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COMPUTABLE LINEARIZATIONS OF WELL-PARTIAL-ORDERINGS

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Abstract. We analyze results on well-partial-orderings from the viewpoint of computability theory, and we answer a question posed by Diana Schmidt. We obtain the following results. De Jongh and Parikh showed that every well-partial-order has a linearization of maximal order type. We show that such a linearization can be found computably. We also show that the process of finding such a linearization is not computably uniform, not even hyperarithmetically.

1. Introduction

We are interested in the following kind of orderings.

Definition 1.1. A *well-quasi-ordering*, or *wqo*, is quasi-ordering which has no infinite strictly descending sequences and no infinite antichains.

This notion has been discovered independently many times, as it is a concept that appears in several areas of mathematics and computer science, and has many equivalent definitions (see [Kru72]). For instance, given a quasi-ordering $\mathcal{Q} = (Q, \leq_Q)$,

$$\mathcal{Q} \text{ is a wqo} \iff \text{for every sequence } \{x_n : n \in \omega\} \subseteq Q,\ \exists i < j\ (x_i \leq_Q x_j).$$

This equivalence can be proven using Ramsey's theorem. Some well known examples of wqo's are as follows: the set of finite strings over a finite alphabet (Higman's theorem [Hig52]), the set of finite trees (Kruskal's theorem [Kru60]), the set of labeled transfinite sequences with finite labels (Nash-Williams [NW65]), the set of scattered linear orderings (Laver's theorem [Lav71], also known as Fraïssé's conjecture), and the set of finite graphs (Robertson and Seymour [RS04]). The ordering in all these examples is some kind of embeddability relation.

If $\mathcal{Q} = (Q, \leq_Q)$ is a quasi-ordering, we consider the partial-ordering associated to it in the usual way: Let $x \equiv_Q y \iff x \leq_Q y \;\&\; y \leq_Q x$, and let $\mathcal{W}$ be the quotient partial-ordering $\mathcal{Q}/{\equiv_Q}$, where the ordering on equivalence classes is defined in the obvious way. Note that $\mathcal{W}$ is also a well-quasi-ordering. We are interested in the following kind of structures.

Definition 1.2. A *well-partial-ordering*, or *wpo*, is a partial-ordering which is well-quasi-ordered.

Associated to each wpo is its length or maximal order type. This is a notion that is frequently used when studying wpo's. First, we notice that every linearization of a wpo is well-ordered. (A *linearization* of a partial-ordering $(P, \leq_P)$ is any linear ordering $\leq_L$ of P such that $x \leq_P y \implies x \leq_L y$.)

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Definition 1.3. Given a wpo $\mathcal{W} = (W, \leq_W)$, we let the *length* of $\mathcal{W}$ be

$$o(\mathcal{W}) = \sup\{\text{ordTy}(W, \leq_L) : \text{ where } \leq_L \text{ is a linearization of } \mathcal{W}\}.$$

We are using $\text{ordTy}(W, \leq_L)$ to denote the order type of $(W, \leq_L)$. So, in this case, $\text{ordTy}(W, \leq_L)$ is always an ordinal.

De Jongh and Parikh [dJP77] show that every wpo $\mathcal{W}$ has a linearization of order type $o(\mathcal{W})$. This is why $o(\mathcal{W})$ is often called the *maximal order type* of $\mathcal{W}$. We call such a linearization, a *maximal linearization* of $\mathcal{W}$. It can be deduced from de Jongh and Parikh's work that $o(\mathcal{W}) + 1 = \text{rk}(\mathbb{B}\text{ad}(\mathcal{W}))$, where

$$\mathbb{B}\text{ad}(\mathcal{W}) = \{\langle x_0, ..., x_{n-1}\rangle \in W^{<\omega} : \forall i < j < n\ (x_i \not\leq_W x_j)\},$$

ordered by reverse inclusion. (See Lemma 2.5 below.) Note that $\mathcal{W}$ is well-quasi-ordered if and only if $\mathbb{B}\text{ad}(\mathcal{W})$ is well-founded. This is one of the reasons why $o(\mathcal{W})$ comes up often in applications of wpo theory.

Schmidt continued the study of maximal order types in her Habilitationsschrift [Sch79]. There, she computed the maximal order type of the wpo investigated by Higman [Hig52], and gave upper bounds for the maximal order types of the wpo's investigated by Kruskal [Kru60] and Nash-Williams [NW65]. In [Sch79, page 9], Schmidt posed two questions: Are there any non-trivial relationships between the height of a wpo and its maximal order type? Is it true that the maximal order type of a computable wpo is always a computable ordinal? We answer the latter question in this paper. A partial answer to the first question has been obtained by Malicki and Rutkowski [MR04]. But there is no known classification of the set of pair of ordinals (α, β) for which there is a wpo $\mathcal{W}$ with $\text{rk}(\mathcal{W}) = \alpha$ and $o(\mathcal{W}) = \beta$.

Our results are part of the program of effective mathematics. The objective of this ongoing program is to analyze theorems and objects that occur in mathematics from a computable viewpoint. Only a very basic knowledge of computability theory is assumed in this paper. The proofs should be of interest to non-computability theorists too because they give us constructive ways of getting maximal linearizations of wpo's, which we did not have before. For an introduction to computability theory see the first chapters of [Soa87] or [AK00].

With respect to Schmidt's second question, we show, in Section 3, that the maximal order type of a computable wpo is indeed a computable ordinal. Moreover, we show that every computable wpo has a maximal linearization which is computable. However, this process is very much non-uniform. We show, in Section 4, that no hyperarithmetic (and in particular, computable) process can produce such maximal linearizations uniformly. The construction of a computable maximal linearization has two steps. First, given a computable wpo $\mathcal{W}$, we define a computable linearization $\preceq^{\mathcal{W}}$ of it in an uniform way. We will show that this linearization is a maximal linearization of $\mathcal{W}$ when $o(\mathcal{W})$ is of the form ω^α. When $o(\mathcal{W})$ is not of the form ω^α, it might not be the maximal linearization of $\mathcal{W}$, but its order type is not too far from the maximal order type of $\mathcal{W}$. Second, given an arbitrary wpo $\mathcal{W}$, we use the Cantor normal form of $o(\mathcal{W})$ to decompose $\mathcal{W}$ into finitely many pieces, each of which has maximal order type of the form ω^β, for some β, and then we apply the algorithm defined before to each of the pieces. It is the second step that is non-uniform. To show that that the process of finding maximal linearizations cannot be done hyperarithmetically, we show that if f is a function that maps computable indexes of wpo's to computable indexes of their maximal linearizations, then f can compute a function which decides whether two computable ordinals are isomorphic or not. It is known that such a function is not hyperarithmetic. Moreover, for a Turing degree **a** to be able to compute such a function, it has to be able to uniformly compute $0^{(\beta)}$, the β-th Turing jump of the empty set, for every computable ordinal β. We show that this is a sufficient and necessary condition for the degree of such a function f.

2. Preliminaries

In this section we define our notation and list some basic results which we will use further on.

2.1. **Orderings.** We start by settling our notation for operations on orderings. Knowledge about basic operations on ordinals, like sum, product, exponentiation and Cantor normal forms, will be assumed. (See [AK00, Chapter 4] or [Ros82, §3.4].)

The *sum*, $\sum_{i\in\alpha}\mathcal{P}_i$, of a set of partial-orderings $\{\mathcal{P}_i\}_{i\in\alpha}$ indexed by $\alpha \leq \omega$, is constructed by taking the disjoint union of the sets P_i and letting $x \leq_{\sum_{i\in\alpha}\mathcal{P}_i} y$ if either for some $i \in \alpha$, $x, y \in P_i$ and $x \leq_{P_i} y$, or $x \in P_i$, $y \in P_j$ and $i < j$.

When $\alpha = m \in \omega$, we sometimes write $\mathcal{P}_0 + ... + \mathcal{P}_{m-1}$, or $\sum_{i<m}\mathcal{P}_i$ instead of $\sum_{i\in m}\mathcal{P}_i$. The *disjoint sum*, $\bigoplus_{i\in I}\mathcal{P}_i$, of a set of partial orderings $\{P_i\}_{i\in I}$ indexed by a set I, is constructed by taking the disjoint union of the sets P_i and letting elements from different P_i's be incomparable.

Given a partial-ordering $\mathcal{P} = (P, \leq_P)$, and $x \in P$, we let $P_{(<x)} = \{y \in P : y <_P x\}$ and $\mathcal{P}_{(<x)} = (P_{(<x)}, \leq_P)$. Analogously we define $\mathcal{P}_{(>x)}$, $\mathcal{P}_{(\leq x)}$, $\mathcal{P}_{(\geq x)}$, $\mathcal{P}_{(\not\geq x)}$, etc..

Given a well-founded-ordering $\mathcal{P}$, the rank function on $\mathcal{P}$ is defined by transfinite recursion: $\mathrm{rk}_{\mathcal{P}}(x) = \sup\{\mathrm{rk}_{\mathcal{P}}(y)+1 : y <_P x\}$. The *rank* of $\mathcal{P}$ is $\mathrm{rk}(\mathcal{P}) = \sup\{\mathrm{rk}_{\mathcal{P}}(x)+1 : x \in P\}$. We note that for well-founded orderings $\mathcal{P}$ and $\mathcal{Q}$, $\mathrm{rk}(\mathcal{P}+\mathcal{Q}) = \mathrm{rk}(\mathcal{P}) + \mathrm{rk}(\mathcal{Q})$ and $\mathrm{rk}(\mathcal{P} \oplus \mathcal{Q}) = \max\{\mathrm{rk}(\mathcal{P}), \mathrm{rk}(\mathcal{Q})\}$.

2.2. **Indecomposable ordinals and commutative sums.** An ordinal δ is said to be *indecomposable* (or *additively indecomposable*) if for every $\alpha, \beta < \delta$, $\alpha + \beta < \delta$. A well known fact is that δ is indecomposable if and only if $\delta = \omega^\gamma$ for some ordinal γ.

The *Cantor normal form* of an ordinal α is a tuple $\langle \alpha_0, ..., \alpha_n\rangle$ such that $\alpha \geq \alpha_0 \geq \alpha_1 \geq ... \geq \alpha_n \geq 0$ and $\alpha = \omega^{\alpha_0} + ... + \omega^{\alpha_n}$. Given two ordinals $\alpha = \omega^{\alpha_0} + ... + \omega^{\alpha_{n-1}}$ and $\beta = \omega^{\beta_0} + ... + \omega^{\beta_{m-1}}$, we define the *commutative sum* between α and β to be

$$\alpha \,\#\, \beta = \omega^{\gamma_0} + \omega^{\gamma_1} + ... + \omega^{\gamma_{n+m-1}},$$

where $\gamma_0, ..., \gamma_{n+m-1}$ are such that $\gamma_0 \geq \gamma_1 \geq ... \geq \gamma_{n+m-1}$, and there exists a partition $\{\{a_0, ..., a_{n-1}\}, \{b_0, ..., b_{m-1}\}\}$ of $\{0, ..., n+m-1\}$ such that $\gamma_{a_i} = \alpha_i$ and $\gamma_{b_i} = \beta_i$. The commutative sum, sometimes called natural sum or Hessenberg sum, was introduced in [Hes06]; see [dJP77, §3] for more information on Hessenberg based operations.

There are only a few well known properties of the commutative sum that we will use:

(CS1) $\alpha + \beta \leq \alpha \,\#\, \beta$,
(CS2) $\alpha \,\#\, \beta = \beta \,\#\, \alpha = o(\alpha \oplus \beta)$,
(CS3) if $\alpha, \beta < \omega^\gamma$, then $\alpha \,\#\, \beta < \omega^\gamma$,
(CS4) if $\alpha_0 \leq \alpha_1$ and $\beta_0 < \beta_1$, then $\alpha_0 \,\#\, \beta_0 < \alpha_1 \,\#\, \beta_1$,
(CS5) if $\beta < \omega^\gamma$, then $\alpha \,\#\, \beta < \alpha + \omega^\gamma$.

The proofs of these facts are not hard.

Given a sequence $\{\alpha_\xi : \xi \in \delta\}$, we define $\#_{\xi\in\delta}\alpha_\xi$ by transfinite induction in the usual way. That is, let $\#_{\xi\in\delta+1}\alpha_\xi = (\#_{\xi\in\delta}\alpha_\xi) \# \alpha_\delta$, and if δ is a limit ordinal, let $\#_{\xi\in\delta}\alpha_\xi = \lim_{\zeta<\delta}\#_{\xi\in\zeta}\alpha_\xi$.

2.3. **Well-partial-orderings and their linearizations.** One of the main results in de Jongh and Parikh [dJP77] is that every wpo $\mathcal{W}$ has a maximal linearization. We cite some lemmas that they used in that proof and some other results that we will use later.

Lemma 2.1. [dJP77, 2.3, 2.14, 2.6 and 2.15] *Let $\mathcal{W}$ and $\mathcal{Q}$ be a wpo's.*

1. *If $\mathcal{W} \subseteq \mathcal{Q}$, then $o(\mathcal{W}) \leq o(\mathcal{Q})$.*
2. *For every $x \in W$, $o(\mathcal{W}_{(\ngeq x)}) < o(\mathcal{W})$.*
3. *$o(\mathcal{W}) = \sup\{o(\mathcal{W}_{(\not> x)}) : x \in W\} = \sup\{o(\mathcal{W}_{(\ngeq x)}) + 1 : x \in W\}$.*
4. *If $o(\mathcal{W}) = \omega^{\alpha_0} + ... + \omega^{\alpha_k}$ with $\alpha_0 \geq ... \geq \alpha_k$, then $\forall x \in W (o(W_{(\geq x)}) \geq \omega^{\alpha_k})$.*

For part (3), de Jongh and Parikh only showed that if $o(\mathcal{W})$ is a limit ordinal, then $o(\mathcal{W}) = \sup\{o(\mathcal{W}_{(\ngeq x)}) : x \in W\}$. If $o(\mathcal{W})$ is not a limit ordinal, then there is a maximal element x in $\mathcal{W}$ (the maximal one in a maximal linearization of $\mathcal{W}$). For that element we have $o(W) = o(\mathcal{W}_{(\ngeq x)}) + 1 = o((\mathcal{W}_{(\not> x)})$.

It is also true that if $o(\mathcal{W})$ is a limit ordinal, then $\mathcal{W}$ has no maximal elements. This follows from part (4).

The following lemma is an extension of [dJP77, 2.17].

Lemma 2.2. *Let $\mathcal{W}$ be a wpo and let $\alpha \leq o(\mathcal{W})$. Then, there exists an ideal $I \subseteq W$ such that $o(I, \leq_W) = \alpha$.*

By *ideal* we mean downward closed subset.

Proof. The proof is by transfinite induction on $o(\mathcal{W})$. Of course, if $\alpha = o(\mathcal{W})$, take $I = W$. So, assume $\alpha < o(\mathcal{W})$. Let x be such that $\alpha \leq o(\mathcal{W}_{\ngeq x}) < o(\mathcal{W})$. By the inductive hypothesis, there exists an ideal $I \subset W_{(\ngeq x)}$ such that $o(I, \leq_W) = \alpha$. □

Lemma 2.3. *Let $\mathcal{W}$ and $\mathcal{Q}$ be wpo's. Then*

1. *$o(\mathcal{W} + \mathcal{Q}) = o(\mathcal{W}) + o(\mathcal{Q})$, and*
2. [dJP77, 3.4] *$o(\mathcal{W} \oplus \mathcal{Q}) = o(\mathcal{W}) \# o(\mathcal{Q})$.*

Corollary 2.4. *Let $\mathcal{W}$, $\mathcal{A}$ and $\mathcal{B}$ be wpo's such that $W = A \sqcup B$ and $\forall x \in A\ \forall y \in B\ (y \nleq_W x)$. Then*

$$o(\mathcal{A}) + o(\mathcal{B}) \leq o(\mathcal{W}) \leq o(\mathcal{A}) \# o(\mathcal{B}).$$

Proof. Any linearization of $\mathcal{A} + \mathcal{B}$ is a linearization of $\mathcal{W}$ and any linearization of $\mathcal{W}$ is a linearization of $\mathcal{A} \oplus \mathcal{B}$. □

Lemma 2.5. *Let $\mathcal{W}$ be a wpo. Then $o(\mathcal{W}) + 1 = \mathrm{rk}(\mathbb{B}\mathrm{ad}(\mathcal{W})) = \mathrm{rk}_{\mathbb{B}\mathrm{ad}(\mathcal{W})}(\emptyset) + 1$.*

Proof. We use transfinite induction on $o(\mathcal{W})$. So, we have that for every $x \in W$, $o(\mathcal{W}_{(\ngeq x)}) = \mathrm{rk}_{\mathbb{B}\mathrm{ad}(\mathcal{W}_{(\ngeq x)})}(\emptyset)$, where $\emptyset$ is the empty string. Then

$$\mathrm{rk}_{\mathbb{B}\mathrm{ad}(\mathcal{W})}(\emptyset) = \sup_{x\in W} \mathrm{rk}_{\mathbb{B}\mathrm{ad}(\mathcal{W})}(\langle x\rangle) + 1 =$$
$$\sup_{x\in W} \mathrm{rk}_{\mathbb{B}\mathrm{ad}(\mathcal{W}_{(\ngeq x)})}(\emptyset) + 1 = \sup_{x\in W} o(\mathcal{W}_{(\ngeq x)}) + 1 = o(\mathcal{W}).$$

□

2.4. **Computable Orderings.** We say that a partial ordering $\mathcal{P} = (P, \leq_P)$ is *computably presented* if $P \subseteq \omega$ and $\leq_P$ is a computable subset of $\omega \times \omega$, or in other words, if there is a computer program that, on imput $\langle p, q\rangle \in \omega \times \omega$, returns `Yes` or `No` depending on whether $p \leq_P q$ or not. We say that $(P, \leq_L)$ is a *computable linearization* of $\mathcal{P}$ if it is a linearization $\leq_P$ and $\leq_L \subseteq P \times P \subseteq \omega \times \omega$ is computable. For the reader familiar with these notions, we remark that this notion depends on the presentation of $\mathcal{P} = (P, \leq_P)$ and not only on its isomorphism type.

We say that an ordinal α is *computable*, if there is a computably presented linear ordering isomorphic to α. It is not hard to see that computable ordinals form a countable initial segment of the class of ordinals.

3. Maximal Order Types

In this section we show that every computable wpo has a computable maximal linearization.

3.1. **Indecomposable norm of an ordinal.** We start by introducing the concept of indecomposable norm.

Definition 3.1. Given an ordinal α, let the *indecomposable norm of* α, $[\![\alpha]\!]$, be the greatest indecomposable ordinal which is less than or equal to α.

Lemma 3.2. *Let α and β be ordinals. Then $[\![\alpha \# \beta]\!] = [\![\alpha + \beta]\!] = \max\{[\![\alpha]\!], [\![\beta]\!]\}$.*

Lemma 3.3. *Let $\{\alpha_i : i \in \omega\}$ be a sequence of ordinals. Then $[\![\#_{i\in\omega} \alpha_i]\!] = [\![\sum_{i\in\omega} [\![\alpha_i]\!]]\!]$.*

Proof. It is clear that the right-hand-side is less than or equal to the left-hand-side:

$$\sum_{i\in\omega} [\![\alpha_i]\!] \leq \sum_{i\in\omega} \alpha_i \leq \#_{i\in\omega} \alpha_i.$$

Let us now prove the other inequality. Let $[\![\#_{i\in\omega} \alpha_i]\!] = \omega^\beta$. We need to show that $\sum_{i\in\omega} [\![\alpha_i]\!] \geq \omega^\beta$. Suppose first that β is a limit ordinal. Then, for every $\delta < \beta$, there exists α_i such that $\omega^\delta \leq [\![\alpha_i]\!]$. Because otherwise, $\#_{i\in\omega} \alpha_i \leq \omega^\delta < \omega^\beta$. So, $\sum_{i\in\omega} [\![\alpha_i]\!] \geq \sup_{i\in\omega} [\![\alpha_i]\!] \geq \omega^\beta$. Suppose now that $\beta = \gamma + 1$. Then, there are infinitely many α_i's such that $\omega^\gamma \leq [\![\alpha_i]\!]$. Because if there were only finitely many such α_is, say m many, then $\#_{i\in\omega} \alpha_i \leq \omega^\gamma \cdot (m+1) < \omega^\beta$. So, $\sum_{i\in\omega} [\![\alpha_i]\!] \geq \omega^\gamma \cdot \omega = \omega^\beta$. □

3.2. **A Computable linearization.** Let $\mathcal{W} = (W, \leq_W)$ be a countable wpo. We assume W comes equipped with an enumeration of itself $\{x_0, x_1, ...\}$. For example, we can assume that always $W \subseteq \omega$ and that $x_0 \leq_{\mathbb{N}} x_1 \leq_{\mathbb{N}} \cdots$. We mention this because we will define a linearization $\preceq^{\mathcal{W}}$ of $\mathcal{W}$, which depends on the enumeration of W.

Construction of $\preceq^{\mathcal{W}}$: Order 2^W (the set of functions from W to $\{0,1\}$) with the lexicographic ordering with respect to the enumeration of W. More precisely, given $\sigma, \tau \in 2^W$, let $\sigma \leq_{lex} \tau$ if and only if either $\sigma = \tau$ or for the first i such that $\sigma(x_i) \neq \tau(x_i)$ we have that $\sigma(x_i) = 0$ and $\tau(x_i) = 1$. Now, for each $y \in W$ define $\sigma_y^{\mathcal{W}} \in 2^W$ as follows. Given $x \in W$, let $\sigma_y^{\mathcal{W}}(x) = 1$ if $x \leq_W y$ and $\sigma_y^{\mathcal{W}}(x) = 0$ otherwise. Finally, let $y \preceq^{\mathcal{W}} z$ if and only if $\sigma_y^{\mathcal{W}} \leq_{lex} \sigma_z^{\mathcal{W}}$. ◊

Note that $\preceq^{\mathcal{W}}$ is uniformly computable from $\mathcal{W}$.

It is not hard to prove that the construction above defines a linearization of $\mathcal{W}$. The reason is that if $y \leq_W z$, then for every i, $\sigma_y^{\mathcal{W}}(x_i) \leq \sigma_z^{\mathcal{W}}(x_i)$.

Lemma 3.4. *If $\mathcal{W} \subseteq \mathcal{Q}$ are wpo's and $\mathcal{W}$ is downward closed in $\mathcal{Q}$, then $\preceq^{\mathcal{W}}$ coincides with the restriction of $\preceq^{\mathcal{Q}}$ to W. We are assuming here that elements of W and the elements of Q are enumerated in the same order.*

Proof. Let $v, w \in W$. Assume that $v \prec^{\mathcal{Q}} w$. There is a first i, such that $\sigma_v^{\mathcal{Q}}(q_i) \neq \sigma_w^{\mathcal{Q}}(q_i)$, and for that i, $\sigma_v^{\mathcal{Q}}(q_i) = 0$ and $\sigma_w^{\mathcal{Q}}(q_i) = 1$, where $\{q_0, q_1, \dots\} = Q$. So $q_i \leq_Q w \in W$, and hence $q_i \in W$. It follows that q_i is also the first place where $\sigma_v^{\mathcal{W}}$ and $\sigma_w^{\mathcal{W}}$ differ, and since $\sigma_v^{\mathcal{W}}(q_i) < \sigma_w^{\mathcal{W}}(q_i)$, $v \prec^{\mathcal{W}} w$. □

Now, we start analyzing the order type of $\preceq^{\mathcal{W}}$. We prove that the linearization $\preceq^{\mathcal{W}}$, which is uniformly computable in $\mathcal{W}$, is relatively close to the maximal linearization of $\mathcal{W}$. However, it will follow from Theorem 4.3 that $\preceq^{\mathcal{W}}$ cannot always be the maximal linearization of $\mathcal{W}$.

Proposition 3.5. *Let $\mathcal{W} \subseteq \mathcal{Q}$ be wpo's. Then $[\![\mathrm{ordTy}(W, \preceq^{\mathcal{Q}})]\!] = [\![o(\mathcal{W})]\!]$.*

Proof. Note that one of the inequalities is immediate: $[\![\mathrm{ordTy}(W, \preceq^{\mathcal{Q}})]\!] \leq [\![o(\mathcal{W})]\!]$.

To prove the other inequality, we will use transfinite induction on $o(\mathcal{W})$. If $o(\mathcal{W})$ is not indecomposable, then, by Lemma 2.2, there is an ideal I of $\mathcal{W}$ such that $o(\mathcal{I}) = [\![o(\mathcal{W})]\!] < o(\mathcal{W})$, where $\mathcal{I} = (I, \leq_W)$. By the induction hypothesis, $[\![\mathrm{ordTy}(I, \preceq^{\mathcal{Q}})]\!] = [\![o(\mathcal{I})]\!] = [\![o(\mathcal{W})]\!]$. So $[\![\mathrm{ordTy}(W, \preceq^{\mathcal{Q}})]\!] \geq [\![o(\mathcal{W})]\!]$.

Assume now that $o(\mathcal{W})$ is indecomposable and infinite. If $o(\mathcal{W})$ is finite the result is trivial. By Lemma 3.4, we can assume without loss of generality that $\mathcal{Q}$ is the downwards closure of $\mathcal{W}$. Let $\{q_0, q_1, ...\}$ be the enumeration of $\mathcal{Q}$. For each i, let

$$R_i = \{x \in W : q_i \not\leq_Q x \ \&\ \forall j < i (q_j \leq_Q x)\}$$

and $R_{<i} = \bigcup_{j<i} R_j$. So, $x \in R_i$ if and only if $\sigma_x^{\mathcal{Q}}$ starts with i many ones and then has a zero. Since $o(\mathcal{W})$ is a limit ordinal, $\mathcal{W}$ has no maximal elements and hence every $x \in W$ belongs to some R_i. Notice that $(W, \preceq^{\mathcal{Q}}) = \sum_{i\in\omega}(R_i, \preceq^{\mathcal{Q}})$. Note that for each $w \in W$, if $w = q_i$, then $\mathcal{W}_{\not\geq w} \subseteq R_{\leq i}$. It follows that $o(\mathcal{W}) = \sup_{w\in W} o(\mathcal{W}_{\not\geq w}) \leq \sup_{i\in\omega} o(R_{<i}) \leq o(\mathcal{W})$. So $o(\mathcal{W}) = \sup_{i\in\omega} o(R_{<i})$. Now, for each i we have that $o(R_{<i}) \leq \#_{j<i}\, o(R_j)$. Therefore, $\#_{j\in\omega}\, o(R_j) \geq o(\mathcal{W})$.

For each $i \in \omega$, let

$$\alpha_i = o(R_i, \leq_W) \quad \text{and} \quad \beta_i = \mathrm{ordTy}(R_i, \preceq^{\mathcal{Q}}).$$

Then $\mathrm{ordTy}(W, \preceq^{\mathcal{Q}}) = \sum_{i\in\omega} \beta_i$ and $o(\mathcal{W}) \leq \#_{i\in\omega}\, \alpha_i$. For every $i \in \omega$, we have that $R_i \subseteq \mathcal{Q}_{(\not\geq q_i)}$ and hence $R_i \subseteq \mathcal{W}_{(\not\geq x)}$ for some $x \in W$, $x \geq_Q q_i$. Using Lemma 2.1.(1) and (2), it follows that $\alpha_i \leq o(\mathcal{W}_{(\not\geq x)}) < o(\mathcal{W})$. So, by the inductive hypothesis, we can assume that $[\![\alpha_i]\!] = [\![\beta_i]\!]$. Then,

$$\mathrm{ordTy}(W, \preceq^{\mathcal{Q}}) = \sum_{i\in\omega} \beta_i \geq [\![\sum_{i\in\omega} [\![\alpha_i]\!]]\!] = [\![\#_{i\in\omega}\, \alpha_i]\!] \geq [\![o(\mathcal{W})]\!] = o(\mathcal{W}),$$

where the middle equality follows from Lemma 3.3. □

Now, we use $\preceq^{\mathcal{W}}$ to construct a computable maximal linearization of $\mathcal{W}$.

Lemma 3.6. *Let $\mathcal{W}$ be a wpo and let $o(\mathcal{W}) = \omega^{\alpha_0} + ... + \omega^{\alpha_{n-1}}$, with $\alpha_0 \geq ... \geq \alpha_{n-1}$. Then, there exists a sequence of ideals $\emptyset = I_0 \subseteq \cdots \subseteq I_n = W$ such that for each $i < n$, $o(I_{i+1} \setminus I_i, \leq_W) = \omega^{\alpha_i}$.*

Proof. By Lemma 2.2, there exists an ideal $I_{n-1} \subseteq W$ such that $o(I_{n-1}, \leq_W) = \omega^{\alpha_0} + ... + \omega^{\alpha_{n-2}}$. Note that $o(W \setminus I_{n-1}, \leq_{\mathcal{W}}) \geq \omega^{\alpha_{n-1}}$, because, by Lemma 2.1.(4), for every $x \in W$, $o(W_{\geq x}) \geq \omega^{\alpha_{n-1}}$. Also, we cannot have $o(W \setminus I_{n-1}, \leq_{\mathcal{W}}) > \omega^{\alpha_{n-1}}$ because $o(\mathcal{W}) \geq o(I_{n-1}, \leq_{\mathcal{W}}) + o(W \setminus I_{n-1}, \leq_{\mathcal{W}})$. So, $o(W \setminus I_{n-1}, \leq_{\mathcal{W}}) = \omega^{\alpha_{n-1}}$. Use the inductive hypothesis to define a sequence of ideals $\emptyset = I_0 \subseteq \cdots \subseteq I_{n-2} \subseteq I_{n-1}$ such that for each $i < n-1$, $o(I_{i+1} \setminus I_i, \leq_W) = \omega^{\alpha_i}$. □

Theorem 3.7. *Every computable wpo has a computable linearization of maximal order type.*

Proof. Let $\mathcal{W}$ be a computable wpo, and let $o(\mathcal{W}) = \omega^{\alpha_0} + ... + \omega^{\alpha_{n-1}}$. Consider a sequence of ideals $\emptyset = I_0 \subseteq \cdots \subseteq I_n = W$ such that for each $i < n$, $o(J_i, \leq_W) = \omega^{\alpha_i}$, where $J_i = I_{i+1} \setminus I_i$. So, by Lemma 3.5, we have that for every $i < n$, $\mathrm{ordTy}(J_i, \preceq^{\mathcal{W}}) = \omega^{\alpha_i} = [\![o(J_i, \leq_W)]\!]$. We define $(W, \trianglelefteq)$ to be $\sum_{i<n}(J_i, \preceq^{\mathcal{W}})$. In other words, we let $y \trianglelefteq x$ if and only if either there exists i such that $y, x \in J_i$ and $y \preceq^{\mathcal{W}} x$, or there exists $i < j$ such that $y \in J_i$ and $x \in J_j$. Observe that the J_i's form a computable partition of W: For each i, let X_i be the set of minimal elements of $W \setminus I_i$. Then, since $\mathcal{W}$ is a wpo, each X_i is finite and for every y in W, $y \in I_i \iff \forall x \in X_i (y \not\geq_W x)$. So, using the X_i's as parameters, we can decide whether $y \in J_i$ or not. Therefore, $(W, \trianglelefteq)$ is a computable linearization of $\mathcal{W}$ of order type $\omega^{\alpha_0} + ... + \omega^{\alpha_{n-1}}$. □

4. Non-uniformity of the linearizations

In this section we study the complexity of the procedure of obtaining maximal linearizations of computable wpo's. In other word we consider the following question: Is there a single computer program that, given a computable wpo (given by the program that computes it), returns (the program for) a computable linearization of it, and if not, how complex is this procedure? It will follow from Theorem 4.3 that this cannot be done uniformly computably, and not even hyperarithmetically. The source of non-uniformity in our constructions of computable linearization is in the proof of Theorem 3.7. In Theorem 3.7, finding the Cantor normal form $o(\mathcal{W}) = \omega^{\alpha_0} + ... + \omega^{\alpha_{n-1}}$ and finding the sets X_i is what we do non-uniformly. We just proved these finite objects exists and used them to construct the program that gives the computable linearization. Theorem 4.3 says that there is no computable procedure to find these finite objects.

We start by giving some basic definitions and stating the facts we will use.

Definition 4.1. Given a set $Z \subseteq \omega$, we use Z' to denote the Turing jump of Z, or equivalently the Halting problem relative to Z. That is Z' is the set of programs (or of numbers that represent computer programs in binary) that when run with oracle Z, halt and do not run for ever. It is well known that Z' can compute Z, but Z cannot compute Z'.

We now consider the iteration of the Turing jump along an ordinal, getting sets which are much more complicated. Given an well-ordering $\mathcal{B} = (B, \leq_B)$, with $B \subseteq \omega$, let $0^{(\mathcal{B})}$

be the unique set $X \subseteq \omega$ such that for every $x \in B$,

$$X^{[x]} = (X^{[B_{(<x)}]})',$$

where $X^{[x]} = \{y : \langle x, y\rangle \in X\}$, $X^{[B_{(<x)}]} = \{\langle z, y\rangle : z \in B_{(<x)} \ \& \ \langle z, y\rangle \in X\}$.

Clearly, since $\mathcal{B}$ is well-ordered, $0^{(\mathcal{B})}$ can be defined by transfinite recursion. It can be shown that if β is a computable ordinal, the the Turing degree of $0^{(\beta)}$ is independent of the computable presentation of β (Spector [Spe55]).

The following lemma is well-known and it follows form the work of Ash and Knight [AK90, AK00].

Lemma 4.2. *Let α be a computable ordinal.*

1. *Given two computable ordinals $\beta, \delta < \alpha$, using $0^{(2\alpha+2)}$ as an oracle, we can decide whether β is less δ and whether they are isomorphic.*
2. *Suppose g is a function that, given two computable ordinals $\beta, \delta \leq \omega^\alpha$, it decides whether β and δ are isomorphic. Then g can compute $0^{(\alpha)}$. Moreover, the reduction $0^{(\alpha)} \leq g$ can be uniformly computed from α.*

Proof. Part (1) follows from [AK00, Proposition 7.2 and Theorem 7.4]. Part (1) follows from [AK90, Example 5]. □

Theorem 4.3. *Let $A \subseteq \omega$. The following are equivalent.*

1. *A computes a function f which, given an index for a computable wpo $\mathcal{W}$, returns an index for a computable maximal linearization of $\mathcal{W}$.*
2. *A computes a function g which, given indexes for two computable ordinals, returns 1 or 0 depending on whether the two ordinals are isomorphic or not.*
3. *A uniformly computes $0^{(\beta)}$ for every computable ordinal β.*

Proof. The equivalence between (2) and (3) follows from Lemma 4.2.

Let us start proving that (1) implies (2). Let α and β be two given computable ordinals. Consider $\mathcal{W} = (\omega^\alpha + \omega^\alpha) \oplus (\omega^\beta + \omega^\beta)$, the disjoint sum of $\omega^\alpha + \omega^\alpha$ and $\omega^\beta + \omega^\beta$. Let a be the first element of the second copy of ω^α in $\omega^\alpha + \omega^\alpha$. Let b be the first element of the second copy of ω^β. Use f to get a maximal linearization $\trianglelefteq$ of $\mathcal{W}$. Let $h(\alpha, \beta) = 0$ if $a \trianglelefteq b$ and $h(\alpha, \beta) = 1$ if $b \trianglelefteq a$. (When we say $h(\alpha, \beta)$, we actually mean $h(e, d)$ where e and d are computable indexes for α and β.) If $\alpha < \beta$, we have that $\mathrm{ordTy}(W, \trianglelefteq) = (\omega^\beta + \omega^\beta) \# (\omega^\alpha + \omega^\alpha) = \omega^\beta + \omega^\beta + \omega^\alpha + \omega^\alpha$, and hence $b \triangleleft a$ and $h(\alpha, \beta) = 1$. Analogously, if $\alpha > \beta$, then $h(\alpha, \beta) = 0$. In the case when $\alpha = \beta$, $h(\alpha, \beta)$ could be either 0 or 1. Now, we use h, and the fact that $\forall \alpha, \beta(\alpha \leq \beta \iff 2\alpha < 2\beta + 1)$, to compute a function g as wanted: Let $g(\alpha, \beta) = 1$ if and only if $h(2\alpha + 1, 2\beta) = 0$ and $h(2\alpha, 2\beta + 1) = 1$ and $g(\alpha, \beta) = 0$ otherwise. Note that $g(\alpha, \beta) = 1$ if and only if $\alpha = \beta$.

To show that (3) implies (1), we have to look at the definition of $\trianglelefteq$ in Theorem 3.7. First, let β be a well-ordered linearization of $\mathbb{B}\mathrm{ad}(\mathcal{W})$. For example, β could be the Kleene-Brouwer ordering of $\mathbb{B}\mathrm{ad}(\mathcal{W})$ (see [AK00, § 5.3]). So, we have that $\beta \geq \mathrm{rk}(\mathbb{B}\mathrm{ad}(\mathcal{W})) > o(\mathcal{W})$, and using A we can compute $0^{(2\beta+\omega)}$, and using $0^{(2\beta+\omega)}$ we can compute $o(\mathcal{W})$: look at all the computable linearizations of $\mathcal{W}$ and take the one of maximal order type. (Recall that $0^{(2\beta+\omega)}$ is able to compare ordinals less than β.) Then, use $0^{(\omega^\beta)}$, also computable from A, to compute the Cantor normal form of $o(\mathcal{W})$: look for a tuple $\langle \alpha_0, ..., \alpha_{n-1} \rangle$ with $o(\mathcal{W}) \geq \alpha_0 \geq ... \geq \alpha_{n-1}$ and $o(\mathcal{W}) = \omega^{\alpha_0} + ... + \omega^{\alpha_{n-1}}$. Then, look for a sequence of ideals $\langle I_0, ..., I_n \rangle$ as the one used in Theorem 3.7. Recall that ideals of a wpo can be coded by finite subsets (the set of minimal elements of the complement of the ideal). Then, linearize each ideal using $\preceq^{\mathcal{W}}$ and construct $\trianglelefteq$ as in Theorem 3.7. □

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