BOOLEAN ALGEBRA APPROXIMATIONS

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Abstract. Knight and Stob proved that every low_4 Boolean algebra is $0^{(6)}$-isomorphic to a computable one. Furthermore, for $n = 1, 2, 3, 4$, every low_n Boolean algebra is $0^{(n+2)}$-isomorphic to a computable one. We show that this is not true for $n = 5$: there is a low_5 Boolean algebra that is not $0^{(7)}$-isomorphic to any computable Boolean algebra.

It is worth remarking that, because of the machinery developed, the proof uses at most a $0''$-priority argument. The technique used to construct this Boolean algebra is new and might be useful in other applications, such as to solve the low_n Boolean algebra problem either positively or negatively.

1. Introduction

Computable structures are one of the main objects of study of effective mathematics. Which mathematical structures have computable presentations is an active area of research in effective mathematics and computability theory. Related to this is the question of what kind of information can be encoded in the isomorphism type of a given structure. Downey and Jockusch [DJ94], proved that every low Boolean algebra has a computable presentation. In other words, if the information encoded in the isomorphism type of a Boolean algebra is low, then there is no information in it at all. This result was extended by Thurber [Thu95] to low_2 Boolean algebras, and by Knight and Stob [KS00] to low_4.

The natural question to follow-up is whether every low_n Boolean algebra has a computable presentation. This problem, originally posed by Downey and Jockusch [DJ94], is still open. One problem here is that the combinatorics of the proofs get exponentially more complicated at each level. We will show that there are also new types of obstacles that appear for the first time at $n = 5$. When $n = 1, 2, 3, 4$, it follows from the earlier results that every low_n Boolean algebra is isomorphic to a computable one via an isomorphism that is computable in $0^{(n+2)}$. (For $n = 2$, the isomorphism found by Thurber is actually computable in $0'''$.) We construct a low_5 Boolean algebra that is not $0^{(5+2)}$-isomorphic to any computable one. Therefore, a proof that every low_5 Boolean algebra is isomorphic to a computable one would have to be, in essence, different than the known proofs for the lower cases.

An interesting feature of our proof is that it does not use more than an infinite injury construction. This is due to the new techniques developed, which are based on the

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authors' work in [HM]. The main notion is that of an n-Z-approximation of a Boolean algebra. We believe that this notion will be useful in a solution of the low_n problem, and more generally, in problems regarding degree spectra or relational spectra of Boolean algebras.

In Section 2, we briefly review the main definitions and results of [HM], which will suffice for the reader to understand the main ideas in our proof. In [HM], we studied the n-back-and-forth relations on Boolean algebras, providing general invariants for the back-and-forth equivalence classes of algebras. These are invariants are constructed, for each n, from a finite set of special types we denote by $\mathbf{BF}_n$ and call the *n-indecomposable* back-and-forth types. (A Boolean algebra is *n-indecomposable* if for any way of expressing the algebra as a finite sum of subalgebras, at least one subalgebra is in the same n-back-and-forth class. See Definition 2.3.) We additionally provided, for each n, a finite set of Boolean algebra unary predicates R_α, one for each $\alpha \in \mathbf{BF}_n$, which are interdefinable with the the sets of predicates used in [DJ94, Thu95, KS00] to solve the low_4 Boolean algebra problem. The main property of our predicates is that given a Boolean algebra $\mathcal{B}$ and a set $Z \geq 0^{(n)}$, the following two statements are equivalent:

- The computably infinitary Π^c_n diagram of $\mathcal{B}$ (to be defined later) is computable in Z;
- $\mathcal{B}$ and the finitely many relations $\mathrm{R}_\alpha(\mathcal{B})$ for $\alpha \in \mathbf{BF}_n$ are computable in Z.

This result motivates the following definition.

Definition 1.1. Given n and a Boolean algebra $\mathcal{A} = (A, \leq, \vee, \wedge, \neg, 0, 1)$, we let $\mathrm{B}_n(\mathcal{A})$ be the structure

$$(A, \leq, \vee, \wedge, \neg, 0, 1, \mathrm{R}_\alpha(\mathcal{A}) : \alpha \in \mathbf{BF}_n),$$

where $\mathrm{R}_\alpha(\mathcal{A}) = \{a \in A : \mathcal{A} \models \mathrm{R}_\alpha(a)\}$. We say that $\mathcal{A}$ is *n-Z-approximable* if $\mathrm{B}_n(\mathcal{A})$ is computable in Z.

The main lemmas in [DJ94, Thu95, KS00] say that for $n = 1, 2, 3, 4$, every n-Z'-approximable Boolean algebra has a copy that is $(n-1)$-Z-approximable. If $\mathcal{A}$ is a computable, or even a low_n, Boolean algebra, then $\mathrm{B}_n(\mathcal{A})$ is $0^{(n)}$-computable. Conversely, if follows from [Mon, Theorem 3.1] that if $\mathrm{B}_n(\mathcal{A})$ is $0^{(n)}$-computable, then $\mathcal{A}$ has a low_n copy. (Using the notation from [Mon], we have that $\mathrm{B}_n(\mathcal{A})$ is the *nth jump of the structure* $\mathcal{A}$.) By [Mon, Theorem 3.5] the low_n Boolean algebra question can be restated as follows.

Question 1. Does every n-$Z^{(n)}$-approximable Boolean algebra have a Z-computable copy?

We will approximate n-Z-approximable Boolean algebras by Z-computable sequences of *finite labeled Boolean algebras.* This type of approximation, which is the main concept of the paper, is introduced in Section 3 and applied in the subsequent sections.

The rest of the paper is dedicated to building a low_5 Boolean algebra that is not $0^{(7)}$-isomorphic to any computable Boolean algebra. In Section 5 we establish some lemmas used in the construction of the low_5 Boolean algebra in Section 6.

2. Background

2.1. **Preliminaries.** Let L be a computable language. We will be considering Σ_n and Π_n formulas in $L_{\omega_1\omega}$ and also *computable* Σ_n and Π_n formulas for $n \in \omega$. We refer to these latter classes as Σ^c_n and Π^c_n for brevity. We note that Σ_0, Π_0, Σ^c_0 and Π^c_0 all denote the class of *finitary* quantifier-free formulas of L. See [AK00, Chapters 6 and 7] for more on these formula classes.

We occasionally use the nonstandard notation $\Sigma^{c,X}_n$ to mean the fragment of Σ_n formulas where conjunctions and disjunctions are required to be X-c.e. So, $\Sigma^{c,0}_n = \Sigma^c_n$.

We consider only *countable* Boolean algebras and we use the signature $\leq, \wedge, \vee, -, 0, 1$, but otherwise we follow the standard reference [Mon89]. We denote Boolean algebras by $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and their elements by a, b, c. We denote the *relative algebra* by $\mathcal{A} \upharpoonright a = \{b \in \mathcal{A} : b \leq a\}$.

A *partition* of an element a in a Boolean algebra $\mathcal{A}$ is a finite sequence $a_0, \dots, a_k$ (which we will write as $(a_i)_{i\leq k}$) of pairwise disjoint elements (that is, $a_i \wedge a_j = 0$ for all $i \neq j$) such that $a = a_0 \vee \dots \vee a_k$; a partition of a Boolean algebra $\mathcal{A}$ is a partition of its unit, $1_{\mathcal{A}}$. We will write $a = a_0 \dot\vee \dots \dot\vee a_k$ to mean that $(a_0, \dots, a_k)$ is a partition of a.

2.2. **Back-and-forth relations.** The main notion studied in [HM] is that of n-back-and-forth relations of Boolean algebras. The purpose behind the use of these relations is to identify the computable Boolean algebras which cannot be distinguished by $0^{(n)}$.

Definition 2.1. [AK00, §15.3.4] Let $\mathcal{A}$ and $\mathcal{B}$ be Boolean algebras. Let $\mathcal{A} \leq_0 \mathcal{B}$ if either both $\mathcal{A}$ and $\mathcal{B}$ are the trivial one-element Boolean algebra, or neither is. For $n > 0$, $\mathcal{A} \leq_n \mathcal{B}$ if for every partition $(b_i)_{i\leq k}$ of $\mathcal{B}$, there is a partition $(a_i)_{i\leq k}$ of $\mathcal{A}$ such that $\mathcal{B} \upharpoonright b_i \leq_{n-1} \mathcal{A} \upharpoonright a_i$ for each $i \leq k$. Let $\mathcal{A} \equiv_n \mathcal{B}$ if $\mathcal{A} \leq_n \mathcal{B}$ and $\mathcal{B} \leq_n \mathcal{A}$.

Theorem 2.2 (Karp; Ash and Knight)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be Boolean algebras. The following are equivalent:*

(1) Given a Boolean algebra $\mathcal{C}$ that is isomorphic to either $\mathcal{A}$ or $\mathcal{B}$, deciding whether $\mathcal{C} \cong \mathcal{A}$ is (boldface) $\boldsymbol{\Sigma}^0_n$-hard.
(2) All the infinitary Σ_n sentences true in $\mathcal{B}$ are true in $\mathcal{A}$.
(3) $\mathcal{A} \leq_n \mathcal{B}$.

Sketch of the proof. The equivalence of (2) and (3) is due to Karp; see [AK00, Proposition 15.1]. For (1) $\Rightarrow$ (2), note that if there is a infinitary Σ_n sentence φ that is true in $\mathcal{B}$ but not in $\mathcal{A}$, then to decide whether $\mathcal{C} \cong \mathcal{A}$, all we have to do is check whether $\mathcal{C} \models \varphi$, and if so, we know that $\mathcal{C} \cong \mathcal{B}$. Checking whether $\mathcal{C} \models \varphi$ holds is $\boldsymbol{\Sigma}^0_n$, so deciding whether $\mathcal{C} \cong \mathcal{A}$ is $\boldsymbol{\Pi}^0_n$, and hence not $\boldsymbol{\Sigma}^0_n$-hard. For (2) $\Rightarrow$ (1) we use [AK00, Theorem 18.6]. Let Z be a set that can compute the n-back-and-forth relations among tuples of elements from $\mathcal{A}$ and $\mathcal{B}$. Relative to Z, we have that $\mathcal{A}$ and $\mathcal{B}$ are n-friendly. Let $\varphi(X)$ be a $\boldsymbol{\Sigma^0_n}$ formula of arithmetic with real parameters and a real free variable $\boldsymbol{X}$, and assume that $\boldsymbol{Z}$ was also chosen to compute all the real parameters in $\boldsymbol{\varphi(X)}$. Using the uniformity in [AK00, Theorem 18.6] we obtain a $\boldsymbol{Z}$-computable procedure (hence a continuous function) that given $\boldsymbol{X}$, produces a Boolean algebra $\boldsymbol{\mathcal{C}_X}$ such that

$$\boldsymbol{\mathcal{C}_X} \cong \begin{cases} \boldsymbol{\mathcal{A}} & \text{if } \boldsymbol{\varphi(X)}, \\ \boldsymbol{\mathcal{B}} & \text{if } \boldsymbol{\neg\varphi(X)}. \end{cases}$$

(See [Kec95, §22B] for more on boldface hard classes.) ∎

In [HM] we studied the family of *ordered monoids*

$$(BAs/\equiv_n\ ,\ \leq_n\ ,\ \oplus)$$

where BAs is the class of all countable Boolean algebras and $\mathcal{A}\oplus\mathcal{B}$ is the direct sum, or Cartesian product, of $\mathcal{A}$ and $\mathcal{B}$. We call the equivalence classes *bf-types*, or *n-bf-types*.

Our aim in this section is to show that there is a computable structure of finite invariants for the n-bf-types: a computable ordered monoid $(\mathbf{INV}_n\ ,\ \leq_n\ ,\ +)$ with

$$(BAs/\equiv_n\ , \leq_n, \oplus) \cong (\mathbf{INV}_n, \leq_n, +),$$

and a map T_n from Boolean algebras to $\mathbf{INV}_n$ such that

$$\mathcal{A} \leq_n \mathcal{B} \iff T_n(\mathcal{A}) \leq_n T_n(\mathcal{B}) \quad \text{and} \quad T_n(\mathcal{A}\oplus\mathcal{B}) = T_n(\mathcal{A}) + T_n(\mathcal{B}).$$

To our knowledge, the back-and-forth equivalence classes through the first five levels (i.e., $\equiv_0$,...,$\equiv_4$) were first described by [Ala04]. Here are the first three levels:

- $\mathcal{A} \leq_0 \mathcal{B}$ if and only if $(|\mathcal{A}| = 1 \iff |\mathcal{B}| = 1)$.
- $\mathcal{A} \leq_1 \mathcal{B}$ if and only if $\mathcal{A} \leq_0 \mathcal{B}$ and $|\mathcal{A}| \geq |\mathcal{B}|$.
- $\mathcal{A} \leq_2 \mathcal{B}$ if and only if $|\mathcal{A}| = |\mathcal{B}|$ and $|\mathrm{At}(\mathcal{A})| \geq |\mathrm{At}(\mathcal{B})|$, where $\mathrm{At}(\mathcal{B})$ is the set of atoms of $\mathcal{B}$.

The key to our investigation of the n-back-and-forth types in [HM] are the n-indecomposable Boolean algebras:

Definition 2.3 ([HM])**.** A Boolean algebra $\mathcal{A}$ is *n-indecomposable* if for every partition $(a_i)_{i\leq k}$ of $\mathcal{A}$, there is some $i \leq k$ with $\mathcal{A} \equiv_n \mathcal{A} \restriction a_i$.

The main results we proved about this class of Boolean algebras are summarized in the following theorem.

Theorem 2.4 ([HM])**.** *For each n, there are only finitely many n-back-and-forth equivalence classes among the n-indecomposable algebras. Furthermore, every Boolean algebra can be decomposed into a finite sum of n-indecomposable algebras.*

We use $\mathbf{BF}_n$ to denote the set of invariants in $\mathbf{INV}_n$ that correspond to n-indecomposable Boolean algebras. So, $\mathbf{INV}_n$ is finitely generated by $\mathbf{BF}_n$ under $+$.

2.3. **Back-and-forth invariants for indecomposables.** We start by defining what we call *back-and-forth invariants* on the Stone space of a Boolean algebra. Recall that the *Stone space* of $\mathcal{A}$ is the set of ultrafilters $\mathrm{Ult}(\mathcal{A})$ with the topology given by the basic clopen sets $\mathcal{O}_a = \{V \in \mathrm{Ult}(\mathcal{A}) : a \in V\}$ for $a \in \mathcal{A}$.

For each n, we define a finite set $\mathbf{BF}_n$ of combinatorial objects that we will use as back-and-forth invariants for ultrafilters. Each $\mathbf{BF}_n$ is a subset of the power set of $\mathbf{BF}_{n-1}$, and $\mathbf{BF}_0 = \{*\}$, where $*$ is just a symbol. On each $\mathbf{BF}_n$, we will define a partial ordering $\leq_n$ in a combinatorial way. (See Definition 2.6 below.) Before defining $\mathbf{BF}_n$ and $\leq_n$, we define the invariant maps.

Definition 2.5. Let $\mathcal{A}$ be a Boolean algebra other than the trivial one-element algebra. For $X \subseteq \mathbf{BF}_n$ we let $\max X$ be the antichain of $\leq_n$-maximal elements of X. Let

$\operatorname{dc} X \subseteq \mathbf{BF}_n$ be the $\leq_n$-downward closure of X. To each ultrafilter U of a Boolean algebra $\mathcal{A}$ and each $n \in \omega$ we assign an n-bf-type as follows.

$$\begin{aligned} t_0(U) &= * \\ \hat{t}_{n+1}(U) &= \big\{\alpha \in \mathbf{BF}_n \,:\, U \text{ is an accumulation point of } \{V \in \operatorname{Ult}(\mathcal{A}) : t_n(V) = \alpha\}\big\}. \\ t_{n+1}(U) &= \max \hat{t}_{n+1}(U). \end{aligned}$$

The accumulation points of a set are determined by the topology on the Stone space. So, $\alpha \in \hat{t}_{n+1}(U)$ if and only if $\forall a \in U\ \exists^\infty V \in \operatorname{Ult}(\mathcal{A})\ (a \in V\ \&\ t_n(V) = \alpha)$. If $t_n(U) = \alpha$, we say that U is an α *ultrafilter*. We define $\mathbf{BF}_n$ to be the set of subsets of $\mathbf{BF}_{n-1}$ which appear in the image of the map t_n. Note that all the elements of $\mathbf{BF}_n$ are $\leq_n$-antichains from $\mathbf{BF}_{n-1}$. In [HM, Section on Realizability] we characterize this image in terms of a combinatorial property.

Notice that $\hat{t}_n(U)$ contains more information about U than $t_n(U)$. The reason why we choose to work with $t_n(U)$ is that (as we will show) the information contained in $t_n(U)$ is exactly what can be decoded with n Turing jumps, whereas the information in $\hat{t}_n(U)$ is more extensive than this.

The idea now is to lift the definition of t_n from ultrafilters to n-indecomposable Boolean algebras. We can view an n-indecomposable element of a Boolean algebra as a small enough neighborhoods of an ultrafilter so that the n-back-and-forth properties of that element are given by those of the ultrafilter.

We still have not defined $\leq_n$. The goal of the definition of $\leq_n$ on $\mathbf{BF}_n$ is to obtain the following result. For any n-indecomposable Boolean algebras $\mathcal{A}$ and $\mathcal{B}$,

$$\mathcal{A} \leq_n \mathcal{B} \iff t_n(\mathcal{A}) \leq_n t_n(\mathcal{B}).$$

We will also define a projection map $(\cdot)_n \colon \mathbf{BF}_{n+1} \to \mathbf{BF}_n$ such that for every n-indecomposable Boolean algebra $\mathcal{A}$,

$$(t_{n+1}(\mathcal{A}))_n = t_n(\mathcal{A}).$$

Definition 2.6. By induction on n, we define a relation $\leq_n$ on $\mathbf{BF}_n$ and a map $(\cdot)_n \colon \mathbf{BF}_{n+1} \to \mathbf{BF}_n$.

- On $\mathbf{BF}_0 = \{*\}$, let $* \leq_0 *$.
- For $\alpha \in \mathbf{BF}_{n+1}$, let $(\alpha)_n = \max\{(\gamma)_{n-1} : \gamma \in \alpha\} \in \mathbf{BF}_n$, unless $n = 0$, in which case $(\alpha)_0 = *$.
- For $\alpha, \beta \in \mathbf{BF}_{n+1}$, let $\alpha \leq_{n+1} \beta$ if $(\alpha)_n \equiv_n (\beta)_n$ and $\operatorname{dc}\beta \subseteq \operatorname{dc}\alpha$.
- For $\alpha, \beta \in \mathbf{BF}_{n+1}$, let $\alpha \equiv_{n+1} \beta$ if $\alpha \leq_{n+1} \beta$ and $\beta \leq_{n+1} \alpha$.

The following theorem shows the connection between ultrafilters and n-indecomposable Boolean algebras.

Theorem 2.7. [HM] *For any Boolean algebra $\mathcal{A}$, the following are equivalent.*

(1) There is an ultrafilter $U \in \operatorname{Ult}(\mathcal{A})$, such that for every $V \in \operatorname{Ult}(A)$ with $V \neq U$, we have $t_{n-1}(V) \in \operatorname{dc} t_n(U)$.
(2) There is an ultrafilter $U \in \operatorname{Ult}(\mathcal{A})$ such that $\mathcal{A} \equiv_n \mathcal{A} \restriction a$ for all $a \in U$.
(3) $\mathcal{A}$ is n-indecomposable.

If $\mathcal{A}$ satisfies any of the conditions of the theorem above we say that $\mathcal{A}$ is *n-indecomposable for U*, and we define $t_n(\mathcal{A})$ to be $t_n(U)$. If $\mathcal{A}$ is any Boolean algebra, $a \in \mathcal{A}$, and $\mathcal{A}\upharpoonright a$ is n-indecomposable for U, then we let $t_n(a) = t_n(U)$. Equivalently: For $n = 0$, let $t_0(a) = *$. Given t_n, let $\hat{t}_{n+1}(a) \subseteq \mathbf{BF}_n$ be the set of $\alpha \in \mathbf{BF}_n$ such that there are infinitely many distinct α ultrafilters $V \in \mathcal{O}_a$; then let $t_{n+1}(a) = \max \hat{t}_{n+1}(a)$. If $\mathcal{A}\upharpoonright a$ is n-indecomposable and $t_n(a) = \alpha$, we say that a has *n-indecomposable type* α, or that a is an *α element*.

For each $\alpha \in \mathbf{BF}_n$, there is a computable n-indecomposable Boolean algebra $\mathcal{A}_\alpha$ of n-bf-type α (see [HM, Section 5]). For each $\alpha \in \mathbf{BF}_n$ we define a Boolean algebra predicate R_α as follows. For a Boolean algebra $\mathcal{B}$ and $b \in \mathcal{B}$, we let

$$\mathcal{B} \models \mathrm{R}_\alpha(b) \iff \mathcal{B}\upharpoonright b \geq_n \mathcal{A}_\alpha.$$

We show in [HM, Lemma 8.9] that each of these predicates is Π^c_n. The predicates R_α for $\alpha \in \mathbf{BF}_n$ capture all the structural information of a Boolean algebra that can be obtained with n Turing jumps, as shown in the next theorem.

Theorem 2.8. *Let $\mathcal{B}$ be a Boolean algebra, $R \subseteq \mathcal{B}$ and $n \in \omega$. The following are equivalent.*

(1) *R relatively intrinsically Σ_{n+1}. That is, if $\mathcal{A} \cong \mathcal{B}$ and $(\mathcal{A}, Q) \cong (\mathcal{B}, R)$ then Q is a $\Sigma^0_{n+1}(\mathcal{A})$ subset of ω.*
(2) *R is explicitly Σ_{n+1}. That is, R can be defined in $\mathcal{B}$ by a computable infinitary Σ^c_{n+1} formula with finitely many parameters from $\mathcal{B}$.*
(3) *There exists a $0^{(n)}$-computable sequence $\{\varphi_i : i \in \omega\}$ of finitary Σ_1 formulas that use the predicates R_α for $\alpha \in \boldsymbol{BF}_n$, together with a finite tuple of parameters $\bar{p} \in \mathcal{B}$ such that*

$$x \in R \iff \bigvee_{i\in\omega} \varphi_i(\bar{p}, x).$$

The equivalence of the first two statements is due to Ash, Knight, Manasse and Slaman, and Chisholm (see [AK00, Theorem 10.1]). The equivalence of the latter two statements is proved in [HM, Theorem 8.11].

The importance of the predicates $\{\mathrm{R}_\alpha : \alpha \in \mathbf{BF}_n\}$ leads to the following definition, stated previously in 1.1,

Definition 2.9. An *n-Boolean algebra* (*n-algebra*, for short), denoted by $\mathrm{B}_n(\mathcal{A})$, is a structure $(A, \leq, \vee, \wedge, \neg, 0, 1, \mathrm{R}_\alpha : \alpha \in \mathbf{BF}_n)$ where $\mathcal{A} = (A, \leq, \vee, \wedge, \neg, 0, 1)$ is a Boolean algebra. If $\mathrm{B}_n(\mathcal{A})$ is Y-computable, we say that $\mathcal{A}$ is *n-Y-approximable*.

Using the notation from Montalbán [Mon], Theorem 2.8 states that $\mathrm{B}_n(\mathcal{A})$ is the *nth jump of the structure $\mathcal{A}$*. The next theorem, a corollary of 2.8 , provides provides alternative characterizations of n-0^n-approximable algebras.

Theorem 2.10. *Let $\mathcal{A}$ be a presentation of a Boolean algebra. The following are equivalent:*

(a) *$B_n(\mathcal{A})$ is computable in $0^{(n)}$.*
(b) *The Σ^c_{n+1}-diagram of $\mathcal{A}$ is a Σ^0_{n+1} set of formulas.*
(c) *The Π^c_n-diagram of $\mathcal{A}$ is computable in $0^{(n)}$,*

The following theorem is a consequence of [Mon, Theorem 3.1].

Theorem 2.11. *Let $\mathcal{A}$ be a Boolean algebra, and let Z be any set. If $B_n(\mathcal{A})$ is $Z^{(n)}$-computable, then $\mathcal{A}$ has a Z-low$_n$ copy. Furthermore, an isomorphism between $\mathcal{A}$ and its Z-low$_n$ copy can be computed by $Z^{(n)}$.*

In our main construction, we will build a 5-$0^{(5)}$-approximable Boolean $\mathcal{A}$ algebra which is not $0^{(7)}$-isomorphic to any computable Boolean algebra. Then using Theorem 2.11, we will get that $\mathcal{A}$ is $0^{(5)}$-isomorphic to a low$_5$ Boolean algebra. This low$_5$ Boolean algebra is then also not $0^{(7)}$-isomorphic to any computable one.

2.4. **General back-and-forth invariants.** Let $(\overline{\mathbf{INV}}_n, +, 0)$ be the free commutative monoid with generators $\mathbf{BF}_n$. Given $\sigma = \alpha_1 + \ldots + \alpha_k \in \overline{\mathbf{INV}}_n$, let $\mathcal{A}_\sigma = \mathcal{A}_{\alpha_1} \oplus \ldots \oplus \mathcal{A}_{\alpha_k}$, where $\mathcal{A}_{\alpha_i}$ is a Boolean algebra of n-bf-type α_i. Given $\sigma, \tau \in \overline{\mathbf{INV}}_n$, let $\sigma \leq_n \tau$ if $\mathcal{A}_\sigma \leq_n \mathcal{A}_\tau$. This induces an equivalence relation $\equiv_n$ on $\overline{\mathbf{INV}}_n$; let $\mathbf{INV}_n$ be the quotient structure $(\mathbf{INV}_n, \leq_n +)$. The invariant map from Boolean algebras to $\mathbf{INV}_n$ is now defined in the obvious way.

Definition 2.12. Given a Boolean algebra $\mathcal{B}$, let $\mathcal{B}_0 \oplus \ldots. \oplus \mathcal{B}_k$ be a partition of $\mathcal{B}$ into n-indecomposable Boolean algebras. Let $T_n(\mathcal{B}) \in \mathbf{INV}_n$ be the $\equiv_n$-equivalence class of

$$t_n(\mathcal{B}_0) + \ldots. + t_n(\mathcal{B}_k) \in \overline{\mathbf{INV}}_n.$$

We proved in [HM, Section 7] that this definition is independent of the choice of the partition $\mathcal{B}_1 \oplus \ldots. \oplus \mathcal{B}_k$ of $\mathcal{B}$ and that

$$T_n \colon (BAs/\equiv_n, \leq_n, \oplus) \to (\mathbf{INV}_n, \leq_n, +)$$

is an isomorphism. We also give a purely combinatorial definition of $\leq_n$ on $\overline{\mathbf{INV}}_n$ in [HM, Section 7].

2.5. **Exclusive bf-types.**

Definition 2.13. We say that $\alpha \in \mathbf{BF}_n$ is *exclusive* if $(\alpha)_{n-1} \notin \operatorname{dc} \alpha$.

In terms of Boolean algebras, if $\mathcal{A}$ is n-indecomposable and has an exclusive n-indecomposable type, then for every partition $a_0, \ldots, a_\ell$ of $\mathcal{A}$ there is a unique i such that $\mathcal{A} \equiv_{n-1} \mathcal{A} \restriction \alpha_i$. On the other hand, if $\mathcal{A}$ is n-indecomposable but its n-bf-type is not exclusive, then there are infinitely many ultrafilters $U \in \operatorname{Ult}(\mathcal{A})$ with $t_{n-1}(U) \geq_{n-1} t_{n-1}(\mathcal{A})$. For example, the 1-atom $\operatorname{Int}(\omega)$ is n-indecomposable for every n, but it has an exclusive n-indecomposable type only when $n \geq 3$. For $n = 2$, $\operatorname{Int}(\omega)$ has the 2-indecomposable type $\{\{\}\}$, corresponding to an algebra bounding infinitely many atoms, as does the algebra $\operatorname{Int}(2 \cdot \eta)$ which is not exclusive for any n. For $n = 1$, there are two bf-types: the exclusive type of the atom $\{\}$ and the type for an infinite algebra $\{*\} = t_1(\operatorname{Int}(\omega))$, where $\{\} \geq_1 \{*\}$.

Up to level four, every exclusive n-indecomposable type is a $\leq_n$-maximal element in $\mathbf{BF}_n$. This is not the case at level five, and we will exploit this in our construction.

3. Boolean algebra approximations

In this section we introduce the notion of an n-approximation of a Boolean algebra. This notion, which is one of the main applications of the work in [HM], is new, although it has roots in the work of others. The idea of using back-and-forth increasing sequences of finite approximations appears in the work of Ash on η-systems, but in a very general setting. (See Chapter 12 of [AK00] on n-systems, and especially Section 15.6, for a motivation of these constructions, which bear a close relationship to our n-approximations.) In [KS00] they used finite Boolean algebras with additional predicates to approximate low$_n$ Boolean algebras in their proof, for $n = 1, 2, 3, 4$, that every low$_n$ Boolean algebra has a computable copy. Their construction in the low$_4$ case corresponds to what we call a 3-approximation, although we use the predicates R_α for $\alpha \in \mathbf{BF}_3$, while they used a finite set of predicates sufficient for defining these relations via Boolean combinations.

3.1. **Finite labeled Boolean algebras.** To approximate an n-Boolean algebra, we use finite n-labeled Boolean algebras.

Definition 3.1. A *finite n-labeled Boolean algebra* is a pair $(\mathcal{C}, t_n^{\mathcal{C}})$, where $\mathcal{C}$ is a finite Boolean algebra and $t_n^{\mathcal{C}} \colon \mathcal{C} \to \mathbf{INV}_n$ is such that

(1) for every minimal element $a \in \mathcal{C}$, $t_n^{\mathcal{C}}(a) \in \mathbf{BF}_n$;
(2) if $b = \bigvee_{i=0}^{k} a_i$, where each a_i is a minimal element in $\mathcal{C}$, then $t_n^{\mathcal{C}}(b) = \sum_{i=0}^{k} t_n^{\mathcal{C}}(a_i)$.

The minimal elements of a finite Boolean algebra are its atoms. We want to avoid using the word "atom" in this context, however, because we already use it to refer to one of the n-bf-types. A minimal element of $\mathcal{C}$ may or may not be labeled with the n-bf-type "atom".

Given two finite n-labeled Boolean algebras $\mathcal{C}_0$ and $\mathcal{C}_1$, we write

$$\mathcal{C}_0 \subseteq_n \mathcal{C}_1$$

if $\mathcal{C}_0$ is a subalgebra of $\mathcal{C}_1$ and for every $x \in \mathcal{C}_0$, $t_n^{\mathcal{C}_0}(x) = t_n^{\mathcal{C}_1}(x)$.

We write

$$\mathcal{C}_0 \leq_n \mathcal{C}_1$$

if $\mathcal{C}_0$ is a subalgebra of $\mathcal{C}_1$ and for every $x \in \mathcal{C}_0$, $t_n^{\mathcal{C}_0}(x) \leq_n t_n^{\mathcal{C}_1}(x)$.

Note that every n-labeled Boolean algebra is also an $(n-1)$-labeled Boolean algebra using the labeling function $t_{n-1}^{\mathcal{C}}(b) = (t_n^{\mathcal{C}}(b))_{n-1}$. We denote this $(n-1)$-labeled Boolean algebra by $(\mathcal{C})_{n-1}$.

We observe that if $\mathcal{C}_0 \leq_n \mathcal{C}_1$, then $(\mathcal{C}_0)_{n-1} \subseteq_{n-1} (\mathcal{C}_1)_{n-1}$, since $\alpha \leq_n \beta$ implies $\alpha \equiv_{n-1} \beta$.

Definition 3.2. Let $\mathcal{B}$ be a an n-labeled Boolean algebra, $b \in \mathcal{B}$ and $\sigma = t_n^{\mathcal{B}}(b) \in \mathbf{BF}_n$. We say that b *splits into a σ-full partition* if there exists a partition $b = \bigvee_{i=0}^{\ell} a_i$ in $\mathcal{B}$ such that

- $t_n^{\mathcal{B}}(a_0) = \sigma$,
- for every $\delta \in \sigma$, there is some j with $0 < j \leq \ell$ such that $t_{n-1}^{\mathcal{B}}(a_j) = \delta$.

For $\sigma \in \mathbf{INV}_n$, we say that b *splits into a σ-full partition* if there exists a partition $b = \bigvee_{i=0}^{\ell} a_i$ in $\mathcal{B}$ such that each a_i has n-indecomposable bf-type and splits into a $t_n^{\mathcal{B}}(a_i)$-full partition.

Remark 3.3. Note that for any Boolean algebra $\mathcal{B}$ and every $b \in \mathcal{B}$, if $T_n^{\mathcal{B}}(b) = \sigma$ then there always exists a σ-full partition of b [HM, Lemma 7.4 (2)]. This need not be the case for finite n-labeled Boolean algebras.

3.2. **n-approximations.** We consider sequences of finite n-labeled Boolean algebras.

Definition 3.4. An n-*approximation* is a sequence $\{(\mathcal{A}_k, t_n^k) : k \in \omega\}$ of finite n-labeled Boolean algebras such that

(1) for each k, $\mathcal{A}_k \leq_n \mathcal{A}_{k+1}$;
(2) for each k and every minimal element b of $\mathcal{A}_k$, b splits in $\mathcal{A}_{k+1}$ into a $t_n^{k+1}(b)$-full partition.

An n-Z-*approximation* is an n-approximation that is computable in Z.

Remark 3.5. If $\mathcal{A}$ has an n-Z-approximation by $\{(\mathcal{A}_k, t_n^k) : k \in \omega\}$ and $m < n$, then it is also has an m-Z-approximation by $\{(\mathcal{A}_k, t_m^k) : k \in \omega\}$, where $t_m^k(x) = \left(t_n^k(x)\right)_m$.

Theorem 3.6. *Let $\{(\mathcal{A}_k, t_n^k) : k \in \omega\}$ be an n-approximation of $\mathcal{A} = \bigcup_k \mathcal{A}_k$. Then for every $x \in \mathcal{A}$, $\{t_n^k(x) : k \in \omega\}$ eventually stabilizes and $T_n^{\mathcal{A}}(x) = \lim_{k\to\infty} t_n^k(x)$.*

Proof. It is clear that $\mathcal{A}$ is a Boolean algebra. What we need to prove is that for every $a \in \mathcal{A}$, $T_n^{\mathcal{A}}(a) = \lim_{k\to\infty} t_n^k(a)$. That is, we need to show that the sequence $\{t_n^k(a) : k \in \omega\}$ stabilizes, and that once it stabilizes at a certain n-bf-type, it ends up building an element of that type. The fact that we are taking full partitions of every minimal element at every step guarantees the latter. In this proof we will cite results and definitions from [HM] which might be difficult for the reader unfamiliar with that source to follow.

We prove, by induction on n, that for every n-indecomposable element $a \in \mathcal{A}$, if $T_n^{\mathcal{A}}(a) = \alpha \in \mathbf{BF}_n$ then $\exists k_0\ \forall k \geq k_0 (t_n^k(a) = \alpha)$. This is sufficient since every $b \in \mathcal{A}$ is a finite sum of n-indecomposables (by Theorem 2.4), and if the property holds for the n-indecomposables in a partition of b, then it also holds for b. Since for every k, $(\mathcal{A}_k)_{n-1} \subseteq_{n-1} (\mathcal{A}_{k+1})_{n-1}$, we may assume, by the induction hypothesis, that for every x and sufficiently large k, $T_{n-1}^{\mathcal{A}}(x) = t_{n-1}^k(x)$.

First, we show that for every k, $t_n^k(a) \leq_n T_n^{\mathcal{A}}(a) = \alpha$. Let $(\alpha_0, \ldots, \alpha_\ell) \in \mathbf{BF}_{n-1}^{<\omega}$ be a partition of α (see [HM, Definition 7.3]), so that (by [HM, Lemma 7.5]) we need to find a partition $(\beta_0, \ldots, \beta_\ell)$ of $t_n^k(a)$ such that $\alpha_i \leq_{n-1} \beta_i$ for every $i \leq \ell$. By [HM, Lemma 7.4], there exists a partition $a = a_0 \dot\vee \ \ldots \ \dot\vee\ a_\ell$ such that $\alpha_i \leq_{n-1} T_{n-1}^{\mathcal{A}}(a_i)$. Let k_1 be such that $a_0, \ldots, a_\ell \in \mathcal{A}_{k_1}$ and $T_{n-1}^{\mathcal{A}}(a_i) = t_{n-1}^{k_1}(a_i)$ for each i. Then we have $\alpha_i \leq_{n-1} T_{n-1}^{\mathcal{A}}(a_i) = t_{n-1}^{k_1}(a_i)$. Now, $t_n^k(a) \leq_n t_n^{k_1}(a)$, so there exists a partition $(\beta_0, \ldots, \beta_\ell)$ of $t_n^k(a)$ such that $t_{n-1}^{k_1}(a_i) \leq_{n-1} \beta_i$. It follows that $\alpha_i \leq_{n-1} \beta_i$ for every $i \leq \ell$, as required.

Now, we need to show that for some k_0, $t_n^{k_0}(a) = T_n^{\mathcal{A}}(a) = \alpha$. Recall that for $\beta \in \mathbf{BF}_n \subseteq \mathcal{P}(\mathbf{BF}_{n-1})$, $\mathrm{dc}(\beta) = \{\gamma \in \mathbf{BF}_{n-1} : \exists \delta \in \beta\ (\gamma \leq_{n-1} \delta)\}$. For $\alpha = \beta_0 + \ldots + \beta_l \in \mathbf{INV}_n$, let $\mathrm{dc}(\alpha) = \bigcup_{i=0}^{l} \mathrm{dc}\,\beta_i$. It follows from [HM, Theorem 7.18.(3a)] that this is well-defined on $\mathbf{INV}_n$. In terms of Boolean algebras, we have that $\delta \in \mathrm{dc}(\alpha)$ if and only if every Boolean algebra of n-bf-type α has infinitely many ultrafilters of $(n-1)$-bf-type $\geq_{n-1} \delta$.

Since the sequence $t_n^k(a)$ is $\leq_n$-increasing, it follows from [HM, Theorem 7.18.(3a)] that for every k, $\mathrm{dc}(t_n^k(a)) \supseteq \mathrm{dc}(t_n^{k+1}(a))$. Let k_0 be such that $\forall k \geq k_0\ \mathrm{dc}(t_n^k(a)) = \mathrm{dc}(t_n^{k_0}(a))$. We will now prove that $t_n^{k_0}(a) \geq_n \alpha$ using the following which is also implied by [HM, Theorem 7.18.(3)].

Given $\alpha \in \mathbf{BF}_n$ and $\beta_0 + ... + \beta_l \in \mathbf{INV}_n$, we have that $\alpha \leq_n \beta_0 + ... + \beta_l$ if and only if

(1) $\mathrm{dc}(\beta_0 + ... + \beta_l) \subseteq \mathrm{dc}\,\alpha$;
(2) there is a partition $(\alpha_0, \dots, \alpha_l) \in \mathbf{INV}_{n-1}^{<\omega}$ of α such that for each i, $(\beta_i)_{n-1} \leq_{n-1} \alpha_i$.

To show (1), that $\mathrm{dc}(t_n^{k_0}(a)) \subseteq \mathrm{dc}\,\alpha$, fix $\delta \in \mathrm{dc}(t_n^{k_0}(a))$. We claim that there are unboundedly many disjoint elements below α of (n-1)-bf-type $\geq_{n-1} \delta$. Suppose not, and that $r \in \omega$ is maximal such that, for some $u > k_0$, there exists a tuple $a_0, ..., a_r$ of disjoint minimal elements in $\mathcal{A}_u \restriction a$ with $t_{n-1}^u(a_i) \geq_{n-1} \delta$. Observe that for every $s > u$ and for each $i \leq r$, there is a minimal element $a_{i,s} \in \mathcal{A}_s \restriction a_i$ with $t_{n-1}^s(a_{i,s}) \geq_{n-1} \delta$ (by [HM, Lemma 7.12]). Furthermore, there is at most one such $a_{i,s}$; otherwise, r would not be maximal. This implies that $\delta \notin \mathrm{dc}\, t_n^{u+1}(a_i)$ for every $i \leq r$; otherwise a $t_n^{u+1}(a_i)$-full partition would add at least a second element of type $\geq_{n-1} \delta$ below a_i, contradicting the maximality of r. Let $b = a - (a_0 \vee ... \vee a_r)$. Since $\delta \in \mathrm{dc}(t_n^{u+1}(a))$, it follows that $\delta \in \mathrm{dc} \bigcup t_n^{u+1}(b)$. But since b splits into a $t_n^{u+1}(b)$-full partition, this would add an $n-1$-indecomposable bf-type $\geq_{n-1} \delta$ below b contradicting the maximality of r. This proves our claim and that $\delta \in \mathrm{dc}\,\alpha$.

To show (2), let $a_0, \dots, a_\ell$ be the minimal elements of $\mathcal{A}_{k_0} \restriction a$, and let $\beta_i = t_n^{k_0}(a_i)$. So, $t_n^{k_0}(a) = \sum_{i=0}^{\ell} \beta_i$ and $\beta_i \equiv_{n-1} T_{n-1}^{\mathcal{A}}(a_i)$. Then $(T_{n-1}^{\mathcal{A}}(a_i))_{i \leq \ell}$ is a partition of α as required by (2). □

Theorem 3.7. *Let $\mathcal{A}$ be a presentation of a Boolean algebra, and let Z be any set. The following are equivalent:*

(1) $\mathcal{A}$ is n-Z-approximable. That is, $\mathrm{B}_n(\mathcal{A})$ is Z-computable (Definition 2.9).
(2) $\mathcal{A}$ has n-Z-approximation.

Proof. First, let us assume that Z computes $\mathrm{B}_n(\mathcal{A})$. We will define an n-Z-approximation $\{(\mathcal{A}_k, t_n^k) : k \in \omega\}$ of $\mathcal{A}$. Let u be the partial function on $\mathcal{A}$ defined so that, for each $a \in \mathcal{A}$, $u(a)$ is the $\leq_n$-greatest n-bf-type $\gamma \in \mathbf{BF}_n$ such that $\mathrm{R}_\gamma(a)$ holds in $\mathcal{A}$, if such γ exists, and $u(a)$ is undefined otherwise. Note that the function u and its domain are Z-computable and that $u(a) \leq_n T_n^{\mathcal{A}}(a)$. Moreover, if $a \in \mathcal{A}$ is n-indecomposable, then $u(a) = T_n^{\mathcal{A}}(a)$.

To define $\mathcal{A}_0$, search for a partition $a_0, \dots, a_\ell$ of $1_{\mathcal{A}}$ such that for every i, $u(a_i)$ is defined (which must exist by Theorem 2.4). Let $\mathcal{A}_0$ be the Boolean algebra generated by these a_i, and let $t_n^0(a_i) = u(a_i)$.

Let $\{b_0, b_1, \dots\}$ be an enumeration of the non-zero elements of $\mathcal{A}$. To ensure that $\mathcal{A} = \bigcup_{k\in\omega} \mathcal{A}_k$, we will make sure that $b_k \in \mathcal{A}_{k+1}$. Suppose we have already defined $(\mathcal{A}_k, t_n^k)$ so that $(\mathcal{A}_k, t_n^k) \leq_n (\mathcal{A}, T_n^{\mathcal{A}})$ and $\{b_0, \dots, b_{k-1}\} \subseteq \mathcal{A}_k$. For each minimal element a of $\mathcal{A}_k$, look for a partition $a_0, \dots, a_\ell \in \mathcal{A}$ of a such that

- $u(a_i)$ is defined for each $i \leq \ell$;
- for some $h \leq \ell$, $a \wedge b_k = \bigvee_{i=0}^{h} a_i$ and $a - b_k = \bigvee_{i=h+1}^{\ell} a_i$;
- $t_n^k(a) \leq_n \sum_{i=0}^{\ell} u(a_i)$;
- the elements $a_0, ..., a_\ell$ can be joined in some way to produce a $\left(\sum_{i=0}^{\ell} u(a_i)\right)$-full partition of a.

There exists such a tuple of elements in $\mathcal{A}$, by applying Remark 3.3 to $a \wedge b_k$ to obtain a partition $a_0, \ldots, a_h$ and to $a - b_k$ to obtain a partition $a_{h+1}, \ldots, a_\ell$, so that a Z-computable search will eventually find such a partition of a.

Let $\mathcal{A}_{k+1}$ be the extension of $\mathcal{A}_k$ generated by the partitions $a_0, \ldots, a_\ell$ chosen below each minimal element a of $\mathcal{A}_k$. For each of these new elements a_i, let $t_n^{k+1}(a_i) = u(a_i)$. For each $b \in \mathcal{A}_{k+1}$ with $b = \bigvee_{i=1}^{j} c_i$, where the c_i's are minimal elements in $\mathcal{A}_{k+1}$, let $t_n^{k+1}(b) = \sum_{i=1}^{j} u(c_i)$.

For every minimal element $d \in \mathcal{A}_{k+1}$, we have $t_n^{k+1}(d) \leq_n T_n^{\mathcal{A}}(d)$, hence $(\mathcal{A}_{k+1}, t_n^{k+1}) \leq_n (\mathcal{A}, T_n^{\mathcal{A}})$. For every minimal element $c \in \mathcal{A}_k$, we have $t_n^k(c) \leq_n t_n^{k+1}(c)$, hence $(\mathcal{A}_k, t_n^k) \leq_n (\mathcal{A}_{k+1}, t_n^{k+1})$. For every minimal element $a \in \mathcal{A}_k$, both $b_k \wedge a$ and $a - b_k$ are in $\mathcal{A}_{k+1}$, hence $b_k \in \mathcal{A}_{k+1}$. This completes the proof of (2).

Suppose now that we have an n-Z-approximation $\{(\mathcal{A}_k, t_n^k) : k \in \omega\}$ of $\mathcal{A}$. By Remark 3.5, we have m-Z-approximations of $\mathcal{A}$ for $m \leq n$ as well. Furthermore, for $m < n$, and for all $\alpha \in \mathbf{BF}_m$ and $a \in \mathcal{A}$, we have that $\mathrm{R}_\alpha(a)$ if and only if $t_m^k(a) \geq_m \alpha$, where k is such that $a \in \mathcal{A}_k$. This is because for all $m < n$, $t_m^k(a) = T_m^{\mathcal{A}}(a)$. Thus, the predicates R_β are computable in Z for $\beta \in \mathbf{BF}_{n-1}$. Given $a \in \mathcal{A}$, we have that $\mathrm{R}_\alpha(a)$ if and only if $\exists k\ (t_n^k(a) \geq_n \alpha)$, so all the predicates $\mathrm{R}_\alpha(\mathcal{A})$ for $\alpha \in \mathbf{BF}_n$ are computably enumerable in Z. On the other hand, from the inductive step in the proof of [HM, Lemma 8.9], we get that R_α is Π_1^c over the predicates R_β for $\beta \in \mathbf{BF}_{n-1}$. Since we are assuming the predicates R_β for $\beta \in \mathbf{BF}_{n-1}$ are Z-computable, we get that $\mathrm{R}_\alpha(\mathcal{A})$ is co-c.e. in Z. Thus, the predicates $\mathrm{R}_\alpha(\mathcal{A})$ for $\alpha \in \mathbf{BF}_n$ are computable in Z. □

Remark 3.8. The proof of Theorem 3.7 is uniform in the following sense. We can uniformly go from a Z-computable index for $\mathrm{B}_n(\mathcal{A})$ to a Z-computable index for the sequence $\{\mathcal{A}_k : k \in \omega\}$, and vice versa. When we refer to an index for an n-Z-approximable Boolean algebra, we mean an index of either of these two kinds.

Remark 3.9. Suppose that $\{(\mathcal{A}_k, t_n^k) : k \in \omega\}$ is an n-approximation of $\mathcal{A}$. Then the ultrafilters U of $\mathcal{A}$ are in one-to-one correspondence with the sequences $\{a_k : k \in \omega\}$, where a_k is a minimal element of $\mathcal{A}_k$ and $a_k \geq a_{k+1}$. (Given U, let a_k be the unique minimal element of $\mathcal{A}_k$ that is in U. Given $\{a_k : k \in \omega\}$, let $U = \{a \in \mathcal{A} : \exists k\ (a \geq a_k)\}$.)

With this in mind, we say that $a \in \mathcal{A}$ *bounds* $V \in \mathrm{Ult}(\mathcal{A})$ if $V \in \mathcal{O}_a$.

4. Some useful 5-bf-types

We now describe the 5-indecomposable types we are going to use in our construction. All the 5-bf-types are described in [HM, Section 6]. There are two 5-indecomposable types that are the principal actors in our construction: $w = f_{23}$ and $u = f_{24}$. The key is that both u and w are exclusive 5-indecomposable types (Definition 2.13) and $w <_5 u$. No such pair of exclusive n-indecomposable types exists for $n = 1, 2, 3, 4$. Another key actor is $w^\infty = f_{12}$, which is not exclusive. The relationship of w^∞ to w and u is given by

$$w^\infty <_5 w <_5 u.$$

These three 5-bf-types have the same parent, which we call $\alpha = e_6 \in \mathbf{BF}_4$. In other words, $(u)_4 = (w)_4 = (w^\infty)_4 = \alpha$. The 5-approximable Boolean algebras we construct will use only seven of the twenty-seven 5-indecomposable types: $f_0, f_1, f_2, f_{12}, f_{23}, f_{24}, f_{26}$.

Let

$$D = \{f_0, f_1, f_2, f_{12}, f_{23}, f_{24}, f_{26}\} \subset \mathbf{BF}_5,$$

and let $\langle D\rangle = \{\sum_{i=0}^k \beta_i : k \in \omega, \beta_i \in D\} \subseteq \mathbf{INV}_5$, the subset of $\mathbf{INV}_5$ generated by D under addition. The first five of the seven 5-indecomposable types listed have the property that all Boolean algebras with that type are isomorphic: f_0 corresponds to atoms, f_1 to 1-atoms, f_2 to 2-atoms, f_{26} to the atomless Boolean algebra, and $u = f_{24}$ to the interval algebra $\mathrm{Int}\big(\omega^2 + \eta\big)$. We will say that such back-and-forth types are *isomorphism types*. The other two 5-bf-types are not isomorphism types, but in our construction we will consider only one Boolean algebra of type $w = f_{23}$, namely, the interval algebra $\mathrm{Int}\big(\omega^3 + \eta\big)$; and we will consider only two Boolean algebras of type $w^\infty = f_{12}$: $\mathrm{Int}\big((\omega^3 + \eta)\cdot\omega\big)$, which we call a 1-$w$-atom and has only one ultrafilter of type w^∞, and $\mathrm{Int}\big((\omega^2 + 1 + \eta)\cdot\eta\big)$, which has densely many ultrafilters of type w^∞. (We will not be using interval algebras in what follows; we just wanted to give an idea of what these types look like.)

We are going to build a sequence $\{\mathcal{A}[s] : s \in \omega\}$ of finite 5-labeled Boolean algebras with the following properties:

- (D1) All the minimal elements of $\mathcal{A}[s]$ are labeled with types in D;
- (D2) Elements of type f_0, f_1, f_2, f_{26}, and u never change their type. Minimal elements in $\mathcal{A}[s]$ of 5-indecomposable type w^∞ can either retain type w^∞ or increase their type to w in $\mathcal{A}[s+1]$. Minimal elements of type w can either retain type w or increase their type to u.
- (D3) Each minimal element a of $\mathcal{A}[s]$ splits in $\mathcal{A}[s+1]$ into a canonical $t_5^{\mathcal{A}[s+1]}(a)$-full partition as indicated in the table below.
- (D4) $t_5^{\mathcal{A}[s]}(1) = w^\infty$ for all s.

Note that these conditions guarantee that $\{\mathcal{A}[s] : s \in \omega\}$ is a 5-approximation of some Boolean algebra $\mathcal{A}$ with $T_5(1_{\mathcal{A}}) = w^\infty$.

In our construction, we will use a particular kind of full partitions that we call *canonical partitions* and we list below. To each $\gamma \in D$, we will assign a tuple $(\gamma_0, \dots, \gamma_k) \in \mathbf{BF}_5^{<\omega}$ such that $\gamma \equiv_5 \sum_{i\leq k} \gamma_i$, $\gamma_0 = \gamma$, and for each i with $1 \leq i \leq k$, $(\gamma_i)_4 \in \gamma$. Note the similarities to the definition of γ-full partition (Definition 3.2). To help the reader picture these types, in the final column we note the isomorphism type of the algebra constructed below an element of a given type, provided the type of the element does not change during the construction.

type	canonical partition	algebra
f_0	f_0	$\mathrm{Int}\big(2\big)$
f_1	f_1, f_0	$\mathrm{Int}\big(\omega\big)$
f_2	f_2, f_0, f_1	$\mathrm{Int}\big(\omega^2\big)$
f_{26}	f_{26}, f_{26}	$\mathrm{Int}\big(\eta\big)$
$u = f_{24}$	u, f_0, f_1, f_{26}	$\mathrm{Int}\big(\omega^2 + \eta\big)$
$w = f_{23}$	w, f_0, f_1, f_2, f_{26}	$\mathrm{Int}\big(\omega^3 + \eta\big)$
$w^\infty = f_{12}$	$w^\infty, w^\infty, f_0, f_1, f_2, f_{26}$	$-$

The following theorem describes $(\langle D\rangle, \leq_5, +)$.

Theorem 4.1. *The following equations hold in $\boldsymbol{INV}_5$:*

(1) $u = u + f_0 = u + f_1 = u + f_{26}$
(2) $w = w + f_0 = w + f_1 = w + f_2 = w + f_{26}$
(3) $w^\infty = w^\infty + f_0 = w^\infty + f_1 = w^\infty + f_2 = w^\infty + f_{26} = w^\infty + u = w^\infty + w = w^\infty + w^\infty$
(4) $w + u = w + w$

Proof. To show (1)-(3), we use the following consequence of [HM, Lemma 7.12] and [HM, Theorem 7.18.(3)]: Given $\beta, \gamma \in \mathbf{BF}_n$,

$$\beta \lhd_n \gamma \implies \beta = \beta + \gamma,$$

where $\gamma \lhd_n \beta$ was defined in [HM, Definition 5.2]. We let the reader verify (using the material in [HM, Section 6]) that $u \lhd_5 f_0, f_1, f_{26}$, that $w \lhd_5 f_0, f_1, f_2, f_{26}$, and that $w^\infty \lhd_5 f_0, f_1, f_2, f_{26}, u, w$. Part (4) follows from [HM, Theorem 7.18.(3)], using the partition (α, α). □

Definition 4.2. Let $\mathcal{A}$ be a Boolean algebra and γ an n-indecomposable type. We let $num(\gamma, \mathcal{A}) \in \omega \cup \{\infty\}$ be the number of ultrafilters in $\mathcal{A}$ of type γ.

For a finite n-labeled Boolean algebra $\mathcal{A}$, we let $num(\gamma, \mathcal{A})$ be the number of minimal elements of $\mathcal{A}$ of type γ.

Definition 4.3. A 1-w-atom is a 5-indecomposable Boolean algebra $\mathcal{A}$ that satisfies the following properties:

(i) $T_5(\mathcal{A}) = w^\infty$.
(ii) For each element $a \in \mathcal{A}$ with $T_4(a) = \alpha$, $T_5(a) \in \{w, w^\infty\}$.
(iii) There is no disjoint pair $a, b \in \mathcal{A}$ with $num(\alpha, \mathcal{A} \restriction a) = num(\alpha, \mathcal{A} \restriction b) = \infty$.

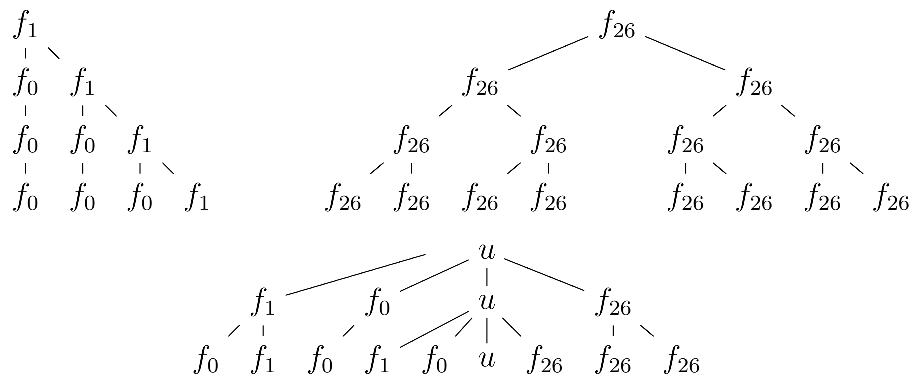
The conditions listed for a 1-w-atom do not determine a unique isomorphism type. However, if we construct a 5-approximation of a Boolean algebra $\mathcal{A}$ which satisfies (D1)-(D4), then every subalgebra of $\mathcal{A}$ which is a 1-w-atom will be isomorphic to the interval algebra $\mathrm{Int}\big((\omega^3 + \eta) \cdot \omega\big)$. (We will not use this fact in our construction.)

Lemma 4.4. *Let $\{\mathcal{A}[s] : s \in \omega\}$ be a 5-approximation of a Boolean algebra $\mathcal{A}$ that satisfies (D1)-(D4). Suppose that for every s_0 there exists $s \geq s_0$ such that $\mathcal{A}[s]$ has exactly one minimal element of 5-indecomposable type w^∞ and no minimal elements of 5-indecomposable type u. Then $\mathcal{A}$ is a 1-w-atom.*

Proof. By condition (D4), $T_5(\mathcal{A}) = w^\infty$. By condition (D1), every 5-indecomposable element $a \in \mathcal{A}$ with $T_4(a) = \alpha$ must satisfy $T_5(a) \in \{u, w, w^\infty\}$. Since there are no minimal elements of type u in any approximation of $\mathcal{A}$, it follows that $T_5(a) \in \{w, w^\infty\}$. Finally, if there were disjoint elements $a, b \in \mathcal{A}$ with $num(\alpha, \mathcal{A} \restriction a) = num(\alpha, \mathcal{A} \restriction b) = \infty$, then we would have $T_n(a) = w^\infty = T_n(b)$, since $T_n(a), T_n(b) \in \langle D \rangle$ and w^∞ is the only 5-bf-type in $\langle D \rangle$ with $\alpha \in \mathrm{dc}\, w^\infty$ by part (3) of Theorem 4.1. By Theorem 3.6, there is an s_0 such that for all $s \geq s_0$, $t_5^s(a) = T_5(a) = w^\infty$ and $t_5^s(b) = T_5(b) = w^\infty$. Thus, at every stage $s \geq s_0$ there are distinct minimal elements $a', b' \in \mathcal{A}[s]$ with $a' \leq a$, $b' \leq b$ and $t_5^s(a') = w^\infty = t_5^r(b')$. This contradicts the hypothesis about $\mathcal{A}$. □

4.1. **Picturing n-approximations.** A good way to picture n-approximations is by the use of labeled trees. Let $\{(\mathcal{A}[s], t_n^s) : s \in \omega\}$ be an n-approximation. At the root of the tree, we put one node labeled $t_n^0(1_\mathcal{A})$. At the sth level of the tree, we draw the minimal elements of $\mathcal{A}[s]$ and label them using t_n^s. The ordering on the nodes of the tree comes from the ordering of the minimal elements of $\mathcal{A}[s+1]$ relative to those of $\mathcal{A}[s]$. So, the labels of the children at level $(s+1)$, of a node a at level s, form a $t_n^{s+1}(a)$-full partition. Notice that the label of a does not need to be $t_n^{s+1}(a)$, but $t_n^s(a)$.

In the picture below we draw the first few levels of the trees corresponding to 5-approximations of the Boolean algebras $\mathcal{A}_{f_1}$, $\mathcal{A}_{f_{26}}$, and $\mathcal{A}_u$, which are of types f_1, f_{26}, and u, respectively. All of these examples satisfy (D1)-(D4), and the 5-bf-types of the elements do not change over time.



5. Necessary tools

In this section we establish several lemmas that will be useful in our construction. First, we look at how to deal with guesses of answers to $0''$ questions. Second, we consider infinite sums of n-Z-approximable Boolean algebras. Third, we show how to enumerate all the 4-0-approximable Boolean algebras, recursively in $0'$. The reader might want to skip the technical proofs in this section in a first read of the paper.

5.1. **Zero double guesses.** The following lemma shows that the guesses to $0''$ questions can be ordered so that the correct one is the limit infimum of all the guesses. This is not new and is essentially what happens when one does infinite-injury priority argument on a tree of strategies.

Lemma 5.1. *Given $e \in \omega$, there are total computable functions $z\colon \omega \to \omega$ and $g\colon \omega \to \omega$ such that there is at most one ℓ with $z(s) = \ell$ for infinitely many s; moreover, there is an ℓ_0 with $z(s) = \ell_0$ for infinitely many s if and only if $\varphi_e^{0''}(0)\downarrow$, in which case $\varphi_e^{0''}(0) = g(\ell_0)$.*

Remark 5.2. We can construct z and g so that, also, for every ℓ, if $z(s) < \ell \leq s$ then $z(t) \neq \ell$ for all $t \geq s$. Thus, if $\varphi_e^{0''}(0) = g(\ell_0)$ then $\ell_0 \leq z(s)$ for all $s \geq \ell_0$.

Proof. Let $K = 0'$ and $Z = K' = 0''$. Fix a computable enumeration $\{k_0, k_1, \dots\}$ of K. We construct a computable approximation of K using finite strings $\{K_s : s \in \omega\}$: $K_s \in \{0,1\}^{k_s+1}$ satisfying $K_s(x) = 1$ if and only if $x = k_i \leq k_s$ for some $i \leq s$. We say that a stage t is a *true stage* if $\forall s > t(k_s > k_t)$; thus, t is a true stage if and only if $K_t = K \restriction k_t + 1$, and also if and only if $K_t \subseteq K_s$ for all s (where "$\subseteq$" means initial segment). Thus, $K = \bigcup_{t \text{ a true stage}} K_t$ (where the union is of partial functions into $\{0,1\}$).

Similarly, we construct a computable approximation $\{Z_s\}_{s\in\omega}$ of Z so that $Z = \lim_{t \text{ a true stage}} Z_t$: Given a string $\sigma \in \{0,1\}^{<\omega}$ and $x < |\sigma|$, we define

$$\sigma'(x) = \begin{cases} 1 & \text{if } \varphi^{\sigma}_{x,|\sigma|}(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

(where φ^{σ}_x is the xth Turing functional) and take $Z_s = K'_s$. If $Z(x) = 1$, then there exists a finite string $\sigma \subseteq K$ with $\sigma'(x) = 1$ and a true stage t with $\sigma \subseteq K_t$; in this case $Z_s(x) = 1$ for all $s \geq t$. If $Z(x) = 0$, then $K'_t(x) = 0$ for every true stage t with $x < |K'_t|$. Thus, $Z = \lim_{t \text{ a true stage}} K'_t$.

We now define g and z as follows. Let

$$g(s) = \varphi^{Z_s}_{e,s}(0),$$

and $g(s) = 0$ if this computation does not converge. If $\varphi^{Z_s}_e(0)\downarrow$, then let $\tilde{Z}_s$ (the use of this computation) be the shortest initial segment of Z_s such that $\varphi^{\tilde{Z}_s}_e(0)\downarrow$, and otherwise let $\tilde{Z}_s = Z_s$ (i.e. if $\varphi^{Z_s}_e(0)\uparrow$). Let $\tilde{K}_s$ be the shortest initial segment of K_s such that $\tilde{K}'_s \supseteq \tilde{Z}_s$. We define $z(s)$ to be the least $t \leq s$ such that $K_t \subseteq K_s$ and $\tilde{K}_t = \tilde{K}_s$. Note that $g(s) = g(z(s))$ because $\tilde{Z}_s = \tilde{Z}_{z(s)}$.

We show that g and z are as claimed. Suppose $z(s) = \ell_0$ for infinitely many s. Then ℓ_0 is a true stage, since $K_{\ell_0} \subseteq K_s$ for infinitely many s. We now claim that $\tilde{Z}_{\ell_0} \subseteq Z$. If not, then since ℓ_0 is a true stage, there is some x such that $\tilde{Z}_{\ell_0}(x) = 0$ and $Z(x) = 1$. In this case there would be another true stage $t > \ell_0$ with $Z_t(x) = 1$. But now for each $s \geq t$ we have $Z_s(x) = 1$, so that $\tilde{Z}_s \neq \tilde{Z}_{\ell_0}$. This contradicts that $z(s) = \ell_0$ for infinitely many s. It follows that $\varphi^{0''}_e(0)\downarrow = g(\ell_0)$.

Suppose $\varphi^{0''}_e(0)\downarrow$, and let t be the least true stage for which $\varphi^{Z_t}_e(0)\downarrow$ and $\tilde{Z}_t \subseteq Z$. It follows that for any true stage $s \geq t$ we must have $\tilde{Z}_t = \tilde{Z}_s$, and of course $K_t \subseteq K_s$, so that $z(s) = z(t)$. Therefore, if $\varphi^{0''}_e(0)\downarrow$ then there is some ℓ with $z(s) = \ell$ for infinitely many s (namely $\ell = z(t)$), and $g(\ell) = \varphi^{0''}_e(0)$.

Finally, we show that z satisfies the condition stated in Remark 5.2: that $z(s) < \ell \leq s$ implies that $z(t) \neq \ell$ for every $t \geq s$. Since $z(s) < s$, $\varphi^{Z_{z(s)}}_e(0)\downarrow$. If $K_\ell \not\subseteq K_s$, then $K_\ell \not\subseteq K_t$ for any $t \geq s$, so that $z(t) \neq \ell$ for all $t \geq s$. If $K_\ell \subseteq K_s$, then $K_{z(s)} \subseteq K_\ell \subseteq K_s$ and $\tilde{K}_{z(s)} = \tilde{K}_\ell = \tilde{K}_s$, so that $z(\ell) = z(s) < \ell$. Since for every t, $z(z(t)) = z(t)$, we can never have $z(t) = \ell$. □

5.2. **Infinite sums of Boolean algebras.** The second tool is ω-sums of Boolean algebras. Given a sequence of Boolean algebras $\{\mathcal{A}_i : i \in \omega\}$, we define

$$\mathcal{A} = \sum_{i\in\omega} \mathcal{A}_i$$

to be the Boolean algebra whose domain is the set of infinite sequences which are eventually constant and equal to either zero or one. In other words, the domain of $\mathcal{A}$ is

$$\mathcal{A} = \{\langle a_i : i \in \omega\rangle : a_i \in \mathcal{A}_i, (\exists n \forall k > n\ (a_k = 0_{\mathcal{A}_k})) \vee (\exists n \forall k > n\ (a_k = 1_{\mathcal{A}_k}))\}$$

and the operations are calculated coordinatewise. (See [Mon89, Ch. 4, §11] for more information on this operation, that they call the *free product*.)

Lemma 5.3. *Let $\{\mathcal{A}_i : i \in \omega\}$ be a Z-computable sequence of uniformly n-Z-approximable Boolean algebras. Let γ be a non-exclusive n-indecomposable type, and suppose that each $\mathcal{A}_i$ has type γ. Then $\sum_{i\in\omega} \mathcal{A}_i$ is also n-Z-approximable and has n-indecomposable type γ.*

Proof. For each $i \in \omega$, let $\{\mathcal{A}_i[s] : s \in \omega\}$ be an n-Z-approximation of $\mathcal{A}_i$. For notational convenience, assume that for every i, $\mathcal{A}_i[0]$ is the two-element Boolean algebra and $t_n^{\mathcal{A}_i[0]}(1_{\mathcal{A}_i}) = \gamma$. Define $\mathcal{A}[s] \subseteq \mathcal{A}$ as follows.

$$\mathcal{A}[s] = \{\langle a_0, a_1, \ldots\rangle : \forall i \leq s\ a_i \in \mathcal{A}_i[s-i]\ \&\ \big((\forall i \geq s\ a_i = 0_{\mathcal{A}_i}) \vee (\forall i \geq s\ a_i = 1_{\mathcal{A}_i})\big)\}.$$

In other words,

$$\mathcal{A}[s] \cong \mathcal{A}_0[s] \oplus \mathcal{A}_1[s-1] \oplus \ldots \oplus \mathcal{A}_{s-1}[1] \oplus \mathcal{A}_s[0].$$

Let $t_n^s(\langle a_0, a_1, \ldots\rangle) = \sum_{i\leq s} t_n^{\mathcal{A}_i[s-i]}(a_i)$. Note that $\mathcal{A}[s]$ is a finite n-labeled Boolean algebra and $\mathcal{A}[s] \leq_n \mathcal{A}[s+1]$. To see that $\{\mathcal{A}[s] : s \in \omega\}$ is an n-Z-approximation, we have to show that every minimal element a of $\mathcal{A}[s]$ splits in $\mathcal{A}[s+1]$ into a $t_n^{s+1}(a)$-full partition. Observe that if $a = \langle a_0, a_1, \ldots\rangle$ is a minimal element of $\mathcal{A}[s]$, then either, for some $i < s$, a_i is a minimal nonzero element of $\mathcal{A}[s-i]$ and $a_j = 0$ for $j \neq i$; or $a_i = 0$ for $i < s$ and $a_i = 1_{\mathcal{A}_i}$ for $i \geq s$. In the former case, we have that a_i splits in $\mathcal{A}_i[s-i+1]$ into a $t_n^{\mathcal{A}_i[s-i+1]}(a_i)$-full partition. In the latter case, we have that $1_{\mathcal{A}_s}$ splits in $\mathcal{A}_s[1]$ into a γ-full partition. In either case, a splits in $\mathcal{A}[s+1]$ into a $t_n^{s+1}(a)$-full partition. □

5.3. **Listing of all n-Z-approximable Boolean algebras.** In this subsection we show that there is a $0'$-computable enumeration of $(n+1)$-$0'$-approximations which lists all the Boolean algebras that have a computable n-approximation.

Lemma 5.4. *Suppose $s > 0$ and $\{(\mathcal{A}_k, t_n^k) : k < s\}$ is a sequence of finite n-labeled Boolean algebras that satisfy the conditions of Definition 3.4. Then there is an n-approximation $\{(\mathcal{A}_k, t_n^k) : k \in \omega\}$ extending $\{(\mathcal{A}_k, t_n^k) : k < s\}$.*

Proof. We show how to build $(\mathcal{A}_s, t_n^s)$. The rest of the sequence is then built in the same way, one step at a time. For each $a \in \mathcal{A}_{s-1}$, let $t_n^s(a) = t_n^{s-1}(a)$. What we need to do now is to build a full partition for each minimal element of $\mathcal{A}[s]$. For each minimal $a \in \mathcal{A}_{s-1}$, we add new disjoint elements $\{a_\gamma : \gamma \in t_n^s(a)\} \cup \{\hat{a}\}$ below a, and let $t_n^s(a_\gamma)$ be such that $(t_n^s(a_\gamma))_{n-1} = \gamma$ and $t_n^s(a) \lhd_n t_n^s(a_\gamma)$, and let $t_n^s(\hat{a}) = t_n^s(a)$. The existence of these $t_n^s(a_\gamma) \in \mathbf{BF}_n$ follows from the following fact: If $\mathcal{B}$ is a Boolean algebra of n-bf-type α that is n-indecomposable for ultrafilter $U \in \mathrm{Ult}(\mathcal{B})$, and $V \in \mathrm{Ult}(\mathcal{B})$ with $V \neq U$, then $t_n(\mathcal{B}) \lhd_n t_n(V)$ (see [HM, Definition 5.2], and the comment immediately thereafter); furthermore, for each $\gamma \in \alpha$ there must be some such $V \in \mathrm{Ult}(\mathcal{B})$ with $t_n(V) = \gamma$. Note that this is a $t_n^s(a)$-full partition of a. Thus, $\{(\mathcal{A}_k, t_n^k) : k \leq s\}$ also satisfies the conditions of Definition 3.4. □

Lemma 5.5. *There is a $0'$-computable list $\{\mathcal{C}_e : e \in \omega\}$ of (indices for) $(n+1)$-$0'$-approximable Boolean algebras such that for every n-0-approximable algebra $\mathcal{D}$, there is an index e with $\mathcal{D} = (\mathcal{C}_e)_n$. Furthermore, the index e can be found $0'$-uniformly from an index for an n-0-approximation of $\mathcal{D}$.*

Proof. Fix $e \in \omega$, the index of a purported n-0-approximation of $\mathcal{D}$. We write $\mathcal{D}[s]$ for $\varphi_e(s)$, a purported finite n-labeled Boolean algebra in the n-0-approximation $\{\mathcal{D}[s] :$

$s \in \omega\}$ of $\mathcal{D}$. Given t, if $\mathcal{D}[s]$ is defined for all $s < t$, then it is computable to check that the conditions of Definition 3.4 are satisfied for $\{\mathcal{D}[s] : s < t\}$ (which we will write as $\{\mathcal{D}[s]\}_{s<t}$).

First, check that the following condition is met: For every t and each stage u, if for each $k < t$, $\mathcal{D}[k]$ converges by stage u, then $\{\mathcal{D}[k]\}_{k<t}$ meets the conditions of Definition 3.4. This is a Π^0_1 condition, so it is computable in $0'$. If the condition does not hold, then φ_e does not give an n-approximation, and hence we need not worry about how to build $\mathcal{C}_e$. Just let $\mathcal{C}_e$ be any $(n+1)$-approximation. Suppose now that this condition holds.

The problem is that $0'$ cannot check whether φ_e is total. So, we will verify this one step at a time; while φ_e looks total, we define $\mathcal{C}_e$ by stages. If we ever discover that φ_e does not converge at some input, then we continue the construction of $\mathcal{C}_e$ using the previous lemma. We now carry out this construction.

To compute the predicates R_γ on $\mathcal{D}$, we use the following observation: For each $\gamma \in \mathbf{BF}_{n+1}$, let $\bar{I}_\gamma$ be the set of sequences $(\tau_0, \dots, \tau_k) \in \mathbf{BF}_n^{<\omega}$ for which there is a partition $\sum_{i \leq k} \rho_i$ of γ with $\tau_i \leq_n \rho_i$ for each i. (See [HM, Section 7.1] for background on partitions.) Note that $\bar{I}_\gamma$ is computable because there are only finitely many partitions of γ of size k. If $\mathcal{B}$ is any Boolean algebra and $a \in \mathcal{B}$, then

$$\mathrm{R}_\gamma(a) \iff \forall a_0 \dot{\vee} \dots \dot{\vee} a_k = a \;\Big[\bigwedge_{i=0}^{k} T_n(a_i) \in \mathbf{BF}_n \implies (T_n(a_0), \dots, T_n(a_k)) \in \bar{I}_\gamma\Big].$$

We will construct an $(n+1)$-$0'$-approximable Boolean algebra $\mathcal{C}_e$ (as in Definition 3.4) by defining the relations R_γ on $\mathcal{C}_e$ for each $\gamma \in \mathbf{BF}_{n+1}$. This approach suffices by Theorem 3.7. Our construction uses a $0'$ oracle. At stage s we will build a finite $(n+1)$-labeled Boolean algebra $\mathcal{C}_e[s]$ and then let $\mathcal{C}_e = \bigcup_s \mathcal{C}_e[s]$. We will now drop the subscript and write just $\mathcal{C}$ for $\mathcal{C}_e$.

At stage 0, if $\mathcal{D}[0]$ does not converge let $\mathcal{C}$ be any $(n+1)$-$0'$-approximable Boolean algebra.

At stage $s+1$, check whether $\mathcal{D}[s+1]$ converges. If it does, let $(\mathcal{C}[s+1])_n = \mathcal{D}[s+1]$. For each $\gamma \in \mathbf{BF}_{n+1}$, we extend the predicate R_γ to each $a \in \mathcal{C}[s+1] \setminus \mathcal{C}[s]$ as follows:

(*) Let $R_\gamma(a)$ if and only if for all $t \geq s$ and all stages u, if $\mathcal{D}[t]$ converges by stage u, then for all partitions $a_1 \dot{\vee} \dots \dot{\vee} a_k = a$ with $a_0, \dots, a_k \in \mathcal{D}[t]$ and $\bigwedge_{i=0}^{k} T_n(a_i) \in \mathbf{BF}_n$, we have that $(T_n(a_0), \dots, T_n(a_k)) \in \bar{I}_\gamma$.

Note that $(*)$ is a Π^0_1 condition. Also, note that $R_{(\gamma)_n}(a)$ holds in $\mathcal{D}[s+1]$. Otherwise, if $\mathcal{D}[s+1]$ does not converge, we have that φ_e is not total. Define $\mathcal{C}$ by extending the sequence $\{\mathcal{C}[t] : t \leq s\}$ constructed thus far using Lemma 5.4. □

6. No zero-triple proof

The following theorem is the heart of our construction of a low$_5$ Boolean algebra which is not isomorphic to a computable Boolean algebra via a $0^{(7)}$-computable isomorphism. We will use the relativization of this theorem to $0^{(4)}$ in the proof of Theorem 6.8.

Theorem 6.1. *There exists a* 5-$0'$-*approximable Boolean algebra which is not* $0'''$-*isomorphic to any* 4-0-*approximable Boolean algebra.*

Proof. Let $\{\mathcal{C}_i : i \in \omega\}$ be a $0'$-computable listing of 5-$0'$-approximations to Boolean algebras which includes all 4-0-approximable Boolean algebras, as in Lemma 5.5. We assume that the listing is such that there is a $0'$-computable function $h : \omega \times \omega \to \omega$ with the property that for each $c \in \mathcal{C}_i$ we have

$$\mathcal{C}_{h(i,c)} = \mathcal{C}_i \upharpoonright c.$$

Let $\{\Phi_e^X : e \in \omega\}$ be a computable listing of Turing functionals.

We are going to build a 5-$0'$-approximable Boolean algebra

$$\mathcal{A} = \sum_{e,i\in\omega} \mathcal{A}_{\langle e,i\rangle},$$

that, for each $e, i \in \omega$, satisfies the requirement:

$$R_{e,i} : \mathcal{A}_{\langle e,i\rangle} \text{ is not isomorphic to } \mathcal{C}_i \upharpoonright \Phi_e^{0'''}(1_{\mathcal{A}_{\langle e,i\rangle}}).$$

Satisfying these requirements suffices to establish the theorem. For suppose, toward a contradiction, that $\Phi_e^{0'''} : \mathcal{A} \to \mathcal{C}_i$ is an isomorphism; then we must have $\mathcal{A}_{\langle e,i\rangle}$ isomorphic to $\mathcal{C}_i \upharpoonright \Phi_e^{0'''}(1_{\mathcal{A}_{\langle e,i\rangle}})$, contradicting $R_{e,i}$.

Fix e and i. By the uniformity of Lemma 5.1, we can compute indices for two $0'$-computable functions

$$z_{e,i} : \omega \to \omega \quad \text{and} \quad g_{e,i} : \omega \to \omega$$

such that

$$g_{e,i}(\ell_0) = h(i, \Phi_e^{0'''}(1_{\mathcal{A}_{\langle e,i\rangle}})),$$

where $\ell_0 \in \omega$ is the only number satisfying $z_{e,i}(s) = \ell_0$ for infinitely many s. If $\Phi_e^{0'''}(1_{\mathcal{A}_{\langle e,i\rangle}}) \uparrow$, then no ℓ occurs infinitely often.

The remainder of the proof is the construction of the restricted algebra $\mathcal{A}_{\langle e,i\rangle}$ using the two $0'$-computable functions $z_{e,i}$ and $g_{e,i}$ to guess the restricted algebra $\mathcal{C}_{h(i,\Phi_e^{0'''}(1_{\mathcal{A}_{\langle e,i\rangle}}))}$. (If $\mathcal{C}_i$ is not the image of $\Phi_e^{0'''}$, then the construction is moot.) We will drop reference to the subscripts e and i in what follows and just write z, g, $\mathcal{A}$, and $\mathcal{C}$. We abbreviate $\mathcal{C}_{h(i,g(\ell))}$ as $\mathcal{B}_\ell$. For simplicity, we will also drop the $0'$ oracle, as we can relativize the proof later. Thus, our goal now is to prove the following proposition.

Proposition 6.2. *Let $\{\mathcal{B}_\ell : \ell \in \omega\}$ be a computable sequence of 5-0-approximable Boolean algebras, and let $z : \omega \to \omega$ be a total computable function with the properties of the one in Lemma 5.1. We can build, uniformly in z and $\{\mathcal{B}_\ell : \ell \in \omega\}$, a 5-0-approximable Boolean algebra $\mathcal{A}$ such that: if there is a number ℓ_0 with the property that $z(s) = \ell_0$ for infinitely many s, then $\mathcal{A}$ is not isomorphic to $\mathcal{B}_{\ell_0}$.*

Let $\mathcal{C} = \mathcal{B}_{\ell_0}$. We will build $\mathcal{A}$ as a uniform sum of 5-0-approximable Boolean algebras

$$\mathcal{A} = \sum_{\ell\in\omega} \mathcal{A}_\ell,$$

where $\mathcal{A}_\ell$ is a subalgebra we build on the guess that $\ell = \ell_0$ and $\mathcal{C} = \mathcal{B}_\ell$.

We construct each $\mathcal{A}_\ell$ to be 5-0-approximable in stages s, so that $\{\mathcal{A}_\ell[s] : s \in \omega\}$ is computable uniformly in ℓ. These approximations will all satisfy properties (D1)-(D4) from Section 4 so that each $\mathcal{A}_\ell$ will have 5-indecomposable type w^∞; it will then follow from Lemma 5.3 that $\mathcal{A}$ is 5-0-approximable with 5-indecomposable type w^∞. Let ℓ_0

be the unique value (if it exists) such that for infinitely many s, $z(s) = \ell_0$. (If ℓ_0 does not exist, think of ℓ_0 as ∞.)

(B1) If $\ell < \ell_0$, then $\mathcal{A}_\ell$ will have infinitely many ultrafilters of type w^∞, and finitely many of type w or u.

(B2) If $\ell = \ell_0$, then $\mathcal{A}_{\ell_0}$ will have either one ultrafilter of type w^∞ and infinitely many of type w, or infinitely many of type w^∞ and finitely many of type w. In either case it will have no ultrafilters of type u.

(B3) If $\ell_0 < \ell$, then $\mathcal{A}_\ell$ will have infinitely many ultrafilters of type w^∞, none of type w and finitely many of type u.

In the table below we list $num(\gamma, \mathcal{A}_\ell)$ for $\gamma = u, w, w^\infty$. The symbol `f` means "finitely many".

	u	w	w^∞
$\ell < \ell_0$	`f`	`f`	∞
$\ell = \ell_0$	0	∞	1
		`f`	∞
$\ell > \ell_0$	`f`	0	∞

Thus, in total, $\mathcal{A}$ will contain either exactly one 1-w-atom and hence infinitely many w ultrafilters, or no 1-w-atom and only finitely many w ultrafilters. In either case it will contain infinitely many u ultrafilters and infinitely many w^∞ ultrafilters.

We will assign a worker G_ℓ to construct the algebra $\mathcal{A}_\ell$ using a strategy which will ensure that the following condition is met:

$$G_\ell : (\exists^\infty s)\, z(s) = \ell \implies \mathcal{A} \text{ is not isomorphic to } \mathcal{B}_\ell.$$

Of course, only G_{ℓ_0} has a real responsibility.

The goal of G_ℓ is achieve one of the following conditions.

(S1) $\mathcal{A}$ has finitely many w ultrafilters, and there are more w ultrafilters in $\mathcal{B}_\ell$ than there are in $\mathcal{A}$.

(S2) $\mathcal{A}_\ell$ is a 1-w-atom and, for every $b \in \mathcal{B}_\ell$, $\mathcal{B}_\ell \restriction b$ is not a 1-w-atom because one of the following applies:
  (a) $t_5^{\mathcal{B}_\ell}(b) \neq w^\infty$;
  (b) b bounds a u ultrafilter in $\mathcal{B}_\ell$;
  (c) b splits into two elements, each of which bounds infinitely many α ultrafilters.

We describe G_ℓ's strategy. First, if $\mathcal{B}_\ell$ does not have 5-bf-type w^∞, then we have nothing to do, as $\mathcal{A}$ will have 5-bf-type w^∞. So suppose it does, and hence from some stage onward $1_{\mathcal{B}_\ell}$ has type w^∞; this is the moment when we start working for G_ℓ. We will attempt to satisfy (S1) as follows: If we see some $b \in \mathcal{B}_\ell$ such that $\mathcal{B}_\ell \restriction b$ has more w elements than $\mathcal{A}$ has so far, then we will restrain the overall production of w elements in $\mathcal{A}$. If all these w elements below b stay with 5-bf-type w forever, we will end up satisfying (S1). However, some of these elements may increase their 5-bf-type to u. If this happens, then b satisfies (S2b), and we can take one step toward making $\mathcal{A}_\ell$ a 1-w-atom. Taking this step implies increasing the 5-bf-type of all but one of the w^∞ minimal elements in $\mathcal{A}_\ell[s]$ to w in $\mathcal{A}_\ell[s+1]$. Notice that this may injure the restraint that other elements b are imposing on the production of w elements, so we will have to order the elements $b \in \mathcal{B}_\ell$ according to some order of priorities. We will argue that if we never manage to satisfy (S1), then we will satisfy (S2), as follows. For every w^∞ element b of $\mathcal{B}_\ell$, there are three possibilities: (i) b splits into two or more w^∞ elements, (ii) there is some u element below b, and (iii) there are too many w elements below b. Hence, we would win by (S2c), (S2b), or (S1), respectively. As long as there are no u elements and too few w elements below b, we believe (S2c) will hold. At stages where that occurs we will wait until a large number of α elements appear in $\mathcal{B}_\ell \restriction b$. The reason is that if (S2c) actually does hold, then all these α elements will have to turn into either u or w elements. If we see any u element below b, then (S2b) holds. Otherwise, we will see a lot of w elements at once - enough to act towards (S1), as mentioned above.

We now describe the construction of the sequence $\{(\mathcal{A}_\ell[s], t_5^{\ell[s]}) : s \in \omega\}$. The construction will be computable uniformly in ℓ. At each stage s, $\mathcal{A}_\ell[s]$ will contain at least one minimal element of 5-indecomposable type w^∞; the other minimal elements which have 4-indecomposable type α will have 5-indecomposable type u, w, or w^∞. The sequence $\{(\mathcal{A}_\ell[s], t_5^{\ell[s]}) : s \in \omega\}$ will satisfy conditions (D1)-(D4) from Section 4.

We write $\mathcal{A}_{\leq \ell}$ to denote $\sum_{i=0}^{l} \mathcal{A}_i$. For each stage s, we will define $r_s \in \omega$, which will tell us how far to look into the approximation of the $\mathcal{B}_\ell$.

Construction of $\mathcal{A}_\ell$:

Stage 0. For every ℓ, let $A_\ell[0]$ be the Boolean algebra with two elements $\{0, 1\}$, where $t_5^{\ell[0]}(1) = w^\infty$. We set $r_0 = 0$.

Stage $s+1$. There are two steps. In Step 1, the construction makes modifications only to the 5-bf-types of elements in $\mathcal{A}_\ell[s]$ (which will change only by increase, if at all). Step 2 is where each subalgebra is extended to $\mathcal{A}_\ell[s+1]$ according to the canonical partitions discussed in Section 4.

Step 1. Each worker G_ℓ is in one of three states: *cancelled*, *inactive*, or *active*; all workers are *inactive* at stage 0.

G_ℓ is cancelled at stage $s+1$. This happens when $z(s) < \ell < s$ and s is the first stage after stage ℓ at which $z(s) < \ell$. By Remark 5.2, we will never have $z(s) = \ell$ after this stage, so we need not worry about G_ℓ anymore. If there are any minimal w elements in $\mathcal{A}_\ell[s]$, increase their 5-bf-type to u. Leave the 5-bf-types of all the other minimal elements unchanged. Define $r_{s+1} = r_s + 1$ and proceed to Step 2.

G_ℓ is inactive at stage $s+1$. This happens when either $\ell \neq z(s)$ or $t_5^{\mathcal{B}_\ell[r_s]}(1) \neq w^\infty$, unless G_ℓ has been cancelled at this stage. No change is made to the values of $t_5^{\ell[s]}$. Define $r_{s+1} = r_s + 1$ and proceed to Step 2.

G_ℓ is active at stage $s+1$. This happens when $z(s) = \ell$. We will call $s+1$ an *ℓ-stage*. There are two possible strategies for active G_ℓ: *restrained* or *unrestrained*.

- *restrained*: Restrain the production of w elements at this stage because condition (S1) currently holds.

- *unrestrained*: It looks as though condition (S2) may hold at the end, so take one step toward building a 1-w-atom.

There are two phases: the *strategy phase*, where the strategy is determined, and the *action phase*, where any actions modifying the types of elements occur.

Strategy phase. The first step is to look far enough into the approximation of $\mathcal{B}_\ell$, or in other words, to define r_{s+1}: For each minimal α element $b \in \mathcal{B}_\ell[r_s]$, we search for a stage $r^b \geq r_s$ in the 5-$\emptyset$-approximation of $\mathcal{B}_\ell$ at which one of the following conditions holds.

(W1) All minimal α elements in $\mathcal{B}_\ell[r^b] \restriction b$ have exclusive 5-indecomposable types.
(W2) $num(\alpha, \mathcal{B}_\ell[r^b] \restriction b) \geq 2 \cdot num(\alpha, \mathcal{A}[s]) + 2$ (i.e. there are many more α elements in $\mathcal{B}_\ell[r^b]$ below b than there are currently in $\mathcal{A}[s]$).
(W3) $t_5^{\mathcal{B}_\ell[r^b]}(1) >_5 w^\infty$ or some minimal element $a \in \mathcal{B}_\ell[r^b]$ has 4-bf-type not in $\{(\sigma)_4 : \sigma \in D\}$.

Once this task has been completed for each minimal element b of $B_\ell[r_s]$, set r_{s+1} to the maximum of all the r^b and $r_s + 1$.

If we find a stage at which (W3) holds, then we know that $\mathcal{A}$ cannot be isomorphic to $\mathcal{B}_\ell$, and so we restrain G_ℓ at every stage at which it is active. In what follows we suppose that (W3) does not hold.

We say that $b \in B_\ell[r_{s+1}]$ *requires attention* at stage $s + 1$ if

$$\begin{aligned} num(w, \mathcal{B}_\ell[r_{s+1}] \restriction b) &> num(w, \mathcal{A}_{\leq \ell}[s]), \qquad \text{and} \\ num(u, \mathcal{B}_\ell[r_{s+1}] \restriction b) &= 0. \end{aligned}$$

If no $b \in \mathcal{B}_\ell[r_{s+1}]$ requires attention at this stage, then declare the strategy of G_ℓ to be *unrestrained*, and move on to the action phase.

Otherwise, let b_0 be the $\leq_\mathbb{N}$-least $b \in \mathcal{B}_\ell[r_{s+1}]$ (i.e., the one of highest priority) that requires attention at this stage. We declare the strategy of G_ℓ to be *restrained by* b_0, unless the previous ℓ-stage was restrained by c for some $c <_\mathbb{N} b_0$, in which case we declare the strategy of G_ℓ to be *unrestrained*. (Notice that in this latter case, c has stopped requiring attention at $s + 1$, as b_0 is the $\leq_\mathbb{N}$-least element that requires attention. The idea here is that we want G_ℓ to have at least one unrestrained stage before being restrained by some element of lower priority.)

Action phase. If G_ℓ has the *unrestrained* strategy, then increase the 5-type of every minimal w^∞ element of $\mathcal{A}_\ell[s]$ to w, except for one element which will remain with 5-type w^∞. (In doing this, we are taking one step towards building a 1-w-atom.)

Otherwise, if G_ℓ has a *restrained* strategy, do not change the 5-type of any element of $\mathcal{A}_\ell[s]$. (In doing this, we avoid building a 1-w-atom and add no new w elements.)

Step 2. We have already defined $t_5^{\ell[s+1]}$ on all the elements of $\mathcal{A}_\ell[s]$. Extend $\mathcal{A}_\ell[s]$ to $\mathcal{A}_\ell[s+1]$ by splitting each minimal element a of $\mathcal{A}_\ell[s]$ into a canonical $t_5^{\ell[s+1]}(a)$-full partition as described in Section 4. $\diamondsuit$

6.1. **Verifications.** We begin by proving that the construction does not get stuck in the *strategy* phase while waiting for one of the conditions (W1), (W2), (W3) to hold.

Lemma 6.3. *Let $s + 1$ be a stage at which G_ℓ is active. For each minimal element $b \in \mathcal{B}_\ell[r_s]$, there is some stage $r^b \geq r_s$ at which one of the conditions (W1)-(W3) is satisfied.*

Proof. Fix a minimal element b in $\mathcal{B}_\ell[r_s]$. Note that it is decidable during the construction whether or not each condition (W1)-(W3) holds for a given stage r^b. Suppose (W3) never occurs. We claim that if there are only finitely many α elements in $\mathcal{B}_\ell \upharpoonright b$, they all have exclusive 5-bf-type. The reason is that if $a \in \mathcal{B}_\ell \upharpoonright b$ has 4-bf-type α but not exclusive 5-bf-type, then $\alpha \in \mathrm{dc}\, t_5^{\mathcal{B}_\ell}(a)$. So, there are infinitely many ultrafilters in $\mathcal{B}_\ell \upharpoonright a$ with 4-bf-type $\geq_4 \alpha$. But $\alpha = e_6$, which is the only 4-bf-type in $\{(\sigma)_4 : \sigma \in D\} = \{e_0, e_1, e_2, e_8, e_6\}$ that is $\geq_4 \alpha$ (see [HM, Section 6]). Thus, if $\mathcal{B}_\ell \upharpoonright b$ bounds only finitely many α elements, they must each have exclusive 5-bf-types. So, either $num(\alpha, \mathcal{B}_\ell[r^b] \upharpoonright b) \geq 2 \cdot num(\alpha, \mathcal{A}[s]) + 2$ and for r^b sufficiently large we will see that (W2) holds; or, we will find a stage $r^b \geq r_s$ at which all the α elements in $\mathcal{B}_\ell \upharpoonright b$ have exclusive 5-bf-type and (W1) holds. □

We have shown that the construction always outputs a Boolean algebra $\mathcal{A}$, independently of the specifics of z and $\{\mathcal{B}_\ell : \ell \in \omega\}$. We now show that if ℓ_0 does exist, then $\mathcal{A} \not\cong \mathcal{B}_{\ell_0}$. This is necessarily true if ($W3$) holds at any stage; so, in what follows, we assume that this condition never holds.

Lemma 6.4. *The algebras $\mathcal{A}_\ell$ have type w^∞ and satisfy conditions (B1)-(B3).*

Proof. For all s and ℓ, $t_5^{\ell[s]}(1) = w^\infty$, so each $\mathcal{A}_\ell$ has type w^∞.

Suppose first that $\ell < \ell_0$. Then for all $s \geq \ell_0$, $z(s) > \ell$, hence G_ℓ is always inactive after stage ℓ_0. Thus, the types of elements of $\mathcal{A}_\ell$ no longer change after stage ℓ_0; so $\mathcal{A}_\ell$ will have densely many w^∞ elements and just as many w or u elements as it had by stage ℓ_0. Therefore, (B1) holds.

Suppose now that $\ell_0 < \ell$. Then G_ℓ will be cancelled at some stage $s+1$, and hence $\mathcal{A}_\ell[s+1]$ has some w^∞ elements, some u elements and no w elements. By remark 5.2, G_ℓ will never be active after this stage, and hence it will always be inactive. $\mathcal{A}_\ell$ will end up having densely many w^∞ elements, just as many u elements as it had at stage $s+1$, and no w elements. Thus, (B3) holds.

Suppose finally that $\ell = \ell_0$. Then there are infinitely many stages at which $z(s) = \ell_0$, and for every $s \geq \ell_0$, $z(s) \geq \ell_0$. Thus, G_ℓ is never cancelled, so it has no u-elements (which can be introduced only at a stage where G_ℓ is cancelled), and G_ℓ is active infinitely often. Since the 5-bf-types do not change when G_ℓ has the *restrained* strategy at ℓ-stages, if G_ℓ has the *restrained* strategy at cofinitely many ℓ-stages, then the 5-bf-types of $\mathcal{A}_\ell$ will also not change at cofinitely many ℓ-stages. In this case $\mathcal{A}_\ell$ will contain only finitely many elements of type w and densely many elements of type w^∞. If G_ℓ has the *unrestrained* strategy at infinitely many ℓ-stages, $\mathcal{A}_\ell$ will have exactly one element of type w^∞, because at each such stage there will be only one minimal element of $\mathcal{A}_\ell[s]$ left in $\mathcal{A}_\ell[s+1]$ with type w^∞, as in Lemma 4.4. □

The following lemma shows how the number of w-elements in $\mathcal{A}[s]$ fluctuates. Keep in mind that for all s and ℓ,

$$num(w, \mathcal{A}[s]) = num(w, \mathcal{A}_{<\ell}[s]) + num(w, \mathcal{A}_\ell[s]) + num(w, \mathcal{A}_{>\ell}[s]).$$

Lemma 6.5. *Let $s_1 + 1$ be an ℓ-stage and let $s_0 + 1$ be the previous ℓ-stage. Then*

$$\begin{aligned} num(w, \mathcal{A}_{<\ell}[s_1+1]) &= num(w, \mathcal{A}_{<\ell}[s_0+1]) = num(w, \mathcal{A}_{<\ell}[s_1]), \\ num(w, \mathcal{A}_{>\ell}[s_1+1]) &= num(w, \mathcal{A}_{>\ell}[s_0+1]) = 0, \\ num(w, \mathcal{A}_\ell[s_1+1]) &\leq num(\alpha, \mathcal{A}_\ell[s_1]). \end{aligned}$$

Thus, $num(w, \mathcal{A}[s_1+1]) \leq num(\alpha, \mathcal{A}[s_1])$. Furthermore, if G_ℓ has a restrained strategy at s_1+1, then

$$num(w, \mathcal{A}[s_1+1]) \quad = \quad num(w, \mathcal{A}[s_0+1]) \quad = \quad num(w, \mathcal{A}_{\leq \ell}[s_1]).$$

Proof. Given that s_1+1 is an ℓ-stage and s_0+1 is the previous ℓ-stage, Remark 5.2 implies that for every stage s with $s_0 < s \leq s_1+1$, $z(s) \geq \ell$. Hence, every $\ell_1 < \ell$ is inactive during these intermediate stages. Thus, $num(w, \mathcal{A}_{<\ell}[s_0+1]) = num(w, \mathcal{A}_{<\ell}[s_1+1]) = num(w, \mathcal{A}_{<\ell}[s_1])$.

Each $\ell_1 > \ell$ which is active before stage s_0, and has not yet been cancelled, will be cancelled at s_0+1. Therefore, $num(w, \mathcal{A}_{>\ell}[s_0+1]) = 0$. And similarly, $num(w, \mathcal{A}_{>\ell}[s_1+1]) = 0$.

The third inequality follows from the fact that every minimal element in $\mathcal{A}_\ell[s_1]$ of type α has, in $\mathcal{A}_\ell[s_1+1]$, at most one minimal element of type w below.

If G_ℓ has a restrained strategy at s_1+1, then $num(w, \mathcal{A}_{\leq \ell}[s_1]) = num(w, \mathcal{A}_{\leq \ell}[s_1+1])$. Since ℓ is not active between stages s_0+2 and s_1, it follows that $num(w, \mathcal{A}_{\leq \ell}[s_0+1]) = num(w, \mathcal{A}_{\leq \ell}[s_1])$. So, $num(w, \mathcal{A}[s_1+1]) = num(w, \mathcal{A}[s_0+1]) = num(w, \mathcal{A}_{\leq \ell}[s_1])$. □

Observation 6.6. The 5-approximation of $\mathcal{B}_\ell$ need not respect the properties (D1)-(D4) from Section 4. However, we can make the following observations:

Since the 4-bf-types do not change in any 5-approximation, we have that, for every $b \in \mathcal{B}$, $num(\alpha, \mathcal{B}[t] \restriction b)$ is non-decreasing on t.

Suppose that $b \in \mathcal{B}[t]$ and that $c_0, ..., c_k$ are the minimal elements which are below b in $\mathcal{B}[t']$ for some $t' \geq t$. Using [HM, Lemma 7.12] we get that for some i_0, $t_5^{\mathcal{B}[t]}(b) \leq t_5^{\mathcal{B}[t']}(c_{i_0})$ and that, for all other i, $t_5^{\mathcal{B}[t]}(b) \triangleleft_5 t_5^{\mathcal{B}[t']}(c_i)$. Therefore, if b has type w^∞ in $\mathcal{B}[t]$, and c_i has 4-bf-type α and exclusive 5-bf-type, then c_i has 5-bf-type either u or w. Also, if b has type w, then there is exactly one i with $t_4^{\mathcal{B}[t']}(c_i) = \alpha$, and $t_5^{\mathcal{B}[t']}(c_i)$ is either u or w.

Lemma 6.7. *One of the following holds.*

1. *There exist a $b_0 \in \mathcal{B}_{\ell_0}$ and some s_0 such that for every ℓ_0-stage $s \geq s_0$, G_{ℓ_0} is restrained by b_0. In this case, we win by (S1).*
2. *There is no such b_0, and we win by (S2).*

Proof. Suppose first that there exists a b_0 as in (1). Then $num(w, \mathcal{B}_\ell[r_{s_0+1}] \restriction b_0) > num(w, \mathcal{A}_{\leq \ell_0}[s_0])$. By Lemma 6.5 the number of w elements does not change between successive ℓ-stages, and since G_{ℓ_0} is restrained b_0 at all ℓ-stages $\geq s_0$, no new w elements are added at ℓ_0-stages. So $num(w, \mathcal{A}_{\leq \ell_0}[s_0]) = num(w, \mathcal{A})$. Since b_0 requires attention at every ℓ_0-stage $s \geq s_0$, we have that $num(u, \mathcal{B}_\ell[r_s] \restriction b_0) = 0$ at all these stages. This implies that no w element in $\mathcal{B}_\ell[r_{s_0+1}] \restriction b$ can change its 5-bf-type, because, by Observation 6.6, some new u element would show up. Hence $num(w, \mathcal{B}_\ell \restriction b_0) \geq num(w, \mathcal{B}_\ell[r_{s_0+1}] \restriction b_0)$. We conclude that

$$num(w, \mathcal{B}_\ell) \geq num(w, \mathcal{B}_\ell[r_{s_0+1}] \restriction b_0) > num(w, \mathcal{A}_{\leq \ell_0}[s_0]) = num(w, \mathcal{A}),$$

and thus (S1).

Suppose now there is no such b_0. We first prove that there are infinitely many unrestrained ℓ_0-stages, and hence that $\mathcal{A}_{\ell_0}$ is a 1-w-atom. Note that if s_0 and s_1 are consecutive ℓ_0-stages at which G_{ℓ_0} is restrained - (say) by c_0 and c_1 respectively - then necessarily $c_0 \geq_{\mathbb{N}} c_1$ (because if c_0 is less than the least c that requires attention at stage

s_1, namely, c_1, then G_{ℓ_0} would have an unrestrained strategy at s_1). Therefore, if G_{ℓ_0} is restrained from some ℓ_0-stage on, then from some point on it will be restrained by the same c. But we are assuming that (1) does not hold. This implies that there are infinitely many unrestrained ℓ_0-stages. Then, by Lemma 4.4, $\mathcal{A}_{\ell_0}$ is a 1-w-atom.

Next, we prove that there is no $b \in \mathcal{B}_{\ell_0}$ such that $\mathcal{B}_{\ell_0} \restriction b$ is a 1-w-atom. At the same time, by induction on $b \in \mathbb{N}$, we show that for each b there is a stage after which G_{ℓ_0} is never restrained by b. Suppose this is the case for all $c <_{\mathbb{N}} b$. Let s_0 be an ℓ_0-stage after which G_{ℓ_0} is never restrained by any $c <_{\mathbb{N}} b$. If G_{ℓ_0} is ever restrained by b at some ℓ_0-stage $s \geq s_0$, then at some stage $s_1 > s$ it has to stop requiring attention since we assuming (1) does not hold. This means that either $num(w, \mathcal{B}_\ell[r_{s_1}] \restriction b) \leq num(w, \mathcal{A}_{\leq \ell}[s_1 - 1])$ or $num(u, \mathcal{B}_\ell[r_{s_1}] \restriction b) > 0$. In the former case, we have that $num(w, \mathcal{A}_{\leq \ell}[s_1]) = num(w, \mathcal{A}_{\leq \ell}[s])$, by using Lemma 6.5 at the ℓ_0-stages between s and s_1 So, $num(w, \mathcal{B}_\ell[r_s] \restriction b) > num(w, \mathcal{B}_\ell[r_{s_1}] \restriction b)$. But the only way the number of w elements can decrease is if some of them increase their 5-bf-type and new u elements appear. Therefore, in either case, some u element had to appear below b at stage s_1. Since the 5-bf-type of u elements never changes, b will never again require attention, and b is not a 1-w-atom. It follows that there is a stage after which G_{ℓ_0} is never again restrained by b, and b never again requires attention. We now show that b cannot be a 1-w-atom.

Let s_1 be an ℓ_0-stage beyond which b never again restrains G_{ℓ_0}. We may as well assume b has 5-bf-type w^∞ and does not bound any u ultrafilters, otherwise b is definitely not a 1-w-atom. At stage s_1 there is some minimal w^∞ element $b_1 < b$ for which r^{b_1} is defined by (W2). It follows that $num(\alpha, \mathcal{B}_\ell[r_{s_1}] \restriction b) \geq 2 \cdot num(\alpha, \mathcal{A}[s_1 - 1]) + 2$. Let $d \in \mathcal{B}_\ell[s_1] \restriction b$ be such that

$$\begin{aligned} num(\alpha, \mathcal{B}_\ell[r_{s_1}] \restriction d) &> num(\alpha, \mathcal{A}[s_1 - 1]) \quad \text{and} \\ num(\alpha, \mathcal{B}_\ell[r_{s_1}] \restriction b - d) &> num(\alpha, \mathcal{A}[s_1 - 1]). \end{aligned}$$

If both d and $b - d$ have type w^∞, then $\mathcal{B}_\ell \restriction b$ cannot be a 1-w-atom and we win by (S2c). Suppose that this is not the case, and that (say) d does not have 5-bf-type w^∞. Since d must have a 5-bf-type in $\langle D \rangle$, it follows that d bounds finitely many α ultrafilters and these must all be w ultrafilters (since b bounds no u ultrafilters). Let $s_3 > s_1$ be the first ℓ_0-stage at which all the minimal α elements below d have 5-bf-type w. We will show that b requires attention at s_3, in contradiction to our assumption that this never happens beyond s_1. At each ℓ_0-stage s with $s_1 \leq s < s_3$ we have

$$num(\alpha, \mathcal{A}[s - 1]) < num(\alpha, \mathcal{B}_\ell[r_s] \restriction d).$$

This is true at $s = s_1$ by hypothesis. If $s > s_1$, then we still believe d has type w^∞ and hence there is some minimal w^∞ element d_1 such that r^{d_1} is defined by (W2), and so

$$num(\alpha, \mathcal{A}[s - 1]) < num(\alpha, \mathcal{B}_\ell[r_s] \restriction d_1) < num(\alpha, \mathcal{B}_\ell[r_s] \restriction d).$$

Let s_2 be the last ℓ_0-stage before s_3. Then

$$num(\alpha, \mathcal{A}[s_2 - 1]) < num(\alpha, \mathcal{B}_\ell[r_{s_2}] \restriction d) \leq num(\alpha, \mathcal{B}_\ell[r_{s_3}] \restriction d).$$

By Lemma 6.5,

$$num(w, \mathcal{A}_{\leq \ell_0}[s_3 - 1]) = num(w, \mathcal{A}[s_2]) \leq num(\alpha, \mathcal{A}[s_2 - 1]).$$

Hence $num(w, \mathcal{A}_{\leq \ell_0}[s_3-1]) < num(w, \mathcal{B}_\ell[r_{s_3}] \upharpoonright b)$ and b requires attention at stage s_3 in contradiction to our assumption that b never requires attention after s_1. □

This proves that G_{ℓ_0} is satisfied and hence finishes the proof of Proposition 6.2 and Theorem 6.1. □

6.2. **Main theorem.** We now come to the main result in this paper.

Theorem 6.8. *There is a low$_5$ Boolean algebra that is not $0^{(7)}$-isomorphic to any computable Boolean algebra.*

Proof. We relativize Theorem 6.1 to $0^{(4)}$: There exists a 5-$0^{(5)}$-approximable Boolean algebra $\mathcal{A}$ which is not $0^{(7)}$-isomorphic to any 4-$0^{(4)}$-approximable algebra. Recall that every computable presentation of a Boolean algebra is 4-$0^{(4)}$-approximable, hence $\mathcal{A}$ is not $0^{(7)}$-isomorphic to any computable Boolean algebra. By Theorem 2.11, there is a low$_5$ copy $\mathcal{B}$ of $\mathcal{A}$ via an isomorphism that is computable in $0^{(5)}$. If $\mathcal{B}$ were $0^{(7)}$-isomorphic to a computable Boolean algebra, then $\mathcal{A}$ would be too. □

Remark 6.9. The Boolean algebra $\mathcal{A}$ constructed in the previous proof is isomorphic to a computable one. The reason is that the Boolean algebra $\mathcal{A}$ (that is $\mathcal{A}_{\langle e,i\rangle}$) constructed in Proposition 6.2 can be shown to be isomorphic to either

$$\bigoplus_{i=1}^{k} \mathcal{A}_w \oplus \sum_{i=1}^{\omega} (\mathcal{A}_u \oplus \mathcal{A}_{w^\infty}) \quad \text{or} \quad \mathcal{A}_{1w} \oplus \sum_{i=1}^{\omega} (\mathcal{A}_u \oplus \mathcal{A}_{w^\infty}),$$

when ℓ_0 exists, and to

$$\sum_{i=1}^{\omega} (\mathcal{A}_u \oplus \mathcal{A}_w \oplus \mathcal{A}_{w^\infty})$$

when it does not, where $\mathcal{A}_u$ is the unique isomorphism type of 5-bf-type u, namely, $\mathrm{Int}(\omega^2+\eta)$; $\mathcal{A}_w$ is the unique isomorphism type of 5-bf-type w which satisfies properties (D1)-(D4) of Section 4, namely, the interval algebra $\mathrm{Int}(\omega^3+\eta)$; $\mathcal{A}_{w^\infty}$ is the isomorphism type of the Boolean algebra of 5-bf-type w^∞, which satisfies (D1)-(D4) and has the property that the 5-bf-types never increase, namely, $\mathrm{Int}((\omega^2+1+\eta)\cdot\eta)$; and $\mathcal{A}_{1w}$ is the 1-w-atom which satisfies (D1)-(D4), namely, $\mathrm{Int}((\omega^3+\eta)\cdot\omega)$. All these algebras are computably presentable.

Furthermore, $0^{(7)}$ can go through the construction in the proof of Proposition 6.2 and decide how the requirements are satisfied (recall that the proof given there is later relativized to $0^{(5)}$), so $0^{(7)}$ can find an isomorphism between $\mathcal{A}_{\langle e,i\rangle}$ and one of these computable algebras. However, $0^{(7)}$ does not know if ℓ_0 exists, so it cannot uniformly compute these isomorphisms. But $0^{(8)}$ can. One can then show that the relativization of the Boolean algebra $\mathcal{A} = \sum_{e,i\in\omega} \mathcal{A}_{\langle e,i\rangle}$ constructed in the proof of Theorem 6.1 is $0^{(8)}$-isomorphic to the following computable Boolean algebra:

$$\sum_{j=1}^{\omega} \Big(\mathcal{A}_w \oplus \mathcal{A}_{1w} \oplus \sum_{i=1}^{\omega} (\mathcal{A}_u \oplus \mathcal{A}_{w^\infty}) \oplus \sum_{i=1}^{\omega} (\mathcal{A}_u \oplus \mathcal{A}_w \oplus \mathcal{A}_{w^\infty})\Big).$$

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