv_T Y$ such that $\mathcal{A}$ has a copy computable in X.*

For instance, from the example above we get that if a linear ordering $\mathcal{A}$ has a copy computable in $0'$ where the successor relation is also computable in $0'$, then $\mathcal{A}$ has a low copy. (This particular case was recently proved, independently, by Frolov [Fro].)

In our discussion thus far, and also in the 6-page paper [Mon09], we have argued that it is useful to have natural complete sets of $\Pi^{\mathtt{c}}_n$ formulas. The question that remains is, for which classes of structures do we have them? Note that we always have at least one countable complete set of $\Pi^{\mathtt{c}}_n$ formulas, namely the set of all $\Pi^{\mathtt{c}}_n$ formulas. However, we are interested in finding sets of $\Pi^{\mathtt{c}}_n$ formulas that are simple and natural. One could argue that a natural complete set of $\Pi^{\mathtt{c}}_n$ formulas should also be complete in the non-effective setting. Thus we introduce the notion of a complete set of infinitary $\Pi^{\mathtt{in}}_n$ formulas, where we look at formulas in $\mathcal{L}_{\omega_1,\omega}$ which are not necessarily computable. (The superscript "in" in $\Pi^{\mathtt{in}}_n$ stands for *infinitary*.)

Definition 1.4. Let $\mathbb{K}$ be a class of $\mathcal{L}$-structures. Let $\{P_0, P_1, ...\}$ be a finite or infinite set of $\Pi^{\mathtt{in}}_n$ formulas. We say that $\{P_0, P_1, ...\}$ is a *complete set of* $\Pi^{\mathtt{in}}_n$ *formulas for* $\mathbb{K}$ if every $\Sigma^{\mathtt{in}}_{n+1}$ $\mathcal{L}$-formula is equivalent in $\mathbb{K}$ to a $\Sigma^{\mathtt{in}}_1$ $(\mathcal{L} \cup \{P_0, ...\})$-formula.

Note that considering the set of all $\Pi^{\mathtt{in}}_n$ formulas as a complete set of $\Pi^{\mathtt{in}}_n$ formulas is not very manageable, as there are continuum many such formulas. We will investigate the question of when a countable complete set of $\Pi^{\mathtt{in}}_n$ formulas exists. We will take the non-existence of such a countable set as an indication of the non-existence of a natural complete set of $\Pi^{\mathtt{c}}_n$ formulas. The reason for this is that one would expect that a *natural* complete set of $\Pi^{\mathtt{c}}_n$ formulas is also $\Pi^{\mathtt{c}}_n$ complete relative to any oracle, and hence $\Pi^{\mathtt{in}}_n$ complete too.

Back-and-forth relations. The back-and-forth relations measure how hard it is to differentiate two structures, or two tuples from the same structure or from different structures. The idea is that two tuples are n-back-and-forth equivalent if we cannot differentiate them using only n Turing jumps. Basic model-theoretic information about these relations may be found in [Bar73], and computability-theoretic information in the work of Ash and Knight [AK00].

Before giving the formal definition, we need a bit of notation. If $\mathcal{L}$ is a language with infinitely many symbols, let $\mathcal{L}\restriction k$ denote only the first k symbols in $\mathcal{L}$. Without loss of generality, assume $\mathcal{L}$ is a relational language. If $\mathcal{A} \in \mathbb{K}$ and $\bar{a}$ is a tuple of elements of $\mathcal{A}$, we abuse notation and write $\bar{a} \in \mathcal{A}$ and also $(\mathcal{A}, \bar{a}) \in \mathbb{K}$.

Definition 1.5. We now define the *n-back-and-forth relations* on tuples of structures of $\mathbb{K}$ by induction on n. Let $\mathcal{A}, \mathcal{B} \in \mathbb{K}$, and let $\bar{a} \in \mathcal{A}$, $\bar{b} \in \mathcal{B}$ be tuples of length k. We say that $(\mathcal{A}, \bar{a}) \leq_0 (\mathcal{B}, \bar{b})$ if $\bar{a}$ and $\bar{b}$ satisfy the same $\mathcal{L}\restriction k$-atomic formulas. We say that $(\mathcal{A}, \bar{a}) \leq_{n+1} (\mathcal{B}, \bar{b})$ if for every $\bar{c} \in \mathcal{B}$ there exists $\bar{d} \in \mathcal{A}$ such that $(\mathcal{A}, \bar{a}\bar{d}) \geq_n (\mathcal{B}, \bar{b}\bar{c})$, where $\bar{c}$ and $\bar{d}$ are of equal length.

The following theorem states three equivalent definitions of these relations showing their naturally. For a tuple $\bar{a} \in \mathcal{A}$ *the* $\Pi^{\mathtt{in}}_n$*-type of* $\bar{a}$ *in* $\mathcal{A}$ (denoted by $\Pi^{\mathtt{in}}_n$-$tp_{\mathcal{A}}(\bar{a})$) is the set of all infinitary $\Pi^{\mathtt{in}}_n$ formulas true of $\bar{a}$ in $\mathcal{A}$.

Theorem 1.6 (Karp; Ash and Knight [AK00, 15.1, 18.6])**.** *For $n \geq 1$, the following are equivalent.*

1. $(\mathcal{A}, \bar{a}) \leq_n (\mathcal{B}, \bar{b})$,
2. $\Pi^{\mathtt{in}}_n$-$tp_{\mathcal{A}}(\bar{a}) \subseteq \Pi^{\mathtt{in}}_n$-$tp_{\mathcal{B}}(\bar{b})$,
3. *If we are given a structure $(\mathcal{C}, \bar{c})$ that we know is isomorphic to either $(\mathcal{A}, \bar{a})$ or $(\mathcal{B}, \bar{b})$, deciding whether it is isomorphic to $(\mathcal{A}, \bar{a})$ is (boldface) $\mathbf{\Sigma}^0_n$-hard. That is, for every $\mathbf{\Sigma}^0_n$ subset $X \subseteq 2^\omega$, there is a continuous operator $F\colon 2^\omega \to \mathbb{K}$ such that, $F(x)$ produces a copy of $(\mathcal{A}, \bar{a})$ if $x \in X$, and a copy of $(\mathcal{B}, \bar{b})$ otherwise.*

(Statement (3) is not exactly [AK00, Theorem 18.6], but it can be derived from it by relativizing; see [HMa].)

The relation $\leq_n$ is a pre-ordering on $\{(\mathcal{A},\bar{a}) : \mathcal{A}\in\mathbb{K}, \bar{a}\in\mathcal{A}\}$, and it induces an equivalence relation and a partial ordering on the quotient as usual. We let $(\mathcal{A},\bar{a})\equiv_n(\mathcal{B},\bar{b})$ if $(\mathcal{A},\bar{a})\leq_n(\mathcal{B},\bar{b})$ and $(\mathcal{A},\bar{a})\geq_n(\mathcal{B},\bar{b})$. We define $\mathbf{bf}_n(\mathbb{K})$ to be the quotient partial ordering:

$$\mathbf{bf}_n(\mathbb{K}) = \frac{\{(\mathcal{A},\bar{a}) : \mathcal{A}\in\mathbb{K}, \bar{a}\in\mathcal{A}\}}{\equiv_n},$$

which is partially ordered by $\leq_n$ in the obvious way. One of the ideas we wish to impart in this paper is that the partial ordering $(\mathbf{bf}_n(\mathbb{K}),\leq_n)$ can give us useful information about $\mathbb{K}$. To start, we will see that the size of $\mathbf{bf}_n(\mathbb{K})$ can tell us quite a bit about $\mathbb{K}$.

Theorem 1.7. *Let $\mathbb{K}$ be a class of structures. The following are equivalent.*

1. *There are countably many $\equiv_n$-equivalence classes of tuples in $\mathbb{K}$.*
2. *There is a countable complete set of $\Pi^{\mathtt{in}}_n$ formulas.*

This theorem will allow us to conclude that for certain $\mathbb{K}$ and n there is no natural complete set of $\Pi^{\mathtt{c}}_n$ formulas. For example, this is the case for linear orderings if $n\geq 3$, because $\mathbf{bf}_3(\mathbb{LO})$ has size $2^{\aleph_0}$. We will see this and other examples in Section 4.

In the countable case, we will see how a good understanding of the structure of $(\mathbf{bf}_n(\mathbb{K}),\leq_n)$ can be useful to derive properties of $\mathbb{K}$. If $\mathbb{K}$ is a somewhat natural class of structures, then one would expect that if $\mathbf{bf}_n(\mathbb{K})$ is countable, the partial ordering $(\mathbf{bf}_n(\mathbb{K}),\leq_n)$ should have a computable description. In Definition 2.3 we will introduce the notion of $\mathbb{K}$ having a *computable n-back-and-forth structure*, and then we will show that if $\mathbb{K}$ has this effectiveness condition, then

- there is a complete set of computable $\Pi^{\mathtt{c}}_n$ formulas for $\mathbb{K}$;
- no non-trivial information can be coded by $(n-1)$ jumps of any structure in $\mathbb{K}$;
- there exists a family of highly effective structures in $\mathbb{K}$, namely an $(n+1)$-friendly family of computable structures in $\mathbb{K}$ with a representative for each n-bftype.

Note that since $\equiv_n$ is a Borel (actually arithmetic) equivalence relation, Silver's theorem [Sil80] implies that if $\mathbb{K}$ is a Borel class of structures (e.g., if it is axiomatizable by countably many $\mathcal{L}_{\omega_1,\omega}$ sentences), then $\mathbf{bf}_n(\mathbb{K})$ either is countable or has size continuum.

Reals coded in isomorphism types. We now look at the information that is coded in the isomorphism type of a structure, possibly by taking a certain number of Turing jumps.

Definition 1.8. We say that a set $D\subseteq\omega$ is *coded by a structure* $\mathcal{A}$ if D is computably enumerable in every presentation of $\mathcal{A}$. We say that a set D is *coded by the nth jump of a structure* $\mathcal{A}$ if D is computably enumerable in the nth Turing jump of every presentation of $\mathcal{A}$. Given $\sigma\in 2^{<\omega}$ and $D\in 2^\omega$, we say that $\sigma\leq_{\mathbb{Q}} D$ if for the largest τ with $\tau\subseteq\sigma$ and $\tau\subseteq D$, we have $D(|\tau|)=1$. We say that a set D is *weakly coded by the nth jump of a structure* $\mathcal{A}$ if $\{\sigma\in 2^{<\omega} : \sigma\leq_{\mathbb{Q}} D\}$ is coded by the nth jump of $\mathcal{A}$.

The question of characterizing the sets D that are coded in a structure was studied by Ash and Knight [AK00, Section 10.6]. Their answer requires the notion of *enumeration-reducibility* that we review in Section 1.1.

Theorem 1.9. [AK00, 10.17] *Let $\mathcal{A}$ be a structure. A set D is coded by $\mathcal{A}$ if and only if D is enumeration-reducible to the finitary-Σ_1-type of some tuple $\bar{a}\in\mathcal{A}$. A set D is coded by nth jump of $\mathcal{A}$ if and only if D is enumeration-reducible to $\Sigma^{\mathtt{c}}_{n+1}$-$tp_{\mathcal{A}}(\bar{a})$ for some $\bar{a}\in\mathcal{A}$.*

Proof. The proof of the first statement is given in [AK00, 10.17]. For the second statement, one could either modify the proof of [AK00, 10.17], or apply the Jump Inversion Theorem (Thorem 1.3) to $\mathcal{A}$ together with all the $\Pi^{\mathtt{c}}_n$ relations and then apply [AK00, 10.17] to the Σ_1-types in the extended language. □

Therefore, the class of sets D which are coded by the nth jump of some structure in $\mathbb{K}$ is exactly the class of sets which are enumeration-reducible to $\Sigma^{\mathtt{c}}_{n+1}$-$tp(\alpha)$ for some $\alpha \in \mathbf{bf}_{n+1}(\mathbb{K})$. The connection between the number of sets that can be coded and the size of $\mathbf{bf}_{n+1}(\mathbb{K})$ is immediate:

Observation 1.10. Let $\mathbb{K}$ be a Borel class of structures. The following are equivalent.

(1) There are countably many $\equiv_n$-equivalence classes.
(2) There are countably many $\Sigma^{\mathtt{c}}_n$-types realized by tuples in $\mathbb{K}$.
(3) There exists an oracle relative to which the only sets of numbers that can be coded by the $(n-1)$st jump of some structure in $\mathbb{K}$ are those that are already c.e. in $0^{(n-1)}$.

That (1) $\Longrightarrow$ (2) and that (2) $\Longrightarrow$ (3) follows from previous observations. That (3) $\Longrightarrow$ (1) is proved in Theorem 3.1. There we will see that if there are uncountably many $\equiv_n$-equivalence classes, every set can be weakly coded by the $(n-1)$st jump of some structure in $\mathbb{K}$, relative to some fixed oracle.

Structures that have Turing degree. With the intention of measuring the complexity of a structure, Jockusch and Richter defined the Turing degree of a structure to be the least degree that can compute a copy of it. When such a least degree exists, we say that the structure *has Turing degree*. It was then shown by Richter [Ric81] that very many structures do not have Turing degree. Nowadays, the *degree spectrum* is the standard measure the computational complexity of a structure, and a good deal of research has been devoted to understanding the possible shapes of degree spectra. However, some structures do have have Turing degree and researchers are still interested in studying these structures because they have the simplest kind of degree spectrums, namely upper-cones, which is sometimes useful for other applications. For instance, it is well known that there are graphs, rings, groups, etc. with any given Turing degree. Calvert, Harizanov, and Shlapentokh [CHS07] have recently shown that there are also fields and torsion-free abelian groups with any given Turing degree, and in their introduction they mention previous work by others.

Definition 1.11. A structure $\mathcal{A}$ has *degree* $D \subseteq \omega$, if for every oracle X, X can compute a copy of $\mathcal{A}$ if and only if X can compute D. A structure $\mathcal{A}$ has *nth jump degree* $D \geq 0^{(n)}$ if, for every oracle $X \geq 0^{(n)}$, X can compute the nth jump of a presentation of $\mathcal{A}$ if and only if X can compute D.

Note that if $\mathcal{A}$ has degree D, then both D and $\overline{D}$ are coded by $\mathcal{A}$ (where $\overline{D}$ is the complement of D). Thus it is not hard to prove that $\mathcal{A}$ has degree D if and only if D can compute a copy of $\mathcal{A}$ and $D \oplus \overline{D}$ is coded by $\mathcal{A}$, in which case it follows from Theorem 1.9 that $D \oplus \overline{D}$ is enumeration-equivalent to the finitary-Σ_1-type of some tuple in $\mathcal{A}$. Furthermore, if D is the nth jump enumeration degree of $\mathcal{A}$, then $D \oplus \overline{D}$ is enumeration-equivalent to the $\Sigma^{\mathtt{c}}_{n+1}$-type of some tuple of $\mathcal{A}$. This implies Richter's old result [Ric81] that if a structure $\mathcal{A}$ has the *computable extensions property*, and it has Turing degree, this degree must be zero. In terms of the notions considered in this paper, the computable extensions property says that each finitary-Σ_1-type realized in $\mathcal{A}$ is computable.

The following definition is due to Jockusch and Soare [JS94]: A class of structures has *Turing ordinal* α if every Turing degree $\mathbf{d} \geq_T 0^{(\alpha)}$ is the αth jump degree of some structure in the class and, for every $\beta < \alpha$, $0^{(\beta)}$ is the only possible βth jump degree of any structure in $\mathbb{K}$. From Observation 1.10, we get the following result.

Theorem 1.12. *Let $\mathbb{K}$ be a class of structures with countably many $\equiv_n$-equivalence classes but uncountably many $\equiv_{n+1}$-equivalence classes. Suppose that $\mathbb{K}$ has Turing ordinal m. Then $n \leq m$.*

For most of the natural classes $\mathbb{K}$ that satisfy they hypothesis of this theorem, we will get that $\mathbb{K}$ has Turing ordinal n.

1.1. **Background and Notation.**

1.1.1. *Infinitary languages.* We will use the infinitary language $\mathcal{L}_{\omega_1,\omega}$ and its effective version throughout this paper. We refer the reader to [AK00, Chapters 6 and 7] for background on infinitary formulas. Let $\mathcal{L}$ be a countable computable language, which we fix for the rest of the paper. Without loss of generality, we assume $\mathcal{L}$ is a relational language. $\mathcal{L}_{\omega_1,\omega}$ is the set of first-order $\mathcal{L}$-formulas where countably infinite disjunctions and conjunctions are allowed, but formulas are allowed to have only a finite number of free variables. These infinitary formulas are arranged in a hierarchy as follows: Given $\alpha < \omega_1$, we say that a formula $\varphi(\bar{x})$ is $\Sigma^{\mathtt{in}}_\alpha$ if it is of the form $\bigvee_{i\in\omega} \exists \bar{y}_i \varphi_i(\bar{x}\bar{y}_i)$ where each $\varphi_i(\bar{x}\bar{y}_i)$ is $\Pi^{\mathtt{in}}_\beta$ for some $\beta < \alpha$. Of course, $\Pi^{\mathtt{in}}_\alpha$ formulas are the negations of $\Sigma^{\mathtt{in}}_\alpha$ formulas, and $\Sigma^{\mathtt{in}}_0$ formulas are the finitary quantifier free formulas to be able to talk about c.e. sets of $\Sigma^{\mathtt{c}}_n$ formulas. When we consider *computable infinitary formulas*, we require the infinite disjunctions and conjunctions to be computably enumerable. The hierarchy of computable infinitary formulas is defined in a similar way as for $\mathcal{L}_{\omega_1,\omega}$, but keeping track of ordinal notations and indices for formulas. In this paper we will deal only with finite ordinals. We denote the classes of formulas in this hierarchy by $\Sigma^{\mathtt{c}}_n$ and $\Pi^{\mathtt{c}}_n$. At times we will consider computable infinitary formulas relative to some oracle X, in which case we allow the infinite disjunctions and conjunctions to be X-computably enumerable. We denote the relativized hierarchies of formulas by $\Sigma^{\mathtt{c},X}_n$ and $\Pi^{\mathtt{c},X}_n$. Note that every $\Sigma^{\mathtt{in}}_n$ formula is $\Sigma^{\mathtt{c},X}_n$ for some X.

1.1.2. *Back-and-forth relations.* Recall that $\mathbf{bf}_n(\mathbb{K})$ is the set of n-back-and-forth equivalence classes of tuples of elements from structures in $\mathbb{K}$. We use the term *n-bftypes* for the elements of $\mathbf{bf}_n(\mathbb{K})$. Given $\alpha \in \mathbf{bf}_n(\mathbb{K})$, we use $|\alpha| = k$ to denote that α is the n-bftype of a k-tuple $\langle a_1, ..., a_k\rangle$, and we define $\mathbf{bf}_{n,k}(\mathbb{K})$ as $\{\alpha \in \mathbf{bf}_n(\mathbb{K}) : |\alpha| = k\}$. We write $\mathbf{bf}_{<n}(\mathbb{K})$ for $\bigcup_{i<n} \mathbf{bf}_i(\mathbb{K})$.

Recall that $\Pi^{\mathtt{in}}_n$-$tp_{\mathcal{A}}(\bar{a}) = \Pi^{\mathtt{in}}_n$-$tp_{\mathcal{B}}(\bar{b})$ whenever $(\mathcal{A}, \bar{a}) \equiv_n (\mathcal{B}, \bar{b})$. Thus given $\alpha \in \mathbf{bf}_n(\mathbb{K})$, we use $\Pi^{\mathtt{in}}_n$-$tp(\alpha)$ to denote the set of $\Pi^{\mathtt{in}}_n$ formulas $\varphi(\bar{x})$ with $|\bar{x}| = |\alpha|$ such that, for every —equivalently, some— $(\mathcal{A}, \bar{a})$ of n-bftype α, $\mathcal{A} \models \varphi(\bar{a})$. This gives us a one-to-one correspondence between $\mathbf{bf}_n(\mathbb{K})$ and the set of $\Pi^{\mathtt{in}}_n$-types realized in $\mathbb{K}$. We define $\Pi^{\mathtt{c}}_n$-$tp(\alpha)$ to be the set of computable infinitary formulas in $\Pi^{\mathtt{in}}_n$-$tp(\alpha)$. For a $\Pi^{\mathtt{in}}_n$ formula $\varphi(\bar{x})$ with $|\bar{x}| = |\alpha|$, we write $\alpha \models \varphi$ if $\varphi \in \Pi^{\mathtt{in}}_n$-$tp(\alpha)$. Given a class of formulas $\Gamma \subseteq \Pi^{\mathtt{in}}_n$, we define Γ-$tp(\alpha)$ as $\{\varphi \in \Gamma | \alpha \models \varphi\}$. If α is a 0-bftype, we define $\alpha \models \varphi$ only for quantifier-free formulas which use only the first $|\alpha|$ symbols of the language. Note that the 0-bftypes of length k are in one-to-one correspondence with the $\mathcal{L}\restriction k$-atomic diagrams of tuples of length k, and hence $\mathbf{bf}_{0,k}(\mathbb{K})$ is finite for every k (recall that $\mathcal{L}$ is relational).

Given $\alpha, \beta \in \mathbf{bf}_n(\mathbb{K})$ with $|\alpha| \leq |\beta|$, we say that $\alpha \subseteq \beta$ if for every —equivalently, some— $(\mathcal{A}, b_1, ..., b_{|\beta|})$ of n-bftype β, $(\mathcal{A}, b_1, ..., b_{|\alpha|})$ has n-bftype α. If τ is a permutation of $\{0, 1, ..., |\alpha| - 1\}$, we use $\alpha =_\tau \beta$ to denote that every —equivalently, some— $(\mathcal{A}, a_0, ..., a_{|\alpha|-1})$ of n-bftype α, $(\mathcal{A}, a_{\tau(0)}, ..., a_{\tau(|\alpha|-1)})$ has n-bftype β.

Given $\alpha \in \mathbf{bf}_n(\mathbb{K})$, we define $ext_n(\alpha) \subseteq \mathbf{bf}_{n-1}(\mathbb{K})$ to be the set of $\delta \in \mathbf{bf}_{n-1}(\mathbb{K})$ such that, for every —equivalently, some— $(\mathcal{A}, \bar{a})$ of n-bftype α, there is some $\bar{d} \in \mathcal{A}$ such that $(\mathcal{A}, \bar{a}\bar{d})$ has $(n-1)$-bftype $\geq_{n-1} \delta$. Observe that $ext_n(\alpha)$ is closed downwards under $\leq_{n-1}$. More importantly, note that $\alpha \leq_n (\mathcal{B}, \bar{b})$ if and only if, for every $\bar{c} \in \mathcal{B}$, $(\mathcal{B}, \bar{b}\bar{c})$ has $(n-1)$-bftype in $ext_n(\alpha)$. It follows that for $\alpha, \beta \in \mathbf{bf}_n(\mathbb{K})$, $\alpha \leq_n \beta$ if and only if $ext_n(\beta) \subseteq ext_n(\alpha)$.

1.1.3. *Enumeration reducibility.* A set D is *enumeration-reducible* to a set E, which we write as $D \leq_e E$, if there is an effective procedure that, given an enumeration of E, produces an enumeration of D. We say that D and E are *enumeration-equivalent* if $D \leq_e E$ and $E \leq_e D$. We call the equivalence classes *e-degrees*. (See [Coo90] for background on enumeration degrees.) We recall Selman's theorem [Sel71] that says that $D \leq_e E$ if and only if whenever E is c.e. in a set X, so is D. The map $D \mapsto D \oplus \overline{D}$ gives an embedding of the Turing degrees into the e-degrees (where $\overline{D}$ is the complement of D). An e-degree is said to be total if it is in the image of this embedding. For instance, the image of $0^{(n)}$ under this embedding is $0^{(n)} \oplus \overline{0^{(n)}}$, which is enumeration-equivalent to $0^{(n+1)}$ and also to $\overline{0^{(n)}}$.

2. Countably many n-bf types

We start this section by proving Theorem 1.7 and Observation 1.10. Then, we will study the case where the back-and-forth relations are computably describable.

Theorem 2.1. *Let $\mathbb{K}$ be a Borel class of structures. The following are equivalent.*

1. *There are countably many $\equiv_n$-equivalence classes of tuples from $\mathbb{K}$.*
2. *There is a countable complete set of $\Pi^{\mathtt{in}}_n$ formulas.*
3. *There exists an oracle relative to which the only sets of numbers that can be coded by the $(n-1)$st jump of some structure in $\mathbb{K}$ are the ones computable in the oracle.*

Before proving this theorem, we prove the following lemma.

Lemma 2.2. *If $\boldsymbol{bf}_{n-1}(\mathbb{K})$ is countable, then for each $\alpha \in \boldsymbol{bf}_n(\mathbb{K})$ there exists a $\Pi^{\mathtt{in}}_n$ formula $\varphi_\alpha(\bar{x})$ with $|\bar{x}| = |\alpha|$ such that, for all $(\mathcal{B}, \bar{b}) \in \mathbb{K}$,*

$$\alpha \leq_n (\mathcal{B}, \bar{b}) \iff \mathcal{B} \models \varphi_\alpha(\bar{b}).$$

Proof. The idea of this proof comes from [HMb].

For every $m < n$, $\mathbf{bf}_{n-1}(\mathbb{K})$ is countable, so by induction we can assume that for each $\delta \in \mathbf{bf}_{n-1}(\mathbb{K})$ there exists such a $\Pi^{\mathtt{in}}_{n-1}$ formula φ_δ. Recall that $ext_n(\alpha) \subseteq \mathbf{bf}_{n-1}(\mathbb{K})$ is the set of δ such that every $(\mathcal{A}, \bar{a})$ of n-bftype α has an extension $(\mathcal{A}, \bar{a}\bar{d})$ of $(n-1)$-bftype $\geq_{n-1} \delta$. Also recall that $ext_n(\alpha)$ is closed downward under $\leq_{n-1}$, and that $\alpha \leq_n (\mathcal{B}, \bar{b})$ if and only if, for every $\bar{c} \in \mathcal{B}$, $(\mathcal{B}, \bar{b}\bar{c})$ has $(n-1)$-bftype in $ext_n(\alpha)$. Therefore $\alpha \leq_n (\mathcal{B}, \bar{b})$ if and only if, for every $\bar{c} \in \mathcal{B}$ and every $\delta \in \mathbf{bf}_{n-1}(\mathbb{K})$, if $\delta \leq_{n-1} (\mathcal{B}, \bar{b}\bar{c})$ then $\delta \in ext_n(\alpha)$. The contrapositive says that if $\delta \notin ext_n(\alpha)$, then $\delta \not\leq_{n-1} (\mathcal{B}, \bar{b}\bar{c})$ for any $\bar{c} \in \mathcal{B}$. We can now let

$$\varphi_\alpha(\bar{x}) = \bigwedge_{\substack{\delta \in \mathbf{bf}_{n-1}(\mathbb{K}),\\ \delta \notin ext_n(\alpha)}} \forall \bar{y} \neg \varphi_\delta(\bar{x}\bar{y}),$$

where the tuple of variables $\bar{y}$ in each disjunct has length $|\delta| - |\alpha|$. □

Now we have that $\varphi_\alpha \in \Pi^{\mathtt{in}}_n\text{-}tp(\alpha)$ — and that, for every $(\mathcal{B}, \bar{b})$, if $\mathcal{B} \models \varphi_\alpha(\bar{b})$ then $\bar{b}$ realizes $\Pi^{\mathtt{in}}_n\text{-}tp(\alpha)$. In other words, we have that

$$\Pi^{\mathtt{in}}_n\text{-}impl_{\mathbb{K}}(\varphi_\alpha) = \Pi^{\mathtt{in}}_n\text{-}tp(\alpha),$$

where $\Pi^{\mathtt{in}}_n\text{-}impl_{\mathbb{K}}(\varphi)$ is the set of $\Pi^{\mathtt{in}}_n$ implications of φ in $\mathbb{K}$. That is, $\Pi^{\mathtt{in}}_n\text{-}impl_{\mathbb{K}}(\varphi)$ is the set of $\Pi^{\mathtt{in}}_n$ formulas $\psi(\bar{x})$ such that, for every $(\mathcal{B}, \bar{b}) \in \mathbb{K}$, if $\mathcal{B} \models \varphi(\bar{b})$ then $\mathcal{B} \models \psi(\bar{b})$. We can read this lemma as saying that if $\mathbf{bf}_{n-1}(\mathbb{K})$ is countable, then every $\Pi^{\mathtt{in}}_n$-type realized in $\mathbb{K}$ is principal.

Proof of Theorem 2.1. Assume (1) holds, and let us prove (2). From (1), we get that $\mathbf{bf}_{n-1}(\mathbb{K})$ is countable. From the lemma, we know that for each $\alpha \in \mathbf{bf}_n(\mathbb{K})$ there exists a $\Pi^{\mathtt{in}}_n$ formula $\varphi_\alpha(\bar{x})$ with $|\bar{x}| = |\alpha|$ such that, for all $(\mathcal{B}, \bar{b}) \in \mathbb{K}$, $\alpha \leq_n (\mathcal{B}, \bar{b}) \iff \mathcal{B} \models \varphi_\alpha(\bar{b})$.

To get (2), we will show that the set of formulas φ_α for $\alpha \in \mathbf{bf}_n(\mathbb{K})$ is a complete set of $\Pi^{\mathtt{in}}_n$ formulas: Given a $\Pi^{\mathtt{in}}_n$ formula $\psi(\bar{x})$ with $|\bar{x}| = k$, we claim that

$$\psi(\bar{x}) \iff \bigvee_{\substack{\alpha\in\mathbf{bf}_{n,k}(\mathbb{K}),\\ \alpha\models\psi}} \varphi_\alpha(\bar{x}).$$

For the direction from left to right, suppose that $\mathcal{A} \models \psi(\bar{a})$. Then if α is the n-bftype of $(\mathcal{A}, \bar{a})$, we have that $\alpha \models \psi$ and $\mathcal{A} \models \varphi_\alpha(\bar{a})$. Thus $(\mathcal{A}, \bar{a})$ satisfies the right-hand side. For the other direction, suppose that $(\mathcal{A}, \bar{a})$ satisfies φ_α for some α with $\alpha \models \psi$. It follows that $\alpha \leq_n (\mathcal{A}, \bar{a})$ and —since ψ is $\Pi^{\mathtt{in}}_n$— $\mathcal{A} \models \psi(\bar{a})$.

Assume (2) holds, and let us prove (1). Let $R_1, R_2, ...$ be a countable complete set of $\Pi^{\mathtt{in}}_n$ formulas. We will not use the fact that the formulas $R_1, R_2, ...$ are $\Pi^{\mathtt{in}}_n$ themselves, but just that every $\Sigma^{\mathtt{in}}_{n+1}$ $\mathcal{L}$-formula is equivalent to a $\Sigma^{\mathtt{in}}_1$ $(\mathcal{L} \cup \{R_1, ...\})$-formula. The proof is by induction on n. Since we know that every $\Sigma^{\mathtt{in}}_n$ $\mathcal{L}$-formulas is equivalent to a $\Sigma^{\mathtt{in}}_1$ $(\mathcal{L} \cup \{R_1, ...\})$-formula, we get, by the induction hypothesis, that there are countably many $(n-1)$-bftypes. Therefore, by Lemma 2.2, for each $\alpha \in \mathbf{bf}_n(\mathbb{K})$ there exists a $\Pi^{\mathtt{in}}_n$ formula $\varphi_\alpha(\bar{x})$ with $|\bar{x}| = |\alpha|$ such that, for all $(\mathcal{B}, \bar{b}) \in \mathbb{K}$, $\alpha \leq_n (\mathcal{B}, \bar{b}) \iff \mathcal{B} \models \varphi_\alpha(\bar{b})$.

The goal is to show that for each $\alpha \in \mathbf{bf}_n(\mathbb{K})$ there is a finitary-Σ_1-$(\mathcal{L} \cup \{R_1, ...\})$-formula ψ_α whose set of $\Pi^{\mathtt{in}}_n$ implications ($\Pi^{\mathtt{in}}_n$-*impl*$_{\mathbb{K}}(\psi_\alpha)$) is $\Pi^{\mathtt{in}}_n$-*tp*(α). This would imply that to each α corresponds a different formula ψ_α, and since there are only countably many finitary-Σ_1-$(\mathcal{L} \cup \{R_1, ...\})$-formulas, we would get that $\mathbf{bf}_n(\mathbb{K})$ is countable. By assumption, φ_α is equivalent to a $\Sigma^{\mathtt{in}}_1$ $(\mathcal{L} \cup \{R_1, ...\})$-formula

$$\bigvee_j \psi_{\alpha,j},$$

where each $\psi_{\alpha,j}$ is a finitary-Σ_1-$(\mathcal{L} \cup \{R_1, ...\})$-formula. Let $(\mathcal{A}, \bar{a}) \in \mathbb{K}$ have n-bftype α. Since $\mathcal{A} \models \varphi_\alpha(\bar{a})$, there is some j (call it $j_{\bar{a}}$) that $\mathcal{A} \models \psi_{\alpha,j_{\bar{a}}}(\bar{a})$; let us write ψ_α for $\psi_{\alpha,j_{\bar{a}}}$. Using the fact that ψ_α implies φ_α, we get that

$$\Pi^{\mathtt{in}}_n\text{-}tp(\alpha) = \Pi^{\mathtt{in}}_n\text{-}impl_{\mathbb{K}}(\varphi_\alpha) \subseteq \Pi^{\mathtt{in}}_n\text{-}impl_{\mathbb{K}}(\psi_\alpha).$$

And using the fact that $\mathcal{A} \models \psi_\alpha(\bar{a})$, we get that

$$\Pi^{\mathtt{in}}_n\text{-}impl_{\mathbb{K}}(\psi_\alpha) \subseteq \Pi^{\mathtt{in}}_n\text{-}tp_{\mathcal{A}}(\bar{a}) = \Pi^{\mathtt{in}}_n\text{-}tp(\alpha).$$

It then follows that $\Pi^{\mathtt{in}}_n$-*impl*$_{\mathbb{K}}(\psi_\alpha) = \Pi^{\mathtt{in}}_n$-*tp*$(\alpha)$, as desired.

To see that (1) implies (3), recall that a set D is coded by the $(n-1)$st jump of some structure $\mathcal{A}$ if, for some $\bar{a} \in \mathcal{A}$, D can be enumerated from $\Sigma^{\mathtt{c}}_n$-*tp*$_{\mathcal{A}}(\bar{a})$. This means that the class of sets coded by the $(n-1)$st jump of some structure in $\mathbb{K}$ is exactly

$$\{D \subseteq \omega : \exists \alpha \in \mathbf{bf}_n(\mathbb{K}) \ \big(D \leq_e \Sigma^{\mathtt{c}}_n\text{-}tp(\alpha)\big)\}.$$

If $\mathbf{bf}_n(\mathbb{K})$ is countable, so is this class of sets, hence there exists some oracle that computes every set in the class.

That (3) implies (1) follows from the implication (1) $\implies$ (3) in the statement of Theorem 3.1. We defer the proof until then. □

We remark that the equivalence between (1) and (2) did not use the hypothesis that $\mathbb{K}$ is Borel.

2.1. **Effective case.** We now look at the statements in Theorem 2.1 in an effective context. If $\mathbf{bf}_n(\mathbb{K})$ is countable and $\mathbb{K}$ is a somewhat natural class of structures, we conjecture that the partial ordering $(\mathbf{bf}_n(\mathbb{K}), \leq_n)$ should have a computable description.

Definition 2.3. We refer to the following family of structures, together with a map that assigns to each 0-bftype α, it's $\mathcal{L}\restriction|\alpha|$-atomic diagram, as the *n-back-and-forth structure of* $\mathbb{K}$:

$$\{(\mathbf{bf}_i(\mathbb{K}); \leq_i, ext_i(\cdot), \subseteq, =_\tau) \ : \ i \leq n\}.$$

We say that $\mathbb{K}$ has a *computable n-back-and-forth structure* if all the structures in this family have computable presentations and the map that assigns to each 0-bftype α, it's $\mathcal{L}\restriction|\alpha|$-atomic diagram is computable.

By $=_\tau$ we mean the ternary relation $\{(\alpha,\beta,\tau) : \alpha,\beta \in \mathbf{bf}_{i,k}(\mathbb{K}), k \in \omega, \tau$ a permutation of $\{0,1,...,k-1\}, \alpha =_\tau \beta\}$. Note that $|\alpha|$ can be defined using $\subseteq$. By ext_i we mean the binary relation $\{(\alpha,\delta) : \delta \in ext_i(\alpha)\}$. Recall that $\delta \in ext_i(\alpha)$ if for every $(\mathcal{A},\bar{a})$ of i-bftype α, there exists $\bar{d} \in \mathcal{A}$ with $(\mathcal{A},\bar{a}\bar{d}) \geq_{i-1} \delta$.

We will show that this property implies the existence of a complete set of $\Pi^{\mathtt{c}}_n$ formulas, the existence of highly effective structures in $\mathbb{K}$, and the non-existence of non-trivial sets coded by the $(n-1)$st jump of the structures in $\mathbb{K}$.

Lemma 2.4. *If $\mathbb{K}$ has a computable n-back-and-forth structure, then for $i \leq n$,*

1. *given a $\Sigma^{\mathtt{c}}_i$ formula ψ and $\alpha \in \boldsymbol{bf}_i(\mathbb{K})$, deciding whether $\alpha \models \psi$ is c.e. in $0^{(i-1)}$, uniformly in α and ψ. For $i = 0$, deciding whether $\alpha \models \psi$ for the appropriate ψ is computable.) ;*
2. *for each $\alpha \in \boldsymbol{bf}_i(\mathbb{K})$ there is a $\Pi^{\mathtt{c}}_i$ formula $\varphi_\alpha(\bar{x})$ such that, for every $(\mathcal{B},\bar{b}) \in \mathbb{K}$, $\mathcal{B} \models \varphi_\alpha(\bar{b})$ if and only if $\alpha \leq_i (\mathcal{B},\bar{b})$. ;*
3. *$\{\varphi_\alpha : \alpha \in \boldsymbol{bf}_n(\mathbb{K})\}$ is a complete set of $\Pi^{\mathtt{c}}_n$ relations for $\mathbb{K}$.*

The proof of this lemma is essentially due to Harris and Montalbán [HMb]. They proved it only for the case of Boolean algebras, but the idea generalizes.

Proof. For (1) and $i = 0$, if α is a 0-bftype and ψ is a finitary, quantifier-free formulas that uses relation symbols in $\mathcal{L}\restriction|\alpha|$, then deciding whether $\alpha \models \psi$ is easily computable since we have a map that assigns to each 0-bftype α, its $\mathcal{L}\restriction|\alpha|$-atomic diagram. The rest of the proof is by induction on i. Consider a $\Sigma^{\mathtt{c}}_{i+1}$ formula $\psi(\bar{x}) = \bigvee_j \exists\bar{y}\psi_j(\bar{x}\bar{y})$, where the ψ_j's are $\Pi^{\mathtt{c}}_i$, and consider an $(i+1)$-bftype α. We claim that

$$\alpha \models \psi \iff \exists j \in \omega \ \exists\beta \in ext_{i+1}(\alpha) \ \big(\beta \models \psi_j\big).$$

This would imply that deciding whether $\alpha \models \psi$ is c.e. in $0^{(i)}$ because, by the induction hypothesis, deciding whether $\beta \models \psi_j$ is co-c.e. in $0^{(i-1)}$. For the direction from left to right, we have that if $(\mathcal{A},\bar{a})$ has $(i+1)$-bftype α and $\mathcal{A} \models \psi(\bar{a})$, then there exists $j \in \omega$ and $\bar{b} \in \mathcal{A}$ such $\mathcal{A} \models \psi_j(\bar{a}\bar{b})$. Let β be the i-bftype of $(\mathcal{A},\bar{a}\bar{b})$; then $\beta \in ext_{i+1}(\alpha)$ and $\beta \models \psi_j$. For the direction from right to left, let $(\mathcal{A},\bar{a})$ have $(i+1)$-bftype α, and suppose that, for some j and some $\beta \in ext_{i+1}(\alpha)$, we have $\beta \models \psi_j$. Since $\beta \in ext_{i+1}(\alpha)$, there exists $\bar{b} \in \mathcal{A}$ such that $\beta \leq_i (\mathcal{A},\bar{a}\bar{b})$. Since ψ_j is $\Pi^{\mathtt{c}}_i$, we get that $\mathcal{A} \models \psi_j(\bar{a}\bar{b})$, and hence that $\mathcal{A} \models \psi(\alpha)$, as desired.

We have already essentially proved (2) in the proof of Lemma 2.2. There, for $\alpha \in \mathbf{bf}_i(\mathbb{K})$ we defined

$$\varphi_\alpha(\bar{x}) = \bigwedge_{\substack{\beta\in\mathbf{bf}_{i-1}(\mathbb{K}),\\ \beta\notin ext_i(\alpha)}} \forall\bar{y}\neg\varphi_\beta(\bar{x},\bar{y}),$$

where φ_β is $\Pi^{\mathtt{in}}_{i-1}$ and was defined inductively. This time we use that $ext_i(\alpha)$ is computable to get a computable conjunction and get a $\Pi^{\mathtt{c}}_i$ formula. In addition we use the fact that $ext_i(\cdot,\cdot)$ is computable to get the definition of φ_α to be uniform in α, which is necessary for the induction step.

For (3) we use the same idea as in the proof of Theorem 2.1. Given a $\Pi^{\mathtt{c}}_n$ formula ψ, we get that

$$\psi(\bar{x}) \iff \bigvee_{\substack{\alpha\in\mathbf{bf}_n(\mathbb{K}),\\ \alpha\models\psi}} \varphi_\alpha(\bar{x}).$$

Note that, by part (1), $0^{(n)}$ can effectively list all the disjuncts on the right-hand side, so the formula is $\Sigma^{\mathtt{c},0^{(n)}}_1$ in the language with relation symbols for $\{\varphi_\alpha : \alpha\in\mathbf{bf}_n(\mathbb{K})\}$. □

Remark 2.5. In Lemma 2.2 and in part (2) of Lemma 2.4 we proved not only that, for every $\alpha\in\mathbf{bf}_i(\mathbb{K})$, φ_α is $\Pi^{\mathtt{c}}_i$, but also that φ_α is a $\Pi^{\mathtt{c}}_1$ formula in the language $\mathcal{L}\cup\{\varphi_\beta : \beta\in\mathbf{bf}_{i-1}(\mathbb{K})\}$.

Also, observe that every $\Pi^{\mathtt{c}}_n$ formula ψ is equivalent to a disjunction of a $0^{(n)}$-computable subset of $\{\varphi_\alpha : \alpha\in\mathbf{bf}_n(\mathbb{K})\}$. Thus, if we want to show that a certain set of formulas $\{R_1, ...\}$ is complete $\Pi^{\mathtt{c}}_n$, we only need to show that each φ_α is equivalent to a $\Sigma^{\mathtt{c},0^{(n)}}_1$ formula in the language with $\{R_1, ...\}$.

The next corollaries show that, with these effectiveness conditions, no non-trivial coding can be done using $n-1$ jumps of any structure of $\mathbb{K}$.

Corollary 2.6. *If $\mathbb{K}$ has a computable n-back-and-forth structure, then $\overline{0^{(n-1)}}$ is the greatest enumeration-degree that can be coded by the $(n-1)$st jump of any structure in $\mathbb{K}$.*

Proof. The reason is that if D is coded by the $(n-1)$st jump of some structure $\mathcal{A}\in\mathbb{K}$, then D is enumeration-reducible to $\Sigma^{\mathtt{c}}_n$-$tp_{\mathcal{A}}(\bar{a})$ for some $\bar{a}\in\mathcal{A}$. But part (1) of the lemma above implies that $\Sigma^{\mathtt{c}}_n$-$tp(\alpha)$ is c.e. in $0^{(n-1)}$ for every $\alpha\in\mathbf{bf}_n(\mathbb{K})$. Therefore, any such D is also c.e. in $0^{(n-1)}$, and hence is enumeration-reducible to $\overline{0^{(n-1)}}$. □

Corollary 2.7. *If $\mathbb{K}$ has a computable n-back-and-forth structure, and $\mathcal{A}\in\mathbb{K}$ has $(n-1)$st jump enumeration degree $\mathbf{d}$, then $\mathbf{d}$ is enumeration-equivalent to $\overline{0^{(n-1)}}$.*

Next, we will show how this effectiveness condition implies the existence of highly effective structures in $\mathbb{K}$.

Definition 2.8. [AK00, Section 15.2] A computable sequence of structures $\{\mathcal{A}_i : i\in\omega\}$ is $(n+1)$-*friendly* if the back-and-forth relations $\leq_j$ for $j\leq n$ are all computably enumerable even between tuples from different structures. That is, the set of quintuples $\{(j, i_0, \bar{a}_0, i_1, \bar{a}_1) : j\leq n, i_0, i_1\in\omega, \bar{a}_0\in\mathcal{A}_{i_0}, \bar{a}_1\in\mathcal{A}_{i_1}$ such that $(\mathcal{A}_{i_0}, \bar{a}_0)\leq_j(\mathcal{A}_{i_1}, \bar{a}_1)\}$ is computably enumerable.

Having a family of $(n+1)$-friendly structures is useful for many applications. For instance, using [AK00, 18.6] we get an effective version of Theorem 1.6: If $\{\mathcal{A}, \mathcal{B}\}$ is a family of $(n+1)$-friendly structures and $\mathcal{A}\leq_n\mathcal{B}$, then for every Σ^0_n set $S\subseteq\omega$, there exists a computable sequence of structures $\{\mathcal{C}_k : k\in\omega\}$ such that for $k\in S$, $\mathcal{C}_k$ is isomorphic to $\mathcal{A}$, and for $k\notin S$, $\mathcal{C}_k$ is isomorphic to $\mathcal{B}$.

For the following three proofs, we use the following terminology and notation: Given $(\mathcal{A}, \bar{a})\in\mathbb{K}$, where $\bar{a}$ has length k, we define the *0-type* of $\bar{a}$ in $\mathcal{A}$ to be the $\mathcal{L}\restriction k$-atomic diagram of $\bar{a}$. If $\mathbb{K}$ is a class of structures, we denote the set of 0-types of tuples in $\mathbb{K}$ by $\mathbb{K}^{fin}$. We think of $\mathbb{K}^{fin}$ as listing the finite substructures A of structures $\mathcal{A}$ in $\mathbb{K}$ where only the first $|A|$ many relations are defined in A. Note that saying that $\mathbb{K}$ has a computable 0-back-and-forth structure is equivalent to saying that $\mathbb{K}^{fin}$ is computably enumerable.

Lemma 2.9. *Let $\mathbb{M}$ be a class of structures that is axiomatizable by a $\Pi^{\mathtt{c}}_2$ sentence in a relational language, and suppose that $\mathbb{M}^{fin}$ is computably enumerable. Then there is a computable structure in $\mathbb{M}$.*

Proof. Let $\mathcal{L} = \{R_1, R_2, ...\}$, and let $\psi = \bigwedge_i \psi_i$ be the Π^c_2 axiom for $\mathbb{M}$, where each ψ_i is of the form $\forall \bar{x} \bigvee_j \exists \bar{y} \psi_{i,j}(\bar{x}, \bar{y})$ and $\psi_{i,j}$ is quantifier free.

We construct a computable structure $\mathcal{A} \in \mathbb{M}$ by stages. At stage s we build $\mathcal{A}_s$ with finite domain where all the relations $R_1, ..., R_{|\mathcal{A}_s|}$ have been decided and $\mathcal{A}_s$ is in $\mathbb{M}^{fin}$. At stage $s+1 = \langle i, k\rangle$, we act to make ψ_i true in $\mathcal{A}$. We know that $\mathcal{A}_s$ is a finite substructure of some $\mathcal{B} \in \mathbb{M}$ and that $\mathcal{B} \models \forall \bar{x} \bigvee_j \exists \bar{y} \psi_{i,j}(\bar{x}, \bar{y})$. There must exist some finite extension $\mathcal{A}_{s+1}$ of $\mathcal{A}_s$ which is in $\mathbb{M}^{fin}$ and has the property that $\forall \bar{x} \in \mathcal{A}_s \bigvee_j \exists \bar{y} \in \mathcal{A}_{s+1} \left(\mathcal{A}_{s+1} \models \psi_{i,j}(\bar{x}, \bar{y})\right)$. Since $\mathbb{M}^{fin}$ is computably enumerable, we will eventually find such an $\mathcal{A}_{s+1}$. Define $\mathcal{A} = \bigcup_s \mathcal{A}_s$. Since we acted for each ψ_i infinitely often, $\mathcal{A} \models \psi_i$ for every i, so $\mathcal{A} \in \mathbb{M}$. □

Proposition 2.10. *Suppose $\mathbb{K}$ is axiomatizable by a Π^c_2 sentence and has a computable n-back-and-forth structure. Then there is a computable, $(n+1)$-friendly sequence of structures $\{\mathcal{A}_i : i \in \omega\}$ in $\mathbb{K}$ such that, for every $\alpha \in \boldsymbol{bf}_n(\mathbb{K})$, there exists $i \in \omega$ and $\bar{a} \in \mathcal{A}_i$ with n-bftype α.*

Proof. Consider the language $\hat{\mathcal{L}} = \mathcal{L} \cup \{\varphi_\alpha : \alpha \in \mathbf{bf}_{\leq n}(\mathbb{K})\} \cup \{\psi_\alpha : \alpha \in \mathbf{bf}_{\leq n}(\mathbb{K})\}$, where φ_α is as in Lemma 2.4 and $\psi_\alpha(\bar{x})$ is the relation that says that $\bar{x}$ has bftype exactly α. First, we note that the $\hat{\mathcal{L}}$-sentences that define the predicates φ_α and ψ_α are Π^c_2: Recall from Lemma 2.4 and Remark 2.5 that, for $\alpha \in \mathbf{bf}_i(\mathbb{K})$, φ_α is equivalent to a Π^c_1 formula in the language $\mathcal{L} \cup \{\varphi_\beta : \beta \in \mathbf{bf}_{<i}(\mathbb{K})\}$. Then, the sentence that says that, for every $\bar{x}$, $\varphi_\alpha(\bar{x})$ is equivalent to this Π^c_1 $\hat{\mathcal{L}}$-formula, is Π^c_2. Also, we add the Π^c_2 $\hat{\mathcal{L}}$-sentence that says that, for every $\bar{x}$, $\psi_\alpha(\bar{x})$ is equivalent to the Π^c_1 $\hat{\mathcal{L}}$-formula $\left(\varphi_\alpha(\bar{x}) \ \& \ \bigwedge_{\beta \not\leq_n \alpha} \neg\varphi_\beta(\bar{x})\right)$. We can now add the definitions of the predicates φ_α and ψ_α to the axioms of $\mathbb{K}$ and stay axiomatizable by a Π^c_2 $\hat{\mathcal{L}}$-sentence.

For each $\alpha \in \mathbf{bf}_n(\mathbb{K})$, let $\mathbb{K}_\alpha$ be the class of $\hat{\mathcal{L}}$-structures in $\mathbb{K}$ which have a tuple of n-bftype α. Note that this is still Π^c_2-axiomatizable, as we have to add only an existential $\hat{\mathcal{L}}$-sentence. Our goal now is to show that $\mathbb{K}_\alpha$ contains a computable structure for which we use the previous lemma. Let $\mathbb{K}^{fin}_\alpha$ be the set of 0-$\hat{\mathcal{L}}$-types of tuples in $\mathbb{K}_\alpha$; we claim that $\mathbb{K}^{fin}_\alpha$ is computably enumerable. First, note that to enumerate $\mathbb{K}^{fin}_\alpha$ it suffices to consider the tuples which contain a sub-tuple of n-bftype α. The key observation is that, given $\beta \in \mathbf{bf}_n(\mathbb{K})$ with $\alpha \subseteq \beta$, we can compute the 0-$\hat{\mathcal{L}}$-type of any tuple $\bar{b}$ of n-bftype β by using our computable n-back-and-forth structure. So, we can enumerate $\mathbb{K}^{fin}_\alpha$ using the set $\{\beta \in \mathbf{bf}_n(\mathbb{K}) : \alpha \subseteq \beta\}$, which is computable.

Therefore, by the previous lemma, we get that $\mathbb{K}_\alpha$ contains a computable structure $\mathcal{A}_\alpha$. Furthermore, the construction of $\mathcal{A}_\alpha$ in the proof of Lemma 2.9 is uniform in α. Let $\mathbb{M} = \{\mathcal{A}_\alpha : \alpha \in \mathbf{bf}_n(\mathbb{K})\}$. Clearly, every n-bftype is represented in $\mathbb{M}$. We claim that $\mathbb{M}$ is $(n+1)$-friendly. For each structure $\mathcal{A}_\alpha$ and each $j \leq n$, we can uniformly define a computable map $f_{\alpha,j} \colon \mathcal{A}^{<\omega}_\alpha \to \mathbf{bf}_j(\mathbb{K})$ by letting $f_{\alpha,j}(\bar{a})$ be the j-bftype of $(\mathcal{A}_\alpha, \bar{a})$. (Define $f_{\alpha,j}(\bar{a})$ computably by searching for $\beta \in \mathbf{bf}_{j,|\bar{a}|}(\mathbb{K})$ such that $\mathcal{A}_\alpha \models \psi_\beta(\bar{a})$ and letting $f_{\alpha,j}(\bar{a}) = \beta$.) Then given $(\mathcal{A}_\alpha, \bar{a})$ and $(\mathcal{A}_\beta, \bar{b})$, we can decide whether $(\mathcal{A}_\alpha, \bar{a}) \leq_j (\mathcal{A}_\beta, \bar{b})$ by checking whether $f_{\alpha,j}(\bar{a}) \leq_j f_{\beta,j}(\bar{b})$ in $\mathbf{bf}_j(\mathbb{K})$. □

Remark 2.11. The assumption that $\mathcal{L}$ is a relational language is without loss of generality, since the sentence that says that a relation symbol represents a function is Π^c_2.

Proposition 2.12. *Let $\mathbb{K}$ be a class of structures that is Π^c_2 axiomatizable and has a computable n-back-and-forth structure. Given $D \geq_T 0^{(n)}$, the following are equivalent:*

(1) *There is a structure in $\mathbb{K}$ which has nth jump degree D.*
(2) *For some $\alpha \in \boldsymbol{bf}_{n+1}(\mathbb{K})$, $D \oplus \overline{D} \equiv_e \Sigma^c_{n+1}$-tp$(\alpha)$.*

Proof. We already have that (1) implies (2), as explained in the paragraph immediately following Definition 1.11.

For the other direction, suppose that $D \oplus \overline{D} \equiv_e \Sigma^{\mathtt{c}}_{n+1}$-$tp(\alpha)$. We need to show that D can compute the nth jump of a structure $\mathcal{A} \in \mathbb{K}$ which has a tuple of $(n+1)$-bftype α. We would then have that $\mathcal{A}$ has degree D.

Extend the language to $\hat{\mathcal{L}}$ by adding the relations φ_β for all $\beta \in \mathbf{bf}_{\leq n}(\mathbb{K})$, and add the $\Pi^{\mathtt{c}}_2$ axioms that define these relations, as in the proof of Proposition 2.10. Note that $\Sigma^{\mathtt{c}}_{n+1}$-$\mathcal{L}$-$tp(\alpha)$ is determined by Σ_1-$\hat{\mathcal{L}}$-$tp(\alpha)$. Furthermore, since $\{\varphi_\alpha : \alpha \in \mathbf{bf}_n(\mathbb{K})\}$ is a complete set of $\Pi^{\mathtt{c}}_n$ formulas, we can computably translate $\Sigma^{\mathtt{c}}_{n+1}$ $\mathcal{L}$-formulas into $\Sigma^{\mathtt{c},0^{(n)}}_1$ $\hat{\mathcal{L}}$-formulas and vice versa. Therefore, relative to $0^{(n)}$, we have that $\Sigma^{\mathtt{c}}_{n+1}$-$\mathcal{L}$-$tp(\alpha)$ is enumeration-equivalent to Σ_1-$\hat{\mathcal{L}}$-$tp(\alpha)$ (i.e., the finitary-Σ_1-$\hat{\mathcal{L}}$-type of α).

We want a $\Pi^{\mathtt{in}}_2$ $\hat{\mathcal{L}}$-sentence that says that a structure has a tuple of $(n+1)$-bftype α. Since we want to keep the language relational, we add one $|\alpha|$-ary relation R_α to $\hat{\mathcal{L}}$, and then we add the sentence that says that R_α is non-empty, as well as the $\Pi^{\mathtt{in}}_2$ sentence that says that, for every $\bar{x}$, if $R_\alpha(\bar{x})$ then $\bar{x}$ has Σ_1-type Σ_1-$\hat{\mathcal{L}}$-$tp(\alpha)$. Since Σ_1-$\hat{\mathcal{L}}$-$tp(\alpha)$ is c.e. in D, this sentence is not $\Pi^{\mathtt{c}}_2$, but it is $\Pi^{\mathtt{c}}_2$ relative to D. Let $\mathbb{K}_\alpha$ be the class of $\hat{\mathcal{L}}$-structures that satisfy these sentences. We need to show that D can compute a structure in $\mathbb{K}_\alpha$. Since from D we can enumerate Σ_1-$\hat{\mathcal{L}}$-$tp(\alpha)$, we can also enumerate all 0-$\hat{\mathcal{L}}$-types of tuples in $\mathbb{K}_\alpha$ (i.e., $\mathbb{K}^{fin}_\alpha$). Now from Lemma 2.9 relativized to D, we get that D can compute an $\hat{\mathcal{L}}$-structure $\hat{\mathcal{A}}$ in $\mathbb{K}_\alpha$. Let $\mathcal{A}$ be the $\mathcal{L}$-structure obtained by restricting $\hat{\mathcal{A}}$ to the language $\mathcal{L}$. By Theorem 1.3, using the fact that $\{\varphi_\alpha : \alpha \in \mathbf{bf}_n(\mathbb{K})\}$ is a complete set of $\Pi^{\mathtt{c}}_n$ formulas, we get that $\mathcal{A}$ has a copy whose nth jump is computable in D, as desired. □

3. Continuum many n-bftypes

We now turn into looking at the other side of the dichotomy: the case where there are uncountably many n-bftypes. Recall from the Introduction that if $\mathbb{K}$ is a Borel class (for instance if $\mathbb{K}$ is axiomatizable by countably many $\mathcal{L}_{\omega_1,\omega}$ sentences) then $\mathbf{bf}_n(\mathbb{K})$ either is countable or has size continuum.

Theorem 3.1. *Let $\mathbb{K}$ be a Borel class of structures, and let $n \in \omega$. The following are equivalent.*

1. *There are continuum many $\equiv_n$-equivalence classes of tuples in $\mathbb{K}$.*
2. *There is no countable complete set of $\Pi^{\mathtt{in}}_n$ formulas for $\mathbb{K}$.*
3. *Relative to some fixed oracle, every set can be weakly coded into the $(n-1)$st jump of some structure in $\mathbb{K}$.*

Proof. By taking negations, we know from Theorem 2.1 that (1) is equivalent to (2), and that (3) implies (1).

Assume (1); we want to prove (3). Suppose that there are countably many $(n-1)$-bftypes. Otherwise, replace the existing n by the least n such that there are continuum many n-bftypes, and note that if (3) is true for the new value of n, it is true for all $m \geq n$. For some $k \in \omega$, we have that $\mathbf{bf}_{n,k}(\mathbb{K})$ has size continuum. We will assume $k = 0$ to simplify the notation needed in the proof; the general case is essentially the same.

Extend the language to $\hat{\mathcal{L}}$ by adding the relations φ_α for all $\alpha \in \mathbf{bf}_{<n}(\mathbb{K})$, and add the axioms that define these relations, as in the proof of Proposition 2.10. If $\hat{\mathcal{L}}$ is not computable, relativize the rest of the proof to the Turing degree of $\hat{\mathcal{L}}$. Also, by relativizing to some oracle if necessary, assume that $\{\varphi_\alpha : \alpha \in \mathbf{bf}_{n-1}(\mathbb{K})\}$ is a complete set of $\Pi^{\mathtt{c}}_{n-1}$ formulas (recall that we do know it is $\Pi^{\mathtt{in}}_{n-1}$-complete). Thus, all the $\Sigma^{\mathtt{c}}_n$ $\mathcal{L}$-formulas are equivalent to $\Sigma^{\mathtt{c},0^{(n-1)}}_1$ $\hat{\mathcal{L}}$-formulas, and the $\Sigma^{\mathtt{in}}_n$-$\mathcal{L}$-types of the tuples in $\mathbb{K}$ are determined by their finitary-Σ_1-$\hat{\mathcal{L}}$-types.

Now we define $t_{\mathcal{A}} \in 2^\omega$ to be the characteristic function of the finitary-Σ_1-$\hat{\mathcal{L}}$ theory of $\mathcal{A}$. More formally: Enumerate all the finitary-Σ_1-$\hat{\mathcal{L}}$ sentences in a list $(\psi_0, \psi_1, ...)$. For every structure $\mathcal{A}$ let $t_{\mathcal{A}} \in 2^\omega$ be such that $t_{\mathcal{A}}(i) = 1$ if $\mathcal{A} \models \psi_i$ and $t_{\mathcal{A}}(i) = 0$ otherwise. Observe that the set $\{i : t_{\mathcal{A}}(i) = 1\}$ can be coded by the $(n-1)$st jump of $\mathcal{A}$ (because the $(n-1)$st jump of any presentation of $\mathcal{A}$ can compute the relations in $\hat{\mathcal{L}}$ and then enumerate Σ_1-$\hat{\mathcal{L}}$-$tp_{\mathcal{A}}$). Let $R = \{t_{\mathcal{A}} : \mathcal{A} \in \mathbb{K}\} \subseteq 2^\omega$. Note that $\Sigma_n^{\mathtt{in}}$-$tp_{\mathcal{A}}$ is determined by $t_{\mathcal{A}}$, and hence $t_{\mathcal{A}} = t_{\mathcal{B}}$ if and only if $\mathcal{A} \equiv_n \mathcal{B}$. Thus, by (1), R has size continuum. Notice that $R \subseteq 2^\omega$ is a $\mathbf{\Sigma}_1^1$ class, because R is the image of $\mathbb{K}$ under t, $\mathbb{K}$ is Borel, and t is arithmetic. Since R is uncountable and $\mathbf{\Sigma}_1^1$, Suslin's theorem (see [Mos80, Corollary 2C.3]) says that R has a perfect closed subset $[T]$, determined by some perfect tree $T \subseteq 2^{<\omega}$ (where $[T]$ is the set of paths through T). In what follows, we relativize our construction to T, so we assume T is computable. Thinking of T as an order-preserving map $2^\omega \to 2^\omega$, for $X \in 2^\omega$ we let $T(X)$ be the path through T obtained as the image of X under this map. For each X, $T(X)$ gives us a Σ_1-$\hat{\mathcal{L}}$-type that is consistent with $\mathbb{K}$ and of Turing degree X (modulo all the relativization we have already done). There is some $\mathcal{A} \in \mathbb{K}$ with Σ_1-$\hat{\mathcal{L}}$-type $t_{\mathcal{A}} = T(X)$, and hence $T(X)$ can be enumerated by the $(n-1)$st jump of any presentation of $\mathcal{A}$. One can show that $\{\sigma \in 2^{<\omega} X\}$ is enumeration reducible to $T(X)$. If follows that X is weakly coded by the $(n-1)$st jump of $\mathcal{A}$. We chose X arbitrarily, so any set can be weakly coded into the $(n-1)$st jump of some structure $\mathcal{A}$ of $\mathbb{K}$. □

4. Examples

In this section we briefly discuss the n-back-and-forth structures of linear orderings for $n = 1, 2, 3$ and of equivalence structures for $n = 1, 2$. We also include references to the work done on Boolean algebras for all n.

4.1. Linear Orderings.

Linear orderings have Turing ordinal 2, as shown by Knight [Kni86]. We will roughly analyze their n-back-and-forth structure for $n = 0, 1, 2$, and include a proof that there are uncountably many 3-bftypes. We will then look at the conclusions obtained by applying the results from the previous sections.

Let $\mathbb{LO}$ be the class of linear orderings. For simplicity we will consider only tuples of distinct elements in the study of $\mathbf{bf}_n(\mathbb{LO})$. We lose no generality with this assumption.

For each k, the 0-bftype of a tuple $\bar{a}$ of length k is given by the order of its elements, so $\mathbf{bf}_{0,k}(\mathbb{K})$ is isomorphic to the set of permutations of $\{0, ..., k-1\}$. A tuple $\bar{a}$ of length k has 0-bftype τ if $a_{\tau(0)} < a_{\tau(1)} < \cdots < a_{\tau(k-1)}$. Of course, given permutations τ_1, τ_2, we have that $\tau_1 \leq_0 \tau_2$ if and only if $\tau_1 = \tau_2$.

The following two lemmas are useful tools to calculate the back-and-forth relations on linear orderings.

Lemma 4.1. [AK00, 15.7] *Suppose $\mathcal{A}$ and $\mathcal{B}$ are linear orderings. Let $\bar{a} = \langle a_0, ..., a_{k-1}\rangle$ and $\bar{b} = \langle b_0, ..., b_{k-1}\rangle$ be increasing tuples from $\mathcal{A}$, $\mathcal{B}$ respectively, and let $\mathcal{A}_i$, $\mathcal{B}_i$ be intervals such that*

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_0 + \{a_0\} + \mathcal{A}_1 + \{a_1\} + \cdots + \mathcal{A}_{k-1} + \{a_{k-1}\} + \mathcal{A}_k, \\ \mathcal{B} &= \mathcal{B}_0 + \{b_0\} + \mathcal{B}_1 + \{b_1\} + \cdots + \mathcal{B}_{k-1} + \{b_{k-1}\} + \mathcal{B}_k. \end{aligned}$$

Then $(\mathcal{A}, \bar{a}) \leq_n (\mathcal{B}, \bar{b})$ if and only if for all $i \leq k$, $\mathcal{A}_i \leq_n \mathcal{B}_i$.

It follows that, for each k, $\mathbf{bf}_{n,k}(\mathbb{LO})$ is isomorphic to $\mathbf{bf}_{0,k}(\mathbb{LO}) \times (\mathbf{bf}_{n,0}(\mathbb{LO}))^{k+1}$ ordered coordinate-wise. Thus, to understand the back-and-forth relations on tuples of size k, it suffices to look at these relations on the empty tuple.

Lemma 4.2. [AK00, 15.8] *Suppose $\mathcal{A}$ and $\mathcal{B}$ are linear orderings. Then $\mathcal{A} \leq_1 \mathcal{B}$ if and only if $\mathcal{A}$ is infinite or at least as large as $\mathcal{B}$. For $n > 1$, $\mathcal{A} \leq_n \mathcal{B}$ if and only if for any partition of $\mathcal{B}$ into intervals $\mathcal{B}_0, ..., \mathcal{B}_k$ with endpoints in $\mathcal{B}$, there is a corresponding partition of $\mathcal{A}$ into intervals $\mathcal{A}_0, ..., \mathcal{A}_k$ with endpoints in $\mathcal{A}$, such that $\mathcal{B}_i \leq_{n-1} \mathcal{A}_i$.*

The 1-back-and-forth relations. We now analyze $\mathbf{bf}_{1,0}(\mathbb{LO})$. All the infinite linear orderings are $\equiv_1$-equivalent to each other. Let us denote this equivalence class of linear orderings by ∞. Note that every finite linear ordering $\mathcal{A}$ is $>_1$-greater than any infinite linear ordering. For each natural number n, let the number n denote the linear ordering with n elements. Two finite linear orderings are $\equiv_1$-equivalent if and only if they have the same size, so we get an $\equiv_1$-equivalence class for each $n \in \omega$. Note that $n \leq_1 m$ if and only if $n \geq_{\mathbb{N}} m$. Thus, the partial ordering of 1-bftypes among empty tuples, $(\mathbf{bf}_{1,0}(\mathbb{LO}), \leq_1)$, is isomorphic to $\omega \cup \{\infty\}$ with the reverse ordering $\geq_{\mathbb{N}}$. The relation ext_1 can easily be computed, so $\mathbb{LO}$ has a computable 1-back-and-forth structure.

From Lemma 2.4, it follows that $\mathbb{LO}$ has a complete set of Π^{c}_1 relations given by $\{\varphi_\alpha : \alpha \in \mathbf{bf}_1(\mathbb{LO})\}$. We can simplify this set of formulas quite a bit. We claim that the formulas $\mathit{first}(x)$, $\mathit{last}(x)$, $\mathit{succ}(x,y)$ form a complete set of Π^{c}_1 formulas, where $\mathit{first}(x)$ says that x is the first element of the linear ordering, $\mathit{last}(x)$ that x is the last element, and $\mathit{succ}(x,y)$ that $x < y$ and there is no element between x and y. From Remark 2.5, it suffices to show that we can express each φ_α as a $\Sigma^{\mathsf{c},0'}_1$ formula that may use $\leq$, $\mathit{first}(x)$, $\mathit{last}(x)$ and $\mathit{succ}(x)$. First, for $m \in \mathbf{bf}_{1,0} = \omega \cup \{\infty\}$, let $\psi^<_m(a)$ be the formula that says that there are at most m elements below a in the linear ordering, let $\psi^>_m(a)$ say that there are at most m elements above a, and let $\psi_m(a,b)$ say that there are at most m elements between a and b. Then $\psi^<_0(x)$ is equivalent to $\mathit{first}(x)$, $\psi^>_0(x)$ to $\mathit{last}(x)$, and $\psi_0(x,y)$ to $\mathit{succ}(x,y)$. Also, note that $\psi_m(a,b)$ says that the linear ordering between a and b is $\geq_1 m$, and similarly for $\psi^<_m(a)$ and $\psi^>_m(a)$. For every $\alpha = (\tau, m_0, ..., m_k) \in \mathbf{bf}_{0,k}(\mathbb{LO}) \times (\mathbf{bf}_{1,0}(\mathbb{LO}))^{k+1} = \mathbf{bf}_{1,k}(\mathbb{LO})$, it is not hard to see (using Lemma 4.1) that $\varphi_\alpha(\bar{x})$ is equivalent to

$$\left(x_{\tau(0)} < x_{\tau(1)} < \cdots < x_{\tau(k-1)}\right) \mathbin{\&} \psi^<_{m_0}(x_{\tau(0)}) \mathbin{\&} \left(\bigwedge_{0<i<k} \psi_{m_i}(x_{\tau(i-1)}, x_{\tau(i)})\right) \mathbin{\&} \psi^>_{m_k}(x_{\tau(k-1)}).$$

Thus the formulas $\psi^<_m(x)$, $\psi^>_m(x)$ and $\psi_m(x,y)$ for $m \in \omega \cup \{\infty\}$ are a complete set of Π^{c}_1 formulas. However, we can do even better than this. First, when $m = \infty$, we have that $\psi^<_\infty(x)$, $\psi^>_\infty(x)$ and $\psi_\infty(x,y)$ are always true, so they are not very useful formulas. Second, for $m \in \omega$, $\psi_m(x,y)$ is equivalent to

$$\bigvee_{k \leq m} \exists x_1 < ... < x_k \left(\mathit{succ}(x, x_1) \mathbin{\&} \left(\bigwedge_{i<k} (\mathit{succ}(x_i, x_{i+1}))\right) \mathbin{\&} \mathit{succ}(x_k, y)\right),$$

$\psi^<_m(x)$ is equivalent to

$$\bigvee_{k \leq m} \exists x_1 < ... < x_k \left(\mathit{first}(x_1) \mathbin{\&} \left(\bigwedge_{i<k} (\mathit{succ}(x_i, x_{i+1}))\right) \mathbin{\&} \mathit{succ}(x_k, y)\right),$$

and analogously for $\psi^>_m(x)$. One can then see how to write each formula $\varphi_\alpha(\bar{x})$ for $\alpha = (\tau, \alpha_0, ..., \alpha_k) \in \mathbf{bf}_{0,k}(\mathbb{LO})$ using only $\leq$, $\mathit{succ}(x)$, $\mathit{first}(x)$, and $\mathit{last}(x)$.

We remark that if we consider the empty linear ordering as a structure in $\mathbb{LO}$, we should also keep the sentence "non-empty" in the complete set of Π^{c}_1 formulas. We also remark that in [Mon09] we showed that $\mathit{succ}(x,y)$ alone was a complete set of Π^{c}_1 relations. The difference is that there we were looking at a single linear ordering and not at the whole class $\mathbb{LO}$, so the first and last elements were given by two elements, and since we were allowing parameters, there was no need to consider the relations *first*(x) and *last*(x).

It follows from Corollary 2.6 that no non-computable degree can be coded in a linear ordering, and from Corollary 2.7 that 0 is the only possible degree a linear ordering could have. This is an old, well-known result by Richter [Ric81].

The 2-back-and-forth relations. The 2-back-and-forth structure of $\mathbb{LO}$ is much richer, though still computable.

Here is a quick sketch of the analysis of $\mathbf{bf}_2(\mathbb{LO})$; we let the reader fill in the details. Consider the set of symbols $S = \{\infty\} \cup \{\infty_n : n \in \omega\} \cup \{n \in \omega\}$. Let $B = S \times S^{<\omega} \times S$. We will define a map $t\colon \mathbb{LO} \to B$ such that $\mathcal{A} \equiv_2 \mathcal{B}$ if and only if $t(\mathcal{A}) = t(\mathcal{B})$, and we will use the image of t as our computable presentation of $\mathbf{bf}_2(\mathbb{LO})$. Consider $\mathcal{A} \in \mathbb{LO}$; we will define $t(\mathcal{A}) = \langle t_0(\mathcal{A}), t_1(\mathcal{A}), t_2(\mathcal{A})\rangle$ as follows. Let $t_0(\mathcal{A}) = n$ if $\mathcal{A} = n + \mathcal{A}_1$ where $\mathcal{A}_1$ has no first element, and let $t_0(\mathcal{A}) = \infty$ if $\mathcal{A} = \omega + \mathcal{A}_1$. Let $t_2(\mathcal{A}) = n$ if $\mathcal{A} = \mathcal{A}_1 + n$ where $\mathcal{A}_1$ has no last element, and let $t_2(\mathcal{A}) = \infty$ if $\mathcal{A} = \mathcal{A}_1 + \omega^*$. One can show that if $\mathcal{A} \leq_2 \mathcal{B}$, then $t_0(\mathcal{A}) \geq_{\mathbb{N}} t_0(\mathcal{B})$ and $t_2(\mathcal{A}) \geq t_2(\mathcal{B})$. If either $t_0(\mathcal{A})$ or $t_2(\mathcal{A})$ is ∞, we let $t_1(\mathcal{A}) = \langle\infty\rangle$; one can prove that, for such $\mathcal{A}$, $\mathcal{A} \leq_2 \mathcal{B}$ if and only if $t_0(\mathcal{A}) \geq_{\mathbb{N}} t_0(\mathcal{B})$ and $t_2(\mathcal{A}) \geq_{\mathbb{N}} t_2(\mathcal{B})$, independently of the value of $t_1(\mathcal{B})$. Now, we restrict ourselves to linear orderings of the form $\mathcal{A} = n_0 + \mathcal{A}_1 + n_2$ where $\mathcal{A}_1$ has no endpoints. For such linear orderings, we have that $n_0 + \mathcal{A}_1 + n_2 \leq_2 m_0 + \mathcal{B}_1 + m_2$ if and only if $n_0 \geq_{\mathbb{N}} m_0$, $\mathcal{A}_1 \leq_2 \mathcal{B}_1$, and $n_2 \geq_{\mathbb{N}} m_2$. We will now define an invariant map t_1 on linear orderings which have no endpoints, and then let $t(n_0 + \mathcal{A} + n_2) = \langle n_0, t_1(\mathcal{A}), n_2\rangle$. If for every n there exists a tuple of n consecutive elements in $\mathcal{A}$, then $\mathcal{A}$ is $\leq_2$-below every other linear ordering without endpoints; we let $t_1(\mathcal{A}) = \langle\infty\rangle$. So suppose that for some m there is no tuple of $m+1$ consecutive elements, and that m is the least such. If there are infinitely many tuples of m consecutive elements, then $\mathcal{A}$ is $\leq_2$-below every other linear ordering with no tuple of $m+1$ consecutive elements; we let $t_1(\mathcal{A}) = \langle\infty_m\rangle$. Otherwise, we can write $\mathcal{A}$ as $\mathcal{A}_0 + m + \mathcal{A}_1 + m + ... + m + \mathcal{A}_k$, where in each $\mathcal{A}_i$ there is no tuple of m consecutive elements; we then let $t_1(\mathcal{A}) = t_1(\mathcal{A}_0)^\frown\langle m\rangle^\frown t_1(\mathcal{A}_1)^\frown\langle m\rangle^\frown \cdots ^\frown \langle m\rangle^\frown t_1(\mathcal{A}_k)$. The recursion works because we know that for each i the maximum number of consecutive elements in $\mathcal{A}_i$ is less than m. It is not hard to see that the image of t is a computable subset of B; let us call it $\mathbf{bf}_2(\mathbb{LO})$. We leave it to the reader to verify that t is as desired.

That $ext_2(x, y)$ is computable requires a little verification, and one would then get that $\mathbb{LO}$ has a computable 2-back-and-forth structure. From Lemma 2.4, it follows that $\mathbb{LO}$ has a complete set of $\Pi^{\mathtt{c}}_2$ formulas given by $\{\varphi_\alpha : \alpha \in \mathbf{bf}_2(\mathbb{LO})\}$. This set of formulas can be simplified. However, we do not know whether there is a finite complete set of $\Pi^{\mathtt{c}}_2$ formulas for $\mathbb{LO}$.

It follows from Corollary 2.6 that no non-Δ^0_2 set can be coded in the jump of a linear ordering, and from Corollary 2.7 that $0'$ is the only possible jump degree a linear ordering could have. This is a well-known result by Knight [Kni86].

The 3-back-and-forth relations. The 3-back-and-forth structure of $\mathbb{LO}$ has size $2^{\aleph_0}$. Here is a proof. For each strictly increasing function $f\colon \omega \to \omega$, let

$$\mathcal{A}_f = \mathbb{Z} + f(0) + \mathbb{Z} + f(1) + \mathbb{Z} + \cdots,$$

where $\mathbb{Z}$ is the ordering of the integers and $f(i)$ represents the linear ordering with $f(i)$ elements. Given $k \in \omega$, there is a $\Sigma^{\mathtt{c}}_3$ sentence ψ_k such that $\mathcal{A}_f \models \psi_k$ if and only if k is in the image of f. Therefore, we get that $\mathcal{A}_f \equiv_3 \mathcal{A}_g$ if and only if $f = g$, and hence $|\mathbf{bf}_3(\mathbb{LO})| = 2^{\aleph_0}$. Every Turing degree $\geq_T 0^{(2)}$ is the 2nd jump degree of some linear ordering: Lerman showed [Ler81] that $\mathcal{A}_f$ has a presentation computable in X if and only if $X^{(2)}$ can enumerate the set $\{\langle x, y\rangle \in \omega^2 : y \leq f(x)\}$. For every $Y \subseteq \omega$, there is a function f such that

$\{\langle x,y\rangle \in \omega^2 : y \leq f(x)\}$ is enumeration-equivalent to $Y \oplus \overline{Y}$. It follows that for every $Y \subseteq \omega$ there is a linear ordering $\mathcal{A}_f$ with 2nd jump degree Y.

Ordinals. The class of ordinals has a computable n-back-and-forth structure for every n. A complete study of the back-and-forth relations on ordinals was done by Ash [Ash86] (see also [AK00, Lemma 15.10].)

4.2. **Equivalence structures.** Let $\mathbb{ES}$ be the class of equivalence structures on an infinite domain; that is, the class of structures (ω, E) where E is an equivalence relation on ω. A partial analysis of the back-and-forth relations on $\mathbb{ES}$ has already been done by Quinn in [Qui08, Section 3.2] with the purpose of characterizing the classes $\mathbb{K}$ which are *Turing computable embeddable* in $\mathbb{ES}$.

The 1-back-and-forth structure of $\mathbb{ES}$ is computable; we quickly describe what $\mathbf{bf}_{1,0}(\mathbb{ES})$ looks like. Let E be an equivalence relation. We define a non-increasing function $K_E\colon \omega \to \omega \cup \{\infty\}$ as follows. For $k \in \omega$, let $K_E(k)$ be the number of E-equivalence classes of size at least k. It is not hard to show that $E_1 \leq_1 E_2$ if and only if, for every k, $K_{E_1}(k) \geq K_{E_2}(k)$. Let $\mathbf{bf}_{1,0}(\mathbb{ES})$ be the set of non-increasing functions $K\colon \omega \to \omega \cup \{\infty\}$. Order $\mathbf{bf}_{1,0}(\mathbb{ES})$ coordinatewise. Notice that this partial ordering is computably presentable, as all such functions are eventually constant.

To consider the 1-back-and-forth relations on non-empty tuples, we get that $(E_1, a_1, ..., a_k) \leq_1 (E_2, b_1, ..., b_k)$ if and only if for each $i \leq k$, the equivalence class of b_i has no more elements than the equivalence class of a_i, and $E_1^- \leq_1 E_2^-$, where E_1^- is obtained by removing the equivalence classes of $a_1, ..., a_k$ from E_1 and similarly for E_2^-. One can then find a computable presentation for $\mathbf{bf}_{1,k}(\mathbb{ES})$ and show that the 1-back-and-forth structure of $\mathbb{ES}$ is computable. From Lemma 2.4, it follows that $\mathbb{ES}$ has a complete set of $\Pi^{\mathtt{c}}_1$ relations.

It follows from Corollary 2.6 that no non-computable set can be coded in an equivalence structure, and from Corollary 2.7 that 0 is the only possible degree an equivalence structure could have.

The 2-back-and-forth structure of $\mathbb{ES}$ is uncountable. Given an equivalence relation E, let $F_E\colon \omega \to \omega \cup \infty$ be defined as follows. $F_E(k)$ is the number of E-equivalence classes of size exactly k. The isomorphism type of E is then determined by F_E, which could be any function $\omega \to \omega \cup \infty$, and the number of infinite equivalence classes (which could be any number or infinity). We claim that $E_1 \equiv_2 E_2$ if and only if $K_{E_1} = K_{E_2}$ and $F_{E_1} = F_{E_2}$, and thus, there are continuum many 2-bftypes.

Every Turing degree $\geq_T 0'$ is the jump degree of some equivalence class: Given $f\colon \omega \to \omega$, let E_f be the equivalence class with $F_{E_f} = f$ and with infinitely many infinite equivalence classes. Ash and Knight [AK00, Thm 9.1] proved that, given a set X, X can compute a copy of E_f if and only if X' can enumerate $\{\langle x,y\rangle \in \omega^2 : y \leq f(x)\}$. It follows that every degree above $0'$ is the jump degree of some equivalence structure.

4.3. **Boolean algebras.** The n-back-and-forth structure of Boolean algebras is a very interesting one. An analysis for every n was done by Harris and Montalbán.

Theorem 4.3. [HMb] *Boolean algebras have a computable n-back-and-forth structure for every n. Moreover, for every n there is a finite complete set of $\Pi^{\mathtt{c}}_n$ formulas.*

Before [HMb], Alaev [Ala04] had already studied the n-back-and-forth relations for $n \leq 4$ but doing a different type of analysis. Complete sets of $\Pi^{\mathtt{c}}_n$ formulas for $n = 1, 2, 3, 4$ were also already known. For the constructions in Downey and Jockusch [DJ94], Thurber [Thu95], and Knight and Stob [KS00] that ended up showing that every low$_4$ Boolean algebra has a computable copy, they considered certain relations that happened to be (surely not by chance) complete sets of $\Pi^{\mathtt{c}}_n$ formulas for $n \leq 4$. The sets $\mathtt{R}_n$ indicated below are the complete sets of $\Pi^{\mathtt{c}}_n$ formulas they considered. The formulas are not all $\Pi^{\mathtt{c}}_n$, but all of them are Boolean combinations of $\Pi^{\mathtt{c}}_n$ formulas.

- $\mathtt{R}_1 = \{\text{atom}\}$.
- $\mathtt{R}_2 = \mathtt{R}_1 \cup \{\text{inf, atomless}\}$.
- $\mathtt{R}_3 = \mathtt{R}_2 \cup \{\text{atomic, 1-atom, atominf}\}$.
- $\mathtt{R}_4 = \mathtt{R}_3 \cup \{\sim\text{-inf}, \text{Int}(\omega + \eta), \text{infatomicless, 1-atomless, nomaxatomless}\}$.

Definitions of these relations can be found in [KS00] and [HMb]. The proof that these are complete sets of $\Pi^{\mathtt{c}}_n$ formulas follows from Harris and Montalbán [HMb].

It follows from Corollary 2.6 that no non-Δ^0_{n+1} set can be coded in the nth jump of a Boolean algebra, and from Corollary 2.7 that $0^{(n)}$ is the only possible nth jump degree a Boolean algebra could have. This is an known result by Jockusch and Soare [JS94], and by Richter [Ric77] for $n = 0$.

If we were to restrict ourselves to particular classes of Boolean algebras, the study of the n-back-and-forth structure might become much simpler. A complete study of the back-and-forth relations on superatomic Boolean algebras was done long ago by Ash [Ash87] (see also [AK00, Proposition 15.14]). A complete study of the n-back-and-forth relations on saturated Boolean algebras was done by Csima, Montalbán and Shore [CMS06] using the Tarski elementary invariants.

References

[AK00] C.J. Ash and J. Knight. *Computable Structures and the Hyperarithmetical Hierarchy*. Elsevier Science, 2000.

[AKMS89] Chris Ash, Julia Knight, Mark Manasse, and Theodore Slaman. Generic copies of countable structures. *Ann. Pure Appl. Logic*, 42(3):195–205, 1989.

[Ala04] P. E. Alaev. Computable homogeneous Boolean algebras and a metatheorem. *Algebra Logika*, 43(2):133–158, 256, 2004.

[Ash86] C. J. Ash. Recursive labelling systems and stability of recursive structures in hyperarithmetical degrees. *Trans. Amer. Math. Soc.*, 298(2):497–514, 1986.

[Ash87] C. J. Ash. Categoricity in hyperarithmetical degrees. *Ann. Pure Appl. Logic*, 34(1):1–14, 1987.

[Bar73] J. Barwise. Back and forth through infinitary logic. In M. D. Morley, editor, *Studies in model theory*, pages 5–34. The Mathematical Association of America, Buffalo, N.Y., 1973.

[Chi90] John Chisholm. Effective model theory vs. recursive model theory. *J. Symbolic Logic*, 55(3):1168–1191, 1990.

[CHS07] Wesley Calvert, Valentina Harizanov, and Alexandra Shlapentokh. Turing degrees of isomorphism types of algebraic objects. *J. Lond. Math. Soc. (2)*, 75(2):273–286, 2007.

[CMS06] Barbara F. Csima, Antonio Montalbán, and Richard A. Shore. Boolean algebras, Tarski invariants, and index sets. *Notre Dame Journal of Formal Logic*, 47(1):1–23, 2006.

[Coo90] S. Barry Cooper. Enumeration reducibility, nondeterministic computations and relative computability of partial functions. In *Recursion theory week (Oberwolfach, 1989)*, volume 1432 of *Lecture Notes in Math.*, pages 57–110. Springer, Berlin, 1990.

[DJ94] Rod Downey and Carl G. Jockusch. Every low Boolean algebra is isomorphic to a recursive one. *Proc. Amer. Math. Soc.*, 122(3):871–880, 1994.

[Fro] Andrey Frolov. Linear orderings with low degree. to appear in the Siberian Mathematical Journal.

[HMa] Kenneth Harris and Antonio Montalbán. Boolean algebra approximations. submitted for publication.

[HMb] Kenneth Harris and Antonio Montalbán. On the n-back-and-forth types of boolean algebras. To appear in the Transactions of the American Math. Soc.

[JS94] Carl G. Jockusch, Jr. and Robert I. Soare. Boolean algebras, Stone spaces, and the iterated Turing jump. *J. Symbolic Logic*, 59(4):1121–1138, 1994.

[Kni86] J.F. Knight. Degrees coded in jumps of orderings. *Journal of Symbolic Logic*, 51(4):pp. 1034–42, December 1986.

[KS00] Julia F. Knight and Michael Stob. Computable Boolean algebras. *J. Symbolic Logic*, 65(4):1605–1623, 2000.

[Ler81] Manuel Lerman. On recursive linear orderings. In *Logic Year 1979–80 (Proc. Seminars and Conf. Math. Logic, Univ. Connecticut, Storrs, Conn., 1979/80)*, volume 859 of *Lecture Notes in Math.*, pages 132–142. Springer, Berlin, 1981.

[Mon09] Antonio Montalbán. Notes on the jump of a structure. *Mathematical Theory and Computational Practice*, pages 372–378, 2009.

[Mos80] Yiannis N. Moschovakis. *Descriptive set theory*, volume 100 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1980.

[Qui08] Sara B. Quinn. *Algorithmic complexity of algebraic structures*. PhD thesis, University of Notre Dame, 2008.

[Ric77] Linda Richter. *Degrees of unsolvability of models*. PhD thesis, University of Illinois at Urbana-Champaign, 1977.

[Ric81] Linda Jean Richter. Degrees of structures. *J. Symbolic Logic*, 46(4):723–731, 1981.

[Sel71] Alan L. Selman. Arithmetical reducibilities. I. *Z. Math. Logik Grundlagen Math.*, 17:335–350, 1971.

[Sil80] Jack H. Silver. Counting the number of equivalence classes of Borel and coanalytic equivalence relations. *Ann. Math. Logic*, 18(1):1–28, 1980.

[Thu95] John J. Thurber. Every low_2 Boolean algebra has a recursive copy. *Proc. Amer. Math. Soc.*, 123(12):3859–3866, 1995.

Department of Mathematics, University of Chicago, 5734 S. University Ave., Chicago, IL 60637, USA

E-mail address: antonio@math.uchicago.edu

URL: www.math.uchicago.edu/∼antonio