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EMBEDDING JUMP UPPER SEMILATTICES INTO THE TURING DEGREES.

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Abstract. We prove that every countable jump upper semilattice can be embedded in $\mathcal{D}$, where a jump upper semilattice (jusl) is an upper semilattice endowed with a strictly increasing and monotone unary operator that we call jump, and $\mathcal{D}$ is the jusl of Turing degrees. As a corollary we get that the existential theory of $\langle \mathbf{D}, \leq_T, \vee, ' \rangle$ is decidable. We also prove that this result is not true about jusls with 0, by proving that not every quantifier free 1-type of jusl with 0 is realized in $\mathcal{D}$. On the other hand, we show that every quantifier free 1-type of jump partial ordering (jpo) with 0 is realized in $\mathcal{D}$. Moreover, we show that if every quantifier free type, $p(x_1, ..., x_n)$, of jpo with 0, which contains the formula $x_1 \leq 0^{(m)}$ & ... & $x_n \leq 0^{(m)}$ for some m, is realized in $\mathcal{D}$, then every every quantifier free type of jpo with 0 is realized in $\mathcal{D}$.

We also study the question of whether every jusl with the c.p.p. and size $\kappa \leq 2^{\aleph_0}$ is embeddable in $\mathcal{D}$. We show that for $\kappa = 2^{\aleph_0}$ the answer is no, and that for $\kappa = \aleph_1$ it is independent of ZFC. (It is true if MA(κ) holds.)

§1. Introduction.

We deal with the following kind of structures.

DEFINITION 1.1. A *partial jump upper semilattice* (pjusl) is a structure

$$\mathcal{J} = \langle J, \leq_{\mathcal{J}}, \cup, \mathfrak{j} \rangle$$

where $\langle J, \leq_{\mathcal{J}} \rangle$ is a partial ordering, $\cup$ is a partial binary operation and $\mathfrak{j}$ are partial unary operation such, that for all $x, y \in J$,

- if $x \cup y$ is defined, it is the least upper bound of x and y, and
- if $\mathfrak{j}(x)$ is defined then $x <_{\mathcal{J}} \mathfrak{j}(x)$; and if $\mathfrak{j}(y)$ is also defined and $x \leq_{\mathcal{J}} y$, then $\mathfrak{j}(x) \leq_{\mathcal{J}} \mathfrak{j}(y)$.

By partial operation we mean that it does not need to be defined everywhere. A *jump upper semilattice* (jusl) is a pjusl where $\mathfrak{j}$ and $\cup$ are total operations. A *jump partial ordering* (jpo) is a pjusl where $\mathfrak{j}$ is total but $\cup$ is undefined.

Given pjusls, $\mathcal{J}_1$ and $\mathcal{J}_2$, an *embedding of* $\mathcal{J}_1$ *into* $\mathcal{J}_2$ is an injective map $f\colon J_1 \to \mathcal{J}_2$ such that for all $x, y \in J_1$:

- $x \leq_{\mathcal{J}_1} y$ if and only if $f(x) \leq_{\mathcal{J}_2} f(y)$;
- if $\mathfrak{j}(x)$ is defined, then $f(\mathfrak{j}(x)) = \mathfrak{j}(f(x))$; and
- if $x \cup y$ is defined, then $f(x \cup y) = f(x) \cup f(y)$.

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Observe that, $\mathcal{D} = \langle \mathbf{D}, \leq_T, \vee, ' \rangle$, the set of Turing degrees together with the Turing reduction, the join operation and the Turing Jump is a jusl.

We address the question of which pjusls can be embedded into $\mathcal{D}$. The first embeddablity result about $\mathcal{D}$ was proved by Kleene and Post in [KP54]. One of the things they proved there is that every finite upper semilattice can be embedded into $\mathcal{D}$. Various others results have been proved. Sacks proved in [Sac61] that every partial ordering of size at most $\aleph_1$ with the c.p.p. can be embedded into $\mathcal{D}$. (Recall that we say that a partial order has the c.p.p. or *countable predecessor property* if every element has at most countably many predecessors.) Abraham and Shore extended this result to upper semilattices in [AS86]. (They even embedded the upper semilattices as initial segments of $\mathcal{D}$.) Hinman and Slaman, proved in [HS91], that every countable jpo is embeddable in $\mathcal{D}$. We prove here that every countable jusl is embeddable in the Turing degrees. We also construct a jpo of size continuum with the c.p.p. which cannot be embedded in $\mathcal{D}$. For cardinals κ between $\aleph_0$ and $2^{\aleph_0}$, we show that, if MA(κ) holds, then every jusl with the c.p.p. and size κ can be embedded in $\mathcal{D}$. (MA(κ) is defined in 6.12.) These two last results imply that whether every jpo (or jusl) of size $\aleph_1$ is embeddable in $\mathcal{D}$ is independent of ZFC.

These kinds of results are always related to decidability results. We know that the elementary theory of $\langle \mathbf{D}, \leq_T \rangle$ is undecidable, as was shown by Lachlan in [Lac68]. However, it is still of interest to know which segments of the theory of $\mathcal{D}$ are decidable. For example, form the results of Kleene and Post in [KP54], we get that the $\exists$-theory of $\langle \mathbf{D}, \leq_T \rangle$ is decidable. Then Jockusch and Slaman, [JS93], showed that the $\forall\exists$-theory of $\langle \mathbf{D}, \leq_T, \vee \rangle$ is decidable. Their result is optimal in the sense that the $\forall\exists\forall$-theory of the same structure is undecidable. This follows from the undecidability of the $\forall\exists\forall$-theory of $\langle \mathbf{D}, \leq_T \rangle$, proved by Schmerl (see [Ler83, Corollary VII.4.6]). Another interesting result, proved by Jockusch and Soare is that the whole elementary theory of $\langle \mathbf{D}, ' \rangle$ is decidable (see [Ler83, Exercise III.4.21]). Here, as a corollary of our main result, we get that the existential theory of $\langle \mathbf{D}, \leq_T, \vee, ' \rangle$ is decidable. This result is optimal too, since the $\forall\exists$-theory was recently proved undecidable by Shore and Slaman, in [SS].

About $\langle \mathbf{D}, \leq_T, ' \rangle$, we know that the $\forall$-theory is decidable and that the $\forall\exists\forall$-theory is undecidable. But, we do not know much about the $\forall\exists$-theory. A sub case of this question, that remains open, is whether the existential theory of $\langle \mathbf{D}, \leq_T, ', 0 \rangle$ is decidable. The best approximation to this question is a result due to Lempp and Lerman [LL96]. They proved that every quantifier free formula, $\varphi(x_1, ..., x_n)$, in the language of $\langle \mathbf{D}, \leq_T, ', 0 \rangle$, that is consistent with the axioms of jpo with 0 (see 5.1 for a definition of jpo with 0) and with the formula $x_1 \leq_T 0'$ & ... & $x_n \leq_T 0'$, is realized by a n-tuple of r.e. degrees. We call a type, $p(x_1, ..., x_n)$ of jpo with 0 *archimedean* if, for some $m \in \omega$, it contains the formula $x_1 \leq_T 0^{(m)}$ & ... & $x_n \leq_T 0^{(m)}$. We prove that if every quantifier free (q.f.) archimedean type of jpo with 0 is realized in $\mathcal{D}$, then every q.f. type of jpo with 0 is realized in $\mathcal{D}$. It seems likely that the hypothesis of every q.f. archimedean type being realized in $\mathcal{D}$ can be proved using iterated trees of strategies, which is a method created by Lempp and Lerman (see, for example, [LL96]). Hinman and Slaman proved in [HS91] and [Hin99] that every q.f. archimedean 1-type of jpo with 0 is realized in $\mathcal{D}$. (Actually they proved something equivalent to this. See the proof of Corollary 5.9 for an explanation of the equivalence.) We extend their result and prove here that every q.f. 1-type of jpo with 0 is realized in $\mathcal{D}$. We also show that this result cannot be extended to jusl with 0. More precisely, we prove that not every quantifier free 1-type of jusl with 0 is realized in $\mathcal{D}$. This also implies that not every countable jusl with 0 can be embedded in $\mathcal{D}$.

Outline. We start by proving that any countable pjusl which supports a jump hierarchy is embeddable in $\mathcal{D}$. (We define jump hierarchies in 2.1.) We do this via a forcing construction that uses some ideas from the one that Hinman and Slaman used in [HS91]. We both simplify the construction in [HS91] and add new features to it. Then, in section 3, we show that certain simple pjusls support jump hierarchies and we deduce that the existential theory of $\langle \mathbf{D}, \leq_T, \vee, ' \rangle$ is decidable. In section 4 we prove our main result: Every countable jusl is embeddable in $\mathcal{D}$. To do this we show that every countable jusl can be embedded into one that supports a jump hierarchy. Part of this proof uses Fraïssé limits which are somewhat similar to the geometric part of the forcing notion used by Hinman and Slaman in [HS91]. In the last two sections we study pjusls with 0 and uncountable pjusls.

§2. The Main construction.

DEFINITION 2.1. Given a structure $\mathcal{P} = \langle P, \leq_P, ...\rangle$, where $\langle P, \leq_P\rangle$ is a partial ordering, a *Jump Hierarchy over* $\mathcal{P}$ is a map $H\colon P \to \omega^\omega$ such that, for all $x, y \in P$,

- $\mathcal{P} \leq_T H(x)$;
- $\bigoplus_{x \leq_P y} H(x) \leq_T H(y)$;
- if $x <_P y$ then $H(x)' \leq_T H(y)$.

When such an H exists, we say that $\mathcal{P}$ *supports* a jump hierarchy.

This section is devoted to proving the following theorem.

THEOREM 2.2. *Every countable partial jump upper semilattice which supports a jump hierarchy can be embedded in* $\mathcal{D}$.

We shall use a forcing construction (see [SW]). We shall also use different kinds of codings. Here is a description of them.

DEFINITION 2.3. For any X, Y, $Z \in \omega^\omega$, and any $n \in \omega$:

1. *X codes Y (directly) in the nth column* if $X^{[n]} = Y$. (Where $X^{[n]}(m) = X(\langle n, m\rangle)$.)
2. *X jump codes Y in the nth column* if for all m,
$$Y(m) = \lim_z X(\langle n, m, z\rangle);$$
that is, for some function S and all m and $z \geq S(m)$, $Y(m) = X(\langle n, m, z\rangle)$. S is called a *Skolem function* for the coding.
3. *X codes Y lazily in the nth column* if for all m and z, either $X(\langle n, m, z\rangle) = 0$ or $X(\langle n, m, z\rangle) = Y(m) + 1$, and for each m there is at least one z such that $Y(m) + 1 = X(\langle n, m, z\rangle)$.

4. *X and Y code Z lazily in the nth column* if for all k, l and $m \in \omega$,

$$X(\langle n,m,l\rangle) = Y(\langle n,m,l\rangle) = k \neq 0 \Longrightarrow Z(m) = k-1,$$

and for each m there is at least one l such that $X(\langle n,m,l\rangle) = Y(\langle n,m,l\rangle) = Z(m)+1$.

OBSERVATION 2.4. *For X, Y and $Z \in \omega^\omega$,*

- *If X codes Y directly or lazily in some column, then $Y \leq_T X$.*
- *If X jump codes Y in some column, then $Y \leq_T X'$.*
- *If X and Y code Z lazily, then $Z \leq_T X \oplus Y$.*

Fix $\mathcal{J} = \langle J, \leq_{\mathcal{J}}, \cup, \mathfrak{j}\rangle$, a countable partial jump upper semilattice. Assume that J, the universe of $\mathcal{J}$, is a recursive subset of ω. Let $H : J \to \omega^\omega$ be a jump hierarchy over $\mathcal{J}$.

In a first reading of this proof, the reader can assume that $\cup$ and $\mathfrak{j}$ are total: there are no essential changes in the proof when we allow $\cup$ and $\mathfrak{j}$ to be partial.

We shall define a function $R_G\colon J \to \omega^\omega$ via a forcing construction. The map $x \mapsto degree(R_G(x))\colon \mathcal{J} \to \mathcal{D}$ is going to be the desired embedding. For each $x \in J$, $R_G(x)$ consists of:

- A direct code of $H(x)$ in the 0th column.
- A jump coding of $R_G(\mathfrak{j}(x))$ in the 2nd column if $\mathfrak{j}(x)\downarrow$. This jump coding has $Sk_G(x)$ as a Skolem function.
- A lazy coding of $R_G(y)$ in the $(3y)$th column for all $y <_{\mathcal{J}} x$.
- A lazy code of the Skolem function $Sk_G(y)$ in the $(3y+1)$st column for all y such that $\mathfrak{j}(y) = x$.
- In the $(3\lfloor x,z\rfloor + 2)$nd column, $R_G(x)$ and $R_G(z)$ code $R_G(x\cup z)$ lazily for each $z \mid_{\mathcal{J}} x$ such that $x \cup y$ is defined, where $\lfloor x,z\rfloor = min(\langle x,z\rangle, \langle z,x\rangle)$ (it is a code for the unordered pair $\{x,z\}$), and $x \mid_{\mathcal{J}} z$ stands for $x \not\leq_{\mathcal{J}} z$ & $z \not\leq_{\mathcal{J}} x$.

2.1. The forcing notion. Now we define a partial ordering $\mathbb{P}$. Then we consider a generic filter G over $\mathbb{P}$, and from it define $R_G\colon J \to \omega^\omega$.

CONSTRUCTION OF $\mathbb{P}$ AND R_G. Let $\overline{\mathbb{P}}$ be the set of pairs $p = \langle R_p, Sk_p\rangle$, where R_p and Sk_p are finite partial functions $J \times \omega \rightharpoonup \omega$. We order $\overline{\mathbb{P}}$ by reverse inclusion in both coordinates. (i.e: $\langle r,s\rangle \leq \langle r',s'\rangle \iff r \supseteq r'$ & $s \supseteq s'$.) For $x \in J$, we write $R_p(x)$ for the partial function $\omega \rightharpoonup \omega$ defined by $R_p(x)(n) = R_p(x,n)$. The same for $Sk_p(x)\colon \omega \rightharpoonup \omega$. Let $\mathbb{P}$ be the set of $p \in \overline{\mathbb{P}}$ such that, for all $x,y,z \in J$, all the following conditions are satisfied:

1. $R_p(x)$ can be consistently extended to code $H(x)$ in the 0th column. i.e:

$$R_p(x)^{[0]} \subset H(x)$$

as partial functions.

2. If $\mathfrak{j}(x)\downarrow$, $Sk_p(x)$ specifies part of a Skolem function for a jump coding of $R_p(\mathfrak{j}(x))$ in the 2nd column of $R_p(x)$. More specifically, if $n \in \omega$ and $Sk_p(x)(n)\downarrow = k$, then $R_p(\mathfrak{j}(x))(n)\downarrow$ and for all $m \geq k$,

$$R_p(x)(\langle 2,n,m\rangle)\downarrow \Longrightarrow R_p(x)(\langle 2,n,m\rangle) = R_p(\mathfrak{j}(x))(n).$$

3. If $y <_{\mathcal{J}} x$, $R_p(x)$ is compatible with coding $R_p(y)$ lazily in the $3y$th column. i.e:
$$\exists m\big(R_p(x)(\langle 3y, n, m\rangle)\downarrow = k \neq 0\big) \Longrightarrow R_p(y)(n)\downarrow = k-1.$$
4. If $\mathfrak{j}(y) = x$, $R_p(x)$ is compatible with lazy coding $Sk(y)$ in the $(3y+1)$st column. i.e:
$$\exists m\big(R_p(x)(\langle 3y+1, n, m\rangle)\downarrow = k \neq 0\big) \Longrightarrow Sk_p(y)(n)\downarrow = k-1.$$
5. If $z \mid_{\mathcal{J}} x$, $y = x \cup z$ and for some $n, m \in \omega$ we have that
$$R_p(x)(\langle 3\lfloor x, z\rfloor + 2, n, m\rangle)\downarrow = R_p(z)(\langle 3\lfloor x, z\rfloor + 2, n, m\rangle) = k \neq 0$$
then $R_p(y)(n)\downarrow = k-1$.

Let G be an arithmetically (in $I\!\!P$) $I\!\!P$-generic filter. Let $R_G(x) = \bigcup\{R_p(x) : p \in G\}$ and $Sk_G(x) = \bigcup\{Sk_p(x) : p \in G\}$. $\diamond$

Lemma 2.5. *(The conditions on $I\!\!P$ are not contradictory.) For each $x \in \mathcal{J}$ and $n \in \omega$ the sets $\{q \in I\!\!P : R_q(x, n)\downarrow\}$ and $\{q \in I\!\!P : Sk_q(x, n)\downarrow\}$ are dense in $I\!\!P$. Hence $R_G(x), Sk_G(x) \in \omega^\omega$.*

Moreover, given $p \in I\!\!P$ and $n = \langle k, m\rangle \in \omega$, there exists $t \in \omega$ such that if we define q just by extending p so that $R_q(x, n)\downarrow = t$ (i.e. $R_q = R_p \cup \{\langle\langle x, n\rangle, t\rangle\}$ and $Sk_q = Sk_p$), then $q \in I\!\!P$. t can be obtained as follows.

1. *If n is in the 0th column, i.e. $k = 0$, let $R_q(x, n) = H(x)(m)$.*
2. *If $k = 2$ and $m = \langle m_1, m_2\rangle$ then, if $\mathfrak{j}(x)\downarrow$, $Sk_p(x, m_1)\downarrow = s \neq 0$ and $s \leq m_2$, define $R_q(x, n) = R_p(\mathfrak{j}(x), m_1)$. (Observe that if $Sk_p(x, m_1)\downarrow$, then $R_p(\mathfrak{j}(x), m_1)\downarrow$.) Otherwise, define $R_q(x, n)$ arbitrarily.*
3. *Now, suppose that $k = 3y$, $y <_{\mathcal{J}} x$ and $m = \langle m_1, m_2\rangle$. We can always set $R_q(x, n) = 0$. But, if we also know that $R_p(y, m_1)\downarrow = l$, then we could set $R_q(x, n) = l + 1$.*
4. *Now suppose that $k = 3y + 1$ with $\mathfrak{j}(y) = x$, and $m = \langle m_1, m_2\rangle$. Then, if $Sk_p(y, m_1)\downarrow = m_3$ set $R_q(x, n) = 0$ or $= m_3 + 1$, otherwise set $R_q(x, n) = 0$.*
5. *If $k = 3\lfloor x, z\rfloor + 2$ with $z \mid_{\mathcal{J}} x$ and $x \cup z$ defined, then we can always set $R_q(x, n) = 0$. Actually, we can set $R_q(x, n)$ to be anything we want as long as $R_q(x, n) \neq R_q(z, n)$ or $R_q(x, n) = R_q(x \cup z, m_1) + 1$ where $n = \langle m_1, m_2\rangle$.*
6. *In any other case we can set $R_q(x, n)$ arbitrarily.*

Sketch of the proof. We have to show that $q \in I\!\!P$. To do this we have to check all the conditions in the definition of $I\!\!P$. For example, suppose that $n = \langle 0, m\rangle$ and that we have set $R_q(x, n) = H(x)(m)$. Since $R_p(x)^{[0]} \subset H(x)$, we also have that $R_q(x)^{[0]} \subset H(x)$. Hence q satisfies condition 1 in the definition of $I\!\!P$. Conditions 2-5 are trivially satisfied. We leave the other cases to the reader.

Since this is true for all p, it implies that $\{q \in I\!\!P : R_q(x, n)\downarrow\}$ is dense in $I\!\!P$.

Now we have to show that $\{q \in I\!\!P : Sk_q(x, n)\downarrow\}$ is dense in $I\!\!P$. Consider $p \in I\!\!P$ and suppose that $Sk_p(x, n)\uparrow$. We want to show that there is an extension p_2 of p such that $Sk_{p_2}(x, n)\downarrow$. Let p_1 be an extension of p such that $R_{p_1}(\mathfrak{j}(x), n)\downarrow$. Let m be such that for all $i \geq m$ $R_{p_1}(z, \langle 3x + 1, n, i\rangle)\uparrow$ and let p_2 be such that $R_{p_2} = R_{p_1}$ and $Sk_{p_2} = Sk_{p_1} \cup \{\langle\langle x, n\rangle, m\rangle\}$. It is easy to verify that $p_2 \in I\!\!P$. $\square$

Lemma 2.6. *For all $x, y, z \in J$:*

1. *$H(x)$ is directly coded in the 0th column of $R_G(x)$.*

2. *if* $\mathfrak{j}(x)\downarrow$, *then* $R_G(\mathfrak{j}(x))$ *is jump coded in the 2nd column of* $R_G(x)$ *with Skolem function* $Sk_G(x)$.
3. *If* $y <_{\mathcal{J}} x$, *then* $R_G(y)$ *is lazily coded in the* $(3y)$*th column of* $R_G(x)$.
4. *If* $\mathfrak{j}(y) = x$, *then,* $R_G(x)$ *codes* $Sk_G(y)$ *lazily in the* $(3y+1)$*st column.*
5. *If* $x\ |_{\mathcal{J}}\ z$ *and* $x \cup z\downarrow$, *then* $R_G(x)$ *and* $R_G(z)$ *code* $R_G(x \cup z)$ *lazily in the* $(3\lfloor x, z\rfloor + 2)$*nd column.*

PROOF. For example, for the third part use Lemma 2.5 and observe that, once $R_p(y,n)\downarrow$, the set

$$\{q : \exists i\big(R_q(x, \langle 3y, n, i\rangle) = R_q(y,n) + 1\big)\}$$

is dense below p. The other parts are similar. □

COROLLARY 2.7. *For all* x *and* y *in* J,

1. $H(x) \leq_T R_G(x)$;
2. *if* $\mathfrak{j}(x)\downarrow$, *then* $R_G(\mathfrak{j}(x)) \leq_T R_G(x)'$;
3. *if* $y \leq_{\mathcal{J}} x$, *then* $R_G(y) \leq_T R_G(x)$;
4. *if* $\mathfrak{j}(y)\downarrow \leq_{\mathcal{J}} x$, *then* $Sk_G(y) \leq_T R_G(x)$;
5. *if* $x\ |_{\mathcal{J}}\ y$ *and* $x \cup y\downarrow$, *then* $R_G(x \cup y) \equiv_T R_G(x) \oplus R_G(y)$.

Moreover, all these Turing reductions are uniform in x *and* y.

PROOF. All the proofs are immediate from the previous lemma and observation 2.4. For (4) observe that $Sk_G(y) \leq_T R_G(\mathfrak{j}(y)) \leq_T R_G(x)$. □

2.2. Preservation of nonorder. We have already proved that $x \leq_{\mathcal{J}} y$ implies that $R_G(x) \leq_T R_G(y)$. In this subsection we prove that if $x \not\leq_{\mathcal{J}} y$, then $R_G(x) \not\leq_T R_G(y)$. To do this we need to analyze $\mathbb{P}$ a little bit more. We shall prove a combinatorial lemma about $\mathbb{P}$ that is going to be useful in the next subsection too.

DEFINITION 2.8. For $x \in J$, define

1. $J_x = \{y \in J : y \leq_{\mathcal{J}} x\}$ and $J_x^{\mathfrak{j}} = \{y \in J : \mathfrak{j}(y)\downarrow \leq_{\mathcal{J}} x\}$;
2. $\overline{\mathbb{P}}_x = \{\langle r, s\rangle : r$ and s are finite partial functions, $r : J_x \times \omega \rightharpoonup \omega$ and $s : J_x^{\mathfrak{j}} \times \omega \rightharpoonup \omega\}$;
3. $p \restriction x = \langle R_p \restriction J_x, Sk_p \restriction J_x^{\mathfrak{j}}\rangle \in \overline{\mathbb{P}}_x$.

DEFINITION 2.9. Say that $p \in \mathbb{P}$ is *nice at* $x \in J$ if for all y, z with $y\ |_{\mathcal{J}}\ z$ and $y \leq_{\mathcal{J}} x \not\geq_{\mathcal{J}} z$, and for all $i \in \omega$

$$R_p(z)^{[3\lfloor y,z\rfloor+2]} \subseteq R_p(y)^{[3\lfloor y,z\rfloor+2]}$$

as partial functions.

OBSERVATION 2.10. *For every* x, *the set of* p *which are nice at* x *is dense.*

PROOF. Use Lemma 2.5. Given y, z with $y\ |_{\mathcal{J}}\ z$ and $y \leq_{\mathcal{J}} x \not\geq_{\mathcal{J}} z$, extend $R_p(y)$ by adding 0's to its $(3\lfloor y, z\rfloor + 2)$nd column at the same places where $R_p(z)^{[3\lfloor y,z\rfloor+2]}$ is defined. □

In the next lemma we need to consider $p \restriction \mathfrak{j}(x)$ even if $\mathfrak{j}(x)$ is undefined. In that case define $p \restriction \mathfrak{j}(x) = p \restriction x \cup \bigcup\{p \restriction \mathfrak{j}(y) : y \leq_{\mathcal{J}} x \ \&\ \mathfrak{j}(y)\downarrow\}$. Where $\cup$ is the union of compatible partial functions.

LEMMA 2.11. *For all $p, q \in \mathbb{P}$ and $x \in J$ such that p is nice at x, we have that*

$$q \leq p \restriction \mathfrak{j}(x) \Longrightarrow q \restriction x \cup p \in \mathbb{P}.$$

PROOF. Let $r = q \restriction x \cup p$. We have to check that all the conditions in the definition of $\mathbb{P}$ are satisfied by r.

Condition 1: $R_r(y)^{[0]} \subseteq R_p(y)^{[0]} \cup R_q(y)^{[0]} \subset H(y)$ as partial functions.

Condition 2: Consider $y \in J$ such that $\mathfrak{j}(y)\downarrow$. We want to show that $Sk_r(y)$ is part of a Skolem function for a jump coding of $R_r(\mathfrak{j}(y))$ in the 2nd column of $R_r(y)$. There are three possible cases: $\mathfrak{j}(y) \leq_{\mathcal{J}} x$; $y \leq_{\mathcal{J}} x$ but $\mathfrak{j}(y) \not\leq_{\mathcal{J}} x$; and $y \not\leq_{\mathcal{J}} x$. If $\mathfrak{j}(y) \leq_{\mathcal{J}} x$, then, since $r \restriction x = q \restriction x$, the condition holds because it does at q. If $y \not\leq_{\mathcal{J}} x$, then the condition holds because it does at p. So, suppose that $y \leq_{\mathcal{J}} x$ but $\mathfrak{j}(y) \not\leq_{\mathcal{J}} x$. We have that $Sk_r(y) = Sk_p(y)$. So, whenever $Sk_r(y,n)\downarrow = k$, $Sk_p(y,n)\downarrow = Sk_q(y,n) = k$, because $q \leq p \restriction \mathfrak{j}(x)$. Then, $R_p(\mathfrak{j}(y), n)\downarrow = R_q(\mathfrak{j}(y), n) = R_r(\mathfrak{j}(y), n)$. Now, if $R_r(y, \langle 2, n, m\rangle)\downarrow = l$ for some $m \geq k$, then $l = R_r(\mathfrak{j}(y), n)$.

Condition 3: Suppose that $y <_{\mathcal{J}} z$ and we want to check that $R_r(z)$ is compatible with lazy coding of $R_r(y)$ in the $(3y)$th column. If $z \leq_{\mathcal{J}} x$, then everything works fine, because it does at q. Otherwise, $R_r(z) = R_p(z)$. So, if for some n, i and k, $R_r(z, \langle 3y, n, i\rangle)\downarrow = k \neq 0$, then $R_p(z, \langle 3y, n, i\rangle)\downarrow = k \neq 0$; and therefore, $R_r(y, n) = R_p(y, n)\downarrow = k - 1$.

Condition 4: Suppose that $\mathfrak{j}(y) = z$ and we want to check that $R_r(z)$ is compatible with lazy coding of $Sk_r(y)$ in the $(3y+1)$th column. If $z \leq_{\mathcal{J}} x$, then everything works fine, because it does at q. Otherwise, $R_r(z) = R_p(z)$. So, if for some n, i and k, $R_r(z, \langle 3y+1, n, i\rangle)\downarrow = k \neq 0$, then $R_p(z, \langle 3y+1, n, i\rangle)\downarrow = k \neq 0$; and therefore, $Sk_r(y, n) = Sk_p(y, n)\downarrow = k - 1$.

Condition 5: Suppose that $y \mid_{\mathcal{J}} z$ and that $y \cup z$ is defined. If both y and z are $\leq_{\mathcal{J}} x$ or neither of them is, then the condition holds: in the former case because it holds at q, and in the later case because it does at p. So assume that $y \leq_{\mathcal{J}} x \not\geq_{\mathcal{J}} z$. Also assume that for some m

$$R_r(y, \langle 3\lfloor y, z\rfloor + 2, m\rangle)\downarrow = R_r(z, \langle 3\lfloor y, z\rfloor + 2, m\rangle)\downarrow = k \neq 0$$

then $R_p(z, \langle 3\lfloor y, z\rfloor + 2, m\rangle)\downarrow = k$ too, because $R_r(z) = R_p(z)$. Thus, we also have that $R_p(y)(\langle 3\lfloor y, z\rfloor + 2, m\rangle)\downarrow$, because p is nice at x. Necessarily $R_p(y)(\langle 3\lfloor y, z\rfloor + 2, m\rangle) = k$. Therefore $R_p(z \cup x)(m)\downarrow = k + 1$, and then $R_r(z \cup x)(m)\downarrow = k + 1$ too. ☐

COROLLARY 2.12. *For all $p \in \mathbb{P}$ and $x \in J$, $p \restriction x \in \mathbb{P}$.*

PROOF. Observe that the empty condition, $\emptyset$, is nice at x and that $p \leq \emptyset \restriction \mathfrak{j}(x) = \emptyset$. Therefore $p \restriction x = p \restriction x \cup \emptyset \in \mathbb{P}$. ☐

COROLLARY 2.13. *If $y \not\leq_{\mathcal{J}} x$, then $R_G(y) \not\leq_T R_G(x)$.*

PROOF. Suppose, toward a contradiction, that for some $p \in G$, $p \Vdash \{e\}^{R_G(x)} = R_G(y)$, where $\{e\}$ is the eth Turing functional and $\Vdash$ is the strong forcing relation, $\Vdash^*_{\mathbb{P}}$, as defined in [SW]. Moreover, by observation 2.10, we can assume that p is nice at x. Let n be of the form $\langle 1, m\rangle$ such that $R_p(y, n)\uparrow$. (Remember that the 1st column is the one that is not coding anything.) Let $q \leq p$ be such that $q \Vdash \{e\}^{R_G(x)}(n) = i$ for some $i \in \omega$. Let $r = q\restriction x \cup p$. Since $q \leq p \leq p\restriction \mathfrak{j}(x)$, we have that $r \in \mathbb{P}$ by the previous lemma. Then, since $R_r(x) = R_q(x)$, $r \Vdash \{e\}^{R_G(x)}(n) = i$. But $R_r(y,n)$ is undefined. Extend r to r^* by setting $R_{r^*}(y,n) = i+1$. From Lemma 2.5 we get that $r^* \in \mathbb{P}$. Then, $r^* \Vdash \{e\}^{R_G(x)}(n) \neq R_G(y)(n)$, contradicting our assumption about p. □

2.3. R_G **preserves the jump.** Fix $x \in J$ such that $\mathfrak{j}(x)$ is defined. We have already proved that $R_G(\mathfrak{j}(x)) \leq_T R_G(x)'$. Now, we want to prove that $R_G(x)' \leq_T R_G(\mathfrak{j}(x))$.

We start by studying the complexity of the statement "p decides $\{e\}^{R_G(x)}(e)\downarrow$".

Definition 2.14. 1. $\mathbb{P}_x = \overline{\mathbb{P}}_x \cap \mathbb{P}$;
2. $\mathbb{P}_{p,x} = \{q\restriction x : q \in \mathbb{P}\ \&\ q \leq p\}$;
3. $G\restriction x = G \cap \mathbb{P}_x$.

Three easy facts about $\mathbb{P}_x$ that we are going to use are:

Observation 2.15. • $\mathbb{P}_{p,x} \subseteq \mathbb{P}_x$.
- *$\mathbb{P}_x$ is recursive in $H(x)$.*
- *Every $p \in \mathbb{P}_x$ is nice at x.*

We are going to show later that some $p \in G\restriction \mathfrak{j}(x)$ decides $\{e\}^{R_G(x)}(e)\downarrow$. First we study the complexity of $G\restriction \mathfrak{j}(x)$.

Lemma 2.16. *$G\restriction x \equiv_T R_G(x)$.*

Proof. Clearly $R_G(x) \leq_T G\restriction x$. Now we prove that $G\restriction x \leq_T R_G(x)$. Observe that $\bar{q} \in \mathbb{P}_x$ is in $G\restriction x$ iff for all y, $R_{\bar{q}}(y) \subset R_G(y)$ and $Sk_{\bar{q}}(y) \subset Sk_G(y)$. We observed in 2.15 that $\mathbb{P}_x \leq_T H(x)$, and by corollary 2.7, $H(x) \leq_T R_G(x)$, $\forall y \in J_x(R_G(y) \leq_T R_G(x))$ and $\forall y \in J_x^{\mathfrak{j}}(Sk_G(y) \leq_T R_G(x))$, uniformly in y. □

Lemma 2.17. *For p nice at x,*
1. *$\mathbb{P}_{p,x} = \mathbb{P}_{p\restriction \mathfrak{j}(x),x} = \{\bar{q} \in \overline{\mathbb{P}}_x : \bar{q} \leq p\restriction x\ \&\ \bar{q} \cup p \in \mathbb{P}\}$;*
2. *$\mathbb{P}_{p,x}$ is recursive in $H(x)$ uniformly in p.*

Proof. We shall prove that

$$\mathbb{P}_{p,x} \subseteq \mathbb{P}_{p\restriction \mathfrak{j}(x),x} \subseteq \{\bar{q} \in \overline{\mathbb{P}}_x : \bar{q} \leq p\restriction x\ \&\ \bar{q} \cup p \in \mathbb{P}\} \subseteq \mathbb{P}_{p,x}.$$

Since $p \leq p\restriction \mathfrak{j}(x)$, we have that $\mathbb{P}_{p,x} \subseteq \mathbb{P}_{p\restriction \mathfrak{j}(x),x}$. Now consider $\bar{q} \in \mathbb{P}_{p\restriction \mathfrak{j}(x),x}$. There is some $q \leq p\restriction \mathfrak{j}(x)$ such that $\bar{q} = q\restriction x$. From Lemma 2.11, we get that $q\restriction x \cup p \in \mathbb{P}$. So $\bar{q} \in \{\bar{q} \in \overline{\mathbb{P}}_x : \bar{q} \leq p\restriction x\ \&\ \bar{q} \cup p \in \mathbb{P}\}$. Now consider $\bar{q} \in \overline{\mathbb{P}}_x$ such that $\bar{q} \leq p\restriction x$ and $r = \bar{q} \cup p \in \mathbb{P}$. Clearly $r \leq p$ and $r\restriction x = \bar{q}$, so $\bar{q} \in \mathbb{P}_{p,x}$. This proves the first part.

For second part, given $\bar{q} \in \overline{\mathbb{P}}_x$, we want to decide, recursively in $H(x)$, whether $\bar{q} \in \mathbb{P}_{p.x}$. Note that checking if $\bar{q} \leq p\restriction x$ is clearly recursive, uniformly in p. To check if $r = \bar{q} \cup p \in \mathbb{P}$ one has to check the conditions in the definition of $\mathbb{P}$. All but the first condition, can be checked recursively in $\mathcal{J}$. For the first condition we already have that, for $y \not\leq_{\mathcal{J}} x$, $R_r(y)^{[0]} = R_p(y)^{[0]} \subseteq H(y)$. So we only have to check if $\forall y \leq_{\mathcal{J}} x(R_r(y)^{[0]} \subseteq H(y))$, which we can do recursively in $H(x)$. □

Lemma 2.18. *For p nice at x, and $e \in \omega$:*
1. *The following are equivalent:*

(a) $p \Vdash \{e\}^{R_G(x)}(e)\uparrow$;
(b) $p \restriction \mathfrak{j}(x) \Vdash \{e\}^{R_G(x)}(e)\uparrow$;
(c) $p \restriction x \Vdash_{\mathbb{P}_{p,x}} \{e\}^{R_G(x)}(e)\uparrow$.

2. *Whether p decides $\{e\}^{R_G(x)}(e)\downarrow$ can be decided recursively in $H(x)'$, uniformly in p and e. Moreover, if p decides $\{e\}^{R_G(x)}(e)\downarrow$, we can also tell whether p forces $\{e\}^{R_G(x)}(e)\downarrow$ or its negation.*

PROOF. By definition of forcing we have that, $p \Vdash \{e\}^{R_G(x)}(e)\uparrow$ if and only if

$$\forall q \leq p \forall s \in \omega(\{e\}_s^{R_q(x)}(e)\uparrow).$$

This is equivalent to

$$\forall \bar{q} \in \mathbb{P}_{p,x} \forall s \in \omega(\{e\}_s^{R_{\bar{q}}(x)}(e)\uparrow), \tag{2.1}$$

which, by definition of $\Vdash_{\mathbb{P}_{p,x}}$, is equivalent to $p \restriction x \Vdash_{\mathbb{P}_{p,x}} \{e\}^{R_G(x)}(e)\uparrow$. We have shown that (1a) is equivalent to (1c). We get that (1b) is equivalent to (1c) in the same way because $\mathbb{P}_{p,x} = \mathbb{P}_{p \restriction \mathfrak{j}(x),x}$. Whether $p \Vdash \{e\}^{R_G(x)}(e)\downarrow$, can be decided recursively, because we only have to check if $\{e\}^{R_p(x)}(e)\downarrow$. Whether $p \Vdash \{e\}^{R_G(x)}(e)\uparrow$, is a $\Pi_1^{H(x)}$ question as shown in (2.1), so $H(x)'$ can answer it. □

COROLLARY 2.19. *if $\mathfrak{j}(x)\downarrow$, then $R_G(x)' \equiv_T R_G(\mathfrak{j}(x))$.*

PROOF. We showed that $R_G(\mathfrak{j}(x)) \leq_T R_G(x)'$ in corollary 2.7. Now we compute $R_G(x)'$ from $R_G(\mathfrak{j}(x))$. Consider $e \in \omega$. Find $p \in G \restriction \mathfrak{j}(x)$ such that p decides $\{e\}^{R_G(x)}(e)\downarrow$. By Lemma 2.16, $G \restriction \mathfrak{j}(x) \leq_T R_G(\mathfrak{j}(x))$, and by Lemma 2.18, $H(x)'$ knows whether p decides $\{e\}^{R_G(x)}(e)\downarrow$. Since H is a jump hierarchy, $H(x)' \leq_T H(\mathfrak{j}(x)) \leq_T R_G(\mathfrak{j}(x))$. So, we can find such a p recursively in $R_G(\mathfrak{j}(x))$. We can also tell whether p forces $\{e\}^{R_G(x)}(e)\downarrow$ or its negation; in the former case we get that $R_G(x)'(e) = 1$ and in the later that $R_G(x)'(e) = 0$. □

This finishes the proof of Theorem 2.2.

§3. Decidability results. As a corollary of Theorem 2.2, we prove that the existential theory of the Turing degrees with $\leq_T$, join and jump is decidable.

Proposition 3.2 is stronger than what we actually need to prove decidability, but we shall use it again later. To prove it we need the following lemma.

LEMMA 3.1. *Given a recursive well founded partial ordering $\mathcal{P}$ of rank α, and a recursive presentation, $\mathcal{A}$, of α, the usual rank map,* $\mathrm{rk} : \mathcal{P} \to \mathcal{A}$, *is recursive in $0^{2\alpha+2}$.*

SKETCH OF THE PROOF. We claim that there is a recursive function f such that, for $\beta < \alpha$, $f(\beta)$ is a $0^{2\beta+2}$-index for the function $\varphi_\beta(x)$ that answers whether $\mathrm{rk}(x) \geq \beta$. The definition of f is by transfinite recursion using that $\mathrm{rk}(x) \geq \beta$ iff for all $\gamma < \beta$, there exists $y \in \mathcal{P}$ such that $y < x$ & $\mathrm{rk}(y) \geq \gamma$. So, $\{f(\beta)\}^{0^{2\beta+2}}(x) = \mathtt{yes}$ if and only if

$$\forall \gamma < \beta \; \exists y \in \mathcal{P}\big(y < x \;\&\; \{f(\gamma)\}^{0^{2\gamma+2}}(y) = \mathtt{yes}\big). \qquad \square$$

PROPOSITION 3.2. *Every well founded partial ordering supports a jump hierarchy.*

Proof. Let $\mathcal{P} = \langle P, \leq_P \rangle$ be a well founded partial ordering. Assume that $\mathcal{P}$ is recursive. Otherwise relativize the proof to the degree of $\mathcal{P}$.

Let $\mathrm{rk}(\mathcal{P})$ be the rank of $\mathcal{P}$ and $\beta = 2\,\mathrm{rk}(\mathcal{P}) + 2$. Let $\{H_a\}_{a \in O}$ be the hyperarithmetic hierarchy, where O is the set of ordinal notations (see [Sac90]). Fix an initial segment of O of length $\beta + \mathrm{rk}(\mathcal{P})$, and think of the ordinals below $\beta + \mathrm{rk}(\mathcal{P})$ as elements of that segment of O. For $x \in P$, let $\mathrm{rk}(x)$ be the usual rank of x in $\mathcal{P}$. Now, for each $x \in P$ define

$$K(x) = H_{\beta + \mathrm{rk}(x)}$$

Clearly $\mathcal{P} \leq_T K(x)$ for all $x \in P$. We get that $x <_T y$ implies $K(x)' \leq_T K(y)$ because $x <_P y$ implies that $\mathrm{rk}(x) < \mathrm{rk}(y)$. We get that $\bigoplus_{x \leq_P y} K(x) \leq_T K(y)$ because given $\langle x, m \rangle$ with $x \leq_P y$ we can compute $\mathrm{rk}(x)$ recursively in H_β, and then compute $H_{\mathrm{rk}(x)}(m)$. Therefore, K is a jump hierarchy over $\mathcal{P}$. □

Remark 3.3. Moreover, for every $X \subseteq \omega$, every well founded partial ordering, $\mathcal{P}$, supports a jump hierarchy, K, such that $\forall x \in P (X \leq_T K(x))$. The construction is the same as above, but now relativize to $X \oplus \mathcal{P}$.

Corollary 3.4. *Every finite pjusl can be embedded into $\mathcal{D}$.*

Proof. Every finite pjusl is well founded, so it supports a jump hierarchy. Therefore, by Theorem 2.2, it can be embedded into $\mathcal{D}$. □

Theorem 3.5. *The $\exists$-theory of $\mathcal{D} = \langle \mathbf{D}, \leq_T, \vee, ' \rangle$ is decidable.*

Proof. Consider an existential sentence φ in the language of $\mathcal{D}$. It is equivalent to a disjunction of sentences of the form

$$\exists x_1 ... \exists x_n (\psi_1 \;\&\; ... \;\&\; \psi_m), \tag{3.1}$$

where each ψ_i has one of the following forms: $x_{j_1} \leq_T x_{j_2}$, $x_{j_1} \not\leq_T x_{j_2}$, $x_{j_1} \neq x_{j_2}$, $x'_{j_1} = x_{j_2}$, or $x_{j_1} \vee x_{j_2} = x_{j_3}$. We have to decide whether one of these disjuncts holds in $\mathcal{D}$. So, suppose that φ is the formula in (3.1). We claim that $\mathcal{D} \models \varphi$ if and only if φ holds in some pjusl with at most n elements. If $\mathcal{D} \models \varphi$, then the degrees $\mathbf{x}_1, ..., \mathbf{x}_n$ which witness φ form the desired pjusl. If $\mathcal{J} \models \varphi$, for some pjusl $\mathcal{J}$ with at most n elements, then, since we can embed $\mathcal{J}$ into $\mathcal{D}$, we have that $\mathcal{D} \models \varphi$. Clearly we can recursively check whether φ holds in some pjusl with at most n elements. □

§4. Jump upper semilattices which support Jump Hierarchies.

Now we show how to embed any countable jusl into one which supports a jump hierarchy.

This section is divided into five subsections. First we show how to define a Harrison Linear Ordering in such a way that we have recursive operations of addition and multiplication. In subsection 4.2 we define, for each $\alpha < \omega_1^{CK}$, a pjusl $\mathcal{P}_\alpha$ which supports a jump hierarchy, and we show that any pjusl with a certain property can be embedded in $\mathcal{P}_\alpha$. In subsection 4.4 we show that every recursive jusl has that property. But first we need to prove that every finitely generated pjusl is well quasiordered (we define well quasiorderings in 4.9); we do this in subsection 4.3. In the last subsection we put all the pieces together and prove that every countable jusl embeds into $\mathcal{D}$.

4.1. Pseudo-well orderings with Jump Hierarchies. In [Har68], Harrison proved that there is a recursive linear ordering of type $\omega_1^{CK} \cdot (1+\eta)$, (i.e: ω_1^{CK} followed by η copies of ω_1^{CK}, where η is the order type of the rational numbers.) which supports a jump hierarchy. Here we show that we can get such a linear ordering also having recursive addition and multiplication. These operations should have the same properties as ordinal addition and multiplication.

DEFINITION 4.1. A *chain of structures of length* α is a sequence $\langle A_i : i < \alpha \rangle$ of structures together with a set of embeddings $\{\varphi_{ij} \colon A_i \to A_j\}_{i<j<\alpha}$, such that $\forall i < j < k < \alpha(\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik})$.

A *recursive chain of length* α $(< \omega_1^{CK})$ is a chain where $\langle A_i : i <_O a\rangle$ and $\langle \varphi_{ij} : i <_O j <_O a\rangle$ are uniformly recursive, and a is an ordinal notation for α.

LEMMA 4.2. *Let $\langle\langle A_i : i <_O a\rangle, \langle \varphi_{ij} : i <_O j <_O a\rangle\rangle$ be a recursive chain. Its direct limit, A_a, and the set of embeddings $\varphi_{ia} \colon A_i \to A_a$, for $i <_O a$, are uniformly recursive. Furthermore, indices for A_a and $\langle \varphi_{ia} \colon A_i \to A_a : i <_O a\rangle$ can be found recursively from an index for $\langle\langle A_i : i <_O a\rangle, \langle \varphi_{ij} : i <_O j <_O a\rangle\rangle$.*

(See [Hod93, page 50] for a general definition of direct limits.)

SKETCH OF THE PROOF. One just has to observe that the usual construction of direct limits is uniformly recursive. What one does is to consider the disjoint union of the A_i:

$$B = \bigcup_{i<_O a} A_i \times \{i\},$$

and define an equivalence relation in B:

$$(x,i) \sim (y,j) \iff y = \varphi_{ij}(x)$$

for $i \leq_O j$. If $i >_O j$ say that $(x,i) \sim (y,j) \iff (y,j) \sim (x,i)$. This equivalence relation is clearly recursive. So $B/{\sim}$ is recursive: for each equivalence class take the element with least index as its representative. It is also easy to see that all the operations on $B/{\sim}$ and the embeddings φ_{ia} are recursive too. □

LEMMA 4.3. *For every $\alpha < \omega_1^{CK}$ there is a recursive well ordering, of order type at least α, in which the operations of addition and multiplications are recursive.*

SKETCH OF THE PROOF. For each $a \in O$ we shall define a recursive chain, c_a, of length $|a|$. c_a consists of recursive well orderings with addition and multiplication such that, for all $i <_O a$, the ith well ordering in the chain has order type at least $|i|$. We also want that if $a <_O b$, c_a is included in c_b. We shall use transfinite recursion. For $|a| = 1$, set c_a to be a chain with only one element consisting of ω with its usual addition and multiplication. If $a = 3 \cdot 5^e$ and we are given $c_{\{e\}(n)}$ for all $n \in \omega$, define c_a to be the union of all the $c_{\{e\}(n)}$. Now suppose that $a = 2^b$ and we are given c_b. If $|b|$ is a limit ordinal, extend c_b by adding its direct limit at the end. We can do this uniformly by the previous lemma. The last case is when $a = 2^b$ and $b = 2^d$, for some $d \in O$. Let l_d be the last well ordering in the chain c_b ($l_d = c_b(d)$). We shall construct a well ordering, l_b, with addition and multiplication, extending l_d. Then we define c_a by putting

l_b at the end of c_b. Let β be a new symbol. (β represents the order type of l_d.) Define l_b as a set of formal sums as follows:

$$l_b = \{\sum_{i=0}^{n} \beta^i x_i : n < \omega, x_i \in l_d, x_n \neq 0\}.$$

The order relation and the addition operation are defined in the obvious way. Define multiplication as follows:

$$(\sum_{i=0}^{n} \beta^i x_i) \cdot (\sum_{j=0}^{m} \beta^j y_j) = \sum_{j=0}^{m} \beta^{n+j} y_i.$$

Is not hard to prove that l_b is a well ordering and that the multiplication defined this way is the usual ordinal multiplication. It is also clear that l_b is recursive. Embed l_d into l_b by mapping x to $\beta^0 x$. $\square$

THEOREM 4.4. *There is a structure $\mathcal{L} = \langle L, \leq, +, \cdot\rangle$ which supports a jump hierarchy, $\mathcal{H}$, such that: $\langle L, \leq\rangle$ is a recursive linear ordering of order type $\omega_1^{CK} \cdot (1 + \eta)$; $+$ and $\cdot$ are recursive and satisfy the axioms of ordinals addition and multiplication; and for all $x \in L$, $\mathcal{H}(x)$ computes every hyperarithmetic set.*

SKETCH OF THE PROOF. We want to get $\langle \mathcal{L}, \mathcal{H}\rangle$ satisfying:

- $\langle L, \leq\rangle$ is a recursive linear ordering;
- for all $a \in O$, there is an x such that the set of predecessors has order type $|a|$;
- for all $a \in O$, there is no infinite descending sequence in $\mathcal{L}$ computable form 0^a;
- $+$ and $\cdot$ are recursive and satisfy the axioms of the inductive definition of addition and multiplication of ordinals;
- $\mathcal{H}$ is a jump hierarchy, and, for all $a \in O$, 0^a is recursive in $\mathcal{H}(x)$ for all $x \in L$.

(We write 0^a for the set corresponding to a in the hyperarithmetical hierarchy. Sometimes we shall write 0^α meaning 0^a for some a, in some fixed path through O, such that $|a| = \alpha$.)

All these axioms can be expressed by a Π^1_1 set, Γ, of computable infinitary formulas as in [AK00]. By the Barwise-Kreisel compactness theorem, as stated in [AK00, Theorem 8.3], if we prove that every Δ^1_1 subset of Γ has a model, then so does Γ. Any Δ^1_1 subset, Λ, of Γ will mention only a Δ^1_1 subset of O, so there has to be a $\beta < \omega_1^{CK}$, which bounds the norm of all of these ordinal notations (see [AK00, Proposition 5.20]). Then, by the previous lemma, we can always get a recursive well ordering with addition and multiplication of length at least β. The hyperarithmetical hierarchy, starting at 0^γ, and going up to $0^{\gamma+\beta}$, would be a jump hierarchy on it, where $\gamma = 2\beta + 2$, as in Proposition 3.2. This well ordering satisfies Λ. So we have that Γ has a model. Harrison proved in [Har68], that every recursive linear ordering with no hyperarithmetic descending sequences has order type either β or $\omega_1^{CK} \cdot (1 + \eta) + \beta$ for some $\beta < \omega_1^{CK}$. The second set of conditions rules out the first case, and the fact that $\mathcal{L}$ is closed under addition makes $\beta = 0$ the only possibility. $\square$

4.2. Partial upper semilattices with level function. Now, we shall construct, for each $\alpha < \omega_1^{CK}$, a pjusl $\mathcal{P}_\alpha$ which supports a jump hierarchy. To define a jump hierarchy on $\mathcal{P}_\alpha$, we assign to each element of $\mathcal{P}_\alpha$ a member of $\mathcal{L}$, where $\mathcal{L}$ is defined in Theorem 4.4.

We work with the following kind of structures.

DEFINITION 4.5. A *partial jump upper semilattice with levels in* $\mathcal{L}$ is a pjusl $\mathcal{J}$ together with a map $\mathrm{lev}\colon J \to \mathcal{L}$ which preserves strict order. (i.e. $x <_{\mathcal{J}} y \Longrightarrow \mathrm{lev}(x) < \mathrm{lev}(y)$.)

Fix $\alpha < \omega_1^{CK}$. Let $\mathcal{K}_\alpha$ be the set of finitely generated pjusl $\mathcal{J}$ with levels in $\mathcal{L}$ which are arithmetic in 0^α and such that $\forall x \in J(\mathfrak{j}(x)\downarrow)$.

LEMMA 4.6. *$\mathcal{K}_\alpha$ has the Uniform Amalgamation Property. i.e: Given A, A_1 and $A_2 \in \mathcal{K}_\alpha$ and embeddings $\varphi_1\colon A \to A_1$ and $\varphi_2\colon A \to A_2$, there are a $C \in \mathcal{K}_\alpha$ and embeddings $\psi_1\colon A_1 \to C$ and $\psi_2\colon A_2 \to C$ such that $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$. Moreover, indices for C, ψ_1 and ψ_2 can be found recursively from indices for A, A_1, A_2, φ_1 and φ_2.*

(An index for an embedding only has to code the embedding restricted to the finitely many generators.)

PROOF. Let $\bar{A}_1 = A_1 \setminus \varphi_1[A]$ and $\bar{A}_2 = A_2 \setminus \varphi_2[A]$. Define the domain of C to be the disjoint union of A, $\bar{A}_1$ and $\bar{A}_2$. Define the embeddings ψ_1 and ψ_2 in the obvious way. Define the jump, join, level and the order relation in $A \cup \bar{A}_1$ as induced by A_1, and in $A \cup \bar{A}_2$ as induced by A_2. Do not define the join between elements of $\bar{A}_1$ and $\bar{A}_2$. (Here is where it is useful to work with pjusl and not with jusl.) To make $\leq$ transitive, define, for $x \in \bar{A}_1$ and $y \in \bar{A}_2$,

$$x \leq y \iff \exists z \in A(x \leq \varphi_1(z) \ \&\ \varphi_2(z) \leq y)$$

and

$$x \geq y \iff \exists z \in A(x \geq \varphi_1(z) \ \&\ \varphi_2(z) \geq y).$$

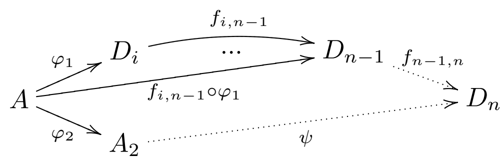
It is not hard to verify that we obtain a partial ordering. We also have to show that what we get is actually a pjusl. The properties for join and level are easily verified too. Let us verify that if $x \in \bar{A}_1$, $y \in \bar{A}_2$, $\mathfrak{j}(x)$ and $\mathfrak{j}(y)$ are defined and $x \leq y$, then $\mathfrak{j}(x) \leq \mathfrak{j}(y)$. Since $x \leq y$, there exists $z \in A$ such that $x \leq \varphi_1(z)$ & $\varphi_2(z) \leq y$. Therefore $\mathfrak{j}(x) \leq \varphi_1(\mathfrak{j}(z))$ and $\varphi_2(\mathfrak{j}(z)) \leq \mathfrak{j}(y)$. So $\mathfrak{j}(x) \leq \mathfrak{j}(y)$. □

Now we shall consider $\mathcal{P}_\alpha$, the Fraïssé limit of $\mathcal{K}_\alpha$ (see [Hod93]). We construct $\mathcal{P}_\alpha$ is such a way that it is recursive in $0^{\alpha+\omega}$.

CONSTRUCTION OF $\mathcal{P}_\alpha$. Enumerate all the tuples $\langle A, A_1, A_2, \varphi_1, \varphi_2\rangle$ such that A, A_1, $A_2 \in \mathcal{K}_\alpha$ and φ_1 and φ_2 are embeddings from A to A_1 and to A_2 respectively. (Actually, enumerate the tuples of indices.) We can get such an enumeration recursively in $0^{\alpha+\omega}$. We shall construct a sequence $\langle D_i : i < \omega\rangle$ together with embeddings $f_{ij}\colon D_i \to D_j$ recursively in $0^{\alpha+\omega}$.

Let $D_0 = \emptyset$. Now, suppose we have defined D_i for all $i < n$. Take the first tuple $\langle A, A_1, A_2, \varphi_1, \varphi_2\rangle$ from the list, not already taken, such that A_1 is equal to some D_i, $i < n$. Using Lemma 4.6, as in the diagram below, construct

$D_n \in \mathcal{K}_\alpha$, and embeddings $f_{n-1,n} \colon D_{n-1} \to D_n$ and $\psi \colon A_2 \to D_n$ such that $f_{n-1,n} \circ f_{i,n-1} \circ \varphi_1 = \psi \circ \varphi_2$. For $j < n$, let $f_{j,n} = f_{n,n-1} \circ f_{j,n-1}$.



Let $\mathcal{P}_\alpha$ be the direct limit of the chain constructed. By Lemma 4.2 relativized to $0^{\alpha+\omega}$, we can get $\mathcal{P}_\alpha \leq_T 0^{\alpha+\omega}$. ◇

Lemma 4.7. *$\mathcal{P}_\alpha$ supports a Jump Hierarchy.*

Proof. $\mathcal{P}_\alpha$ is a pjusl with a level function to $\mathcal{L}$ and $\mathcal{L}$ supports a jump hierarchy, $\mathcal{H}$. So, for each $x \in \mathcal{P}_\alpha$, we can define

$$R(x) = \mathcal{H}(\mathrm{lev}(x)).$$

We claim that R is a jump hierarchy over $\mathcal{P}_\alpha$. Since $\mathcal{P}_\alpha$ is hyperarithmetic, we have that $\mathcal{P}_\alpha \leq_T R(x)$, for all $x \in \mathcal{P}_\alpha$. We also have that

$$\bigoplus_{x \leq y} R(x) \leq_T R(y),$$

because, given $x \leq y$, we can compute $\mathrm{lev}(x)$ recursively in $R(y)$, and then, compute $\mathcal{H}(\mathrm{lev}(y))$. The third thing that needs to be verified is that $x < y$ implies $H(x)' \leq_T H(y)$. This is true because $\mathcal{H}$ is a jump hierarchy over $\mathcal{L}$ and lev preserves strict order. □

Lemma 4.8. *Let $\mathcal{J}$ be a pjusl with levels in $\mathcal{L}$ such that there is a sequence*

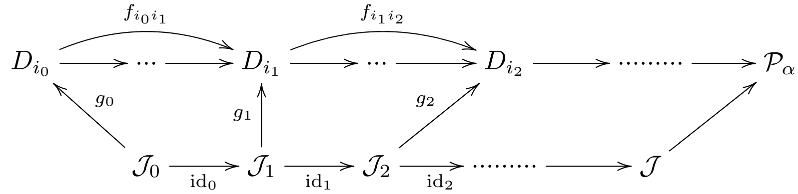
$$\mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq .... \subseteq \mathcal{J},$$

with $\mathcal{J} = \bigcup_{i<\omega} \mathcal{J}_i$, and for all i, $\mathcal{J}_i \in \mathcal{K}_\alpha$. Then $\mathcal{J}$ can be embedded in $\mathcal{P}_\alpha$.

Proof. We have constructed $\mathcal{P}_\alpha$ as the direct limit of $\langle D_i : i < \omega \rangle$ with embeddings f_{ij}. We shall get a subsequence $\{D_{i_k}\}_{k<\omega}$ such that for each k there is an embedding $g_k \colon J_k \to D_{i_k}$ such that if $k_1 \leq k_2$, then

$$f_{i_{k_1} i_{k_2}} \circ g_{k_1} = g_{k_2} \restriction \mathcal{J}_{k_1}.$$

This would imply that $\mathcal{J}$, the direct limit of $\langle \mathcal{J}_i : i < \omega \rangle$, embeds into $\mathcal{P}_\alpha$.



Let $\mathcal{J}_{-1} = \emptyset$, $i_{-1} = 0$ and $g_{-1} : \emptyset \to D_0$ be the empty map. Now suppose we have defined i_n and $g_n \colon \mathcal{J}_n \to D_{i_n}$. Consider the tuple $\langle \mathcal{J}_n, D_{i_n}, \mathcal{J}_{n+1}, g_n, \mathrm{id}_n \rangle$, where id_n is the inclusion map $\mathcal{J}_n \hookrightarrow \mathcal{J}_{n+1}$. Eventually, say at step i_{n+1}, this tuple is going to be considered in the construction of $\mathcal{P}_\alpha$. So, $D_{i_{n+1}}$ is going to be defined, together with a map $g_{n+1}\colon J_{n+1} \to D_{i_{n+1}}$, so that $f_{i_n i_{n+1}} \circ g_n = g_{n+1} \restriction \mathcal{J}_n$. □

4.3. Well quasiorderings. Now we move into the direction of proving that every recursive pjusl embeds in some $\mathcal{P}_\alpha$.

DEFINITION 4.9. A *well quasiordering* is a set Q together a transitive and reflexive relation $\leq$ such that for every sequence $\{x_i\}_{i\in\omega}$, there are $i < j$ with $x_i \leq x_j$.

OBSERVATION 4.10. 1. *A partial ordering which is well quasiordered, is well founded.*
2. *The image of a well quasiordering under an order preserving map is well quasiordered too.*

PROOF. The first observation is trivial. For the second one consider: Q, a well quasiordering; $f\colon Q \to P$, an order preserving map; and a sequence $\{x_i\}_{i<\omega} \subseteq P$. Let $\{y_i\}_{i<\omega} \subseteq Q$ be such that for all i, $f(y_i) = x_i$. There exist $i < j$ with $y_i \leq y_j$. Then $x_i \leq x_j$. □

DEFINITION 4.11. 1. Given a set F of variables, let $\mathcal{T}_F$ be the set of terms over the language with j, $\cup$, and variables from F.
2. For $t \in \mathcal{T}_F$, the *Jump Rank* of t is defined by recursion:

$$\operatorname{jrk}(t) = \begin{cases} 0 & \text{if } t \text{ is a variable;} \\ \max(\operatorname{jrk}(t_1), \operatorname{jrk}(t_2)) & \text{if } t = t_1 \cup t_2; \\ \operatorname{jrk}(t_1) + 1 & \text{if } t = \mathsf{j}(t_1). \end{cases}$$

3. The *support* of t, $\operatorname{supp}(t)$, is the set of variables that actually occur in t.
4. For t with $\operatorname{supp}(t) \subseteq F$, we define the *Jump Rank of t over F* by recursion:

$$\operatorname{jrk}_F(t) = \begin{cases} -\infty & \text{if } \operatorname{supp}(t) \subset F; \\ 0 & \text{if } t \text{ is a variable } x_i \text{ and } F = \{x_i\}; \\ \max(\operatorname{jrk}_F(t_1), \operatorname{jrk}_F(t_2), 0) & \text{if } t = t_1 \cup t_2 \text{ and } \operatorname{supp}(t) = F; \\ \operatorname{jrk}(t_1) + 1 & \text{if } t = \mathsf{j}(t_1). \end{cases}$$

5. For terms $t_1(\bar{x})$ and $t_2(\bar{x})$, say that $t_1 \leq t_2$ if for every jusl U
$$U \models \forall \bar{x}\big(t_1(\bar{x}) \leq t_2(\bar{x})\big).$$
6. We say that t_1 is equivalent to t_2, and write $t_1 \equiv t_2$, if $t_1 \leq t_2$ and $t_2 \leq t_1$.

We shall write $\mathsf{j}^m(b)$ for $\overbrace{\mathsf{j}(\mathsf{j}(...\mathsf{j}}^{m}(x)..))$ and $\bigcup\{b_1, ..., b_n\}$ for $b_1 \cup b_2 \cup ... \cup b_n$.

LEMMA 4.12. *For every term $t \in \mathcal{T}_F$;*
1. $\bigcup \operatorname{supp}(t) \leq t$*;*
2. $t \leq \mathsf{j}^{\operatorname{jrk}(t)}(\bigcup F)$*;*
3. *if* $\operatorname{supp}(t) = F$ *then* $\mathsf{j}^{\operatorname{jrk}_F(t)}(\bigcup F) \leq t$.

PROOF. The first two parts are straightforward by induction on t. The third part can be proved by induction on $\operatorname{jrk}_F(t)$ as follows. If $\operatorname{jrk}_F(t) = 0$ then $\mathsf{j}^0(\bigcup F) = (\bigcup F) \leq t$ by the first part. Now suppose that $\operatorname{jrk}_F(t) > 0$. If $t = \mathsf{j}(t_1)$, then $\mathsf{j}^{\operatorname{jrk}_F(t)}(\bigcup F) = \mathsf{j}(\mathsf{j}^{\operatorname{jrk}_F(t_1)}(\bigcup F)) \leq \mathsf{j}(t_1) = t$ by inductive hypothesis. If

$t = t_1 \cup t_2$, then either $\mathrm{jrk}_F(t_1)$ or $\mathrm{jrk}_F(t_2)$ is equal to $\mathrm{jrk}_F(t)$. Say the first one. Then $\mathfrak{j}^{\mathrm{jrk}_F(t)}(\bigcup F) = \mathfrak{j}^{\mathrm{jrk}_F(t_1)}(\bigcup F) \leq t_1 \leq t$. □

LEMMA 4.13. *For finite F, $\mathcal{T}_F$ is a well quasiordering.*

PROOF. We use induction on $|F|$, so we can assume that $\mathcal{T}_G$ is a well quasiordering for every $G \subset F$. (Note that the empty set is well quasiordered.) Now consider a sequence $\{t_i\}_{i\in\omega} \subseteq \mathcal{T}_F$. We want to show that there are $i < j$, such that $t_i \leq t_j$. Let $m_0 = \mathrm{jrk}(t_0)$. If for some $i \neq 0$, $\mathrm{jrk}_F(t_i) \geq m_0$, we are done because, by Lemma 4.12,

$$t_0 \ \leq\ \mathfrak{j}^{m_0}(\textstyle\bigcup F) \ \leq\ \mathfrak{j}^{\mathrm{jrk}_F(t_i)}(\bigcup F) \ \leq\ t_i.$$

So, assume that there is some $m \in \omega$ such that for all i, $\mathrm{jrk}_F(t_i) < m$. Let $\mathcal{T}_{F,m} = \{t \in \mathcal{T}_F : \mathrm{jrk}_F(t) < m\}$. We shall prove, by induction on m, that $\mathcal{T}_{F,m}$ is well quasiordered. This will imply that there are $i < j$ as we want, and hence that $\mathcal{T}_F$ is a well quasiordering.

For $m = 0$ we have that

$$\mathcal{T}_{F,0} = \textstyle\bigcup\{\mathcal{T}_G : G \subset F\}.$$

It is not hard to see that a finite union of well quasiordering is a well quasiordering. So, since we are assuming that each $\mathcal{T}_G$ is well quasiordered, $\mathcal{T}_{F,0}$ is well quasiordered. Now assume that $\mathcal{T}_{F,m}$ is well quasiordered and consider $\{t_i\}_{i\in\omega} \subseteq \mathcal{T}_{F,m+1}$. Suppose, toward a contradiction, that for all $i < j$, $t_i \not\leq t_j$. There cannot be infinitely many terms in $T_{F,0}$ because of the base case we have just proved. If we eliminate the terms in $T_{F,0}$, we can assume that $\{t_i\}_{i\in\omega}$ is a sequence where all the terms have support F. First observe that every t_i in the sequence can be written, up to equivalence, as

$$\textstyle\bigcup_{j<r_i} \mathfrak{j}(s_{ij}) \cup \bigcup G_i,$$

where $G_i \subseteq F$ and $s_{ij} \in \mathcal{T}_{F,m}$. For each $i > 0$, since $t_0 \not\leq t_i$ and $\bigcup G_i \leq \bigcup F \leq t_i$, we have that for some $j < r_0$, $\mathfrak{j}(s_{0j}) \not\leq t_i$. Therefore

$$\exists j < r_0 \exists^\infty i \in \omega \big(\mathfrak{j}(s_{0j}) \not\leq t_i\big).$$

Let j_0 be one of those j's. Let $s_0 = s_{0j_0}$, and $I_0 = \{i \in \omega : \mathfrak{j}(s_0) \not\leq t_i\}$. Now consider i_1, the first element in I_0. For the same reason,

$$\exists j < r_{i_1} \exists^\infty i \in I_0 \big(\mathfrak{j}(s_{i_1 j}) \not\leq t_i\big).$$

Let j_1 be one of those j's. Let $s_1 = s_{i_1 j_1}$, and $I_1 = \{i \in I_0 : \mathfrak{j}(s_1) \not\leq t_i\}$. Repeat this procedure to get a sequence $\{s_i\}_{i\in\omega} \subseteq \mathcal{T}_{F,m}$ such that

$$\forall i < j \big(\mathfrak{j}(s_i) \not\leq \mathfrak{j}(s_j)\big).$$

But, by inductive hypothesis, there are $i < j$ such that $s_i \leq s_j$. Which implies that $\mathfrak{j}(s_i) \leq \mathfrak{j}(s_j)$. Contradiction. □

COROLLARY 4.14. *Every finitely generated pjusl is well quasiordered.*

PROOF. Every finitely generated pjusl, $\mathcal{J}$, is the image of a subset of $\mathcal{T}_F$, for some finite F, under an order preserving map. Therefore, since $\mathcal{T}_F$ is well quasiordered, so is $\mathcal{J}$ by observation 4.10. □

4.4. The decomposition of $\mathcal{J}$. Consider $\mathcal{J} = \langle J, \leq_{\mathcal{J}}, \cup, \mathsf{j}\rangle$, a recursive pjusl such that j is a total function. We want to show that we can define a level function to $\mathcal{L}$ on it and a sequence

$$\mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq .... \subseteq \mathcal{J},$$

with $\mathcal{J} = \bigcup_{i<\omega} \mathcal{J}_i$, such that for some $\alpha < \omega_1^{CK}$, $\mathcal{J}_i \in \mathcal{K}_\alpha$ for all $i \in \omega$.

Enumerate J as $\{a_0, a_1, ..., a_n, ...\}$. Let $\mathcal{J}_n = \langle a_i : i < n\rangle_{\mathcal{J}}$, the pjusl generated by $a_0, ..., a_{n-1}$. Let J_n be the domain of $\mathcal{J}_n$. Note that for each n, $\mathcal{J}_n \leq_T 0'$. Let $\preceq$ be a recursive linear ordering extending the ordering of $\mathcal{J}$. In other words, $\langle J, \preceq\rangle$ is a recursive linear ordering and $\leq_{\mathcal{J}} \subseteq \preceq$. A proof of the fact that every recursive partial ordering has a recursive linear extension can be found in [Dow98, Obs. 6.1].

Let $\preceq_n$ be $\preceq$ restricted to J_n. Since $\langle J_n, \leq_{\mathcal{J}}\rangle$ is well quasiordered, $\preceq_n$ is well quasiordered too. Since $\preceq_n$ is linear, it is actually a well ordering. Let γ be the supremum of the order types of $\preceq_n$, for $n < \omega$. We know that $\gamma < \omega_1^{CK}$ because $\langle \preceq_n : n < \omega\rangle$ is a an arithmetic sequence of well orderings. Think of γ as an initial segment of $\mathcal{L}$. The rank function of $\langle J_n, \leq_{\mathcal{J}}\rangle$, $\mathrm{rk}_{\preceq_n} : J_n \to \gamma$ is recursive in $0^{2\gamma+2}$ by Lemma 3.1. Let $\alpha = 2\gamma + 2$.

LEMMA 4.15. *There is a level function* $\mathrm{lev}\colon \mathcal{J} \to \mathcal{L}$ *such that for each* n, $\mathrm{lev} \restriction J_n$ *is recursive in* 0^α.

PROOF. To simplify the definitions, add to $\preceq$ an element, ∞, on top: Let $\bar{J}_n = J_n \cup \{\infty\}$ and for all $x \in J_n$ set $x \preceq \infty$. Together with lev we define a sequence $\{\sigma_n\}_{n\in\omega} \subseteq \mathcal{L}$ and for each $y \in J_n$ an element $b^n_y \in \mathcal{L}$. We require that each $\sigma_n \notin \omega_1^{CK}$ (we identify ω_1^{CK} with the initial segment of $\mathcal{L}$ of order type ω_1^{CK}), that

$$\forall x \in J_n (x < y \Longrightarrow \mathrm{lev}(x) < b^n_y),$$

and that

$$\mathrm{lev}(y) \geq b^n_y + \sigma_n.$$

CONSTRUCTION OF lev. The construction is done by recursion on n. For $n = 0$, we have that $J_0 = \{\infty\}$. Let $b^0_\infty = 0$ and let σ_0 be anything in $\mathcal{L} \smallsetminus \omega_1^{CK}$. Define $\mathrm{lev}(\infty) = \sigma_0$. Now suppose we have defined $\mathrm{lev}(y)$, and b^n_y for all $y \in J_n$, recursively in 0^α and the finite sequence $\langle \sigma_0, ..., \sigma_n\rangle$. Let σ_{n+1} be such that $\sigma_{n+1} \notin \omega_1^{CK}$ and $\sigma_{n+1} \cdot (\alpha + 1) < \sigma_n$. Such a σ_{n+1} exists because the set $\{\beta \in \mathcal{L} : \beta \cdot (\alpha + 1) < \sigma_n\}$ is recursive and contains ω_1^{CK}, but since ω_1^{CK} is not recursive, there is some σ_{n+1} in that set which is not in ω_1^{CK}. For $x \in J_{n+1} \smallsetminus J_n$, define:

$$\begin{aligned} \beta_x &= \mathrm{rk}_{\preceq_{n+1}}(x) \in \omega_1^{CK} \subset \mathcal{L}; \\ y_x &= \mu y \in J_n (x \prec y); \\ \mathrm{lev}(x) &= b^n_{y_x} + \sigma_{n+1} \cdot (\beta_x + 1); \\ b^{n+1}_x &= b^n_{y_x} + \sigma_{n+1} \cdot \beta_x. \end{aligned}$$

For $y \in J_n$, let $b^{n+1}_y = b^n_y + \sigma_{n+1} \cdot \alpha$. ◇

Since $\mathrm{rk}_{\preceq_{n+1}}(x)$ can be found recursively in 0^α, we can find β_x recursively in 0^α. Also note that $y = \mu y \in J_n(x \prec y)$ can be found recursively in $0''$, because such a y always exists and $\langle J_n, \preceq\rangle \leq_T 0'$. Is easy to verify, by induction on n, that the construction does what we want. □

4.5. Putting the pieces together.

PROPOSITION 4.16. *Let $\mathcal{J}$ be a countable pjusl such that its jump operation is total. There exists a countable pjusl $\mathcal{P}$ which extends $\mathcal{J}$ and supports a jump hierarchy R.*

PROOF. Assume that $\mathcal{J}$ is recursive. Otherwise we can relativize the proof. Let α be as defined in the beginning of subsection 4.4 and let $\mathcal{P} = \mathcal{P}_\alpha$. By Lemma 4.15, $\mathcal{J}_i \in \mathcal{K}_\alpha$ for all $i \in \omega$, so, from Lemma 4.8 we get that $\mathcal{J}$ embeds into $\mathcal{P}_\alpha$. By Lemma 4.7, $\mathcal{P}_\alpha$ supports a jump hierarchy. □

THEOREM 4.17. *Every countable jump upper semilattice can be embedded into $\mathcal{D}$.*

PROOF. Immediate from the previous proposition and Theorem 2.2. □

§5. Adding 0 to the Language. In this section we add 0 to the structure and we ask the same kind of questions we asked for jump upper semilattices. We are concerned with the following kind of structures.

DEFINITION 5.1. A *partial jump upper semilattice with 0* is a structure $\mathcal{J} = \langle J, \leq_{\mathcal{J}}, \cup, \mathsf{j}, 0\rangle$ such that $\langle J, \leq_{\mathcal{J}}, \cup, \mathsf{j}\rangle$ is a pjusl, 0 is the least element of $\langle J, \leq_{\mathcal{J}}\rangle$, and for all $n \in \omega$, $\mathsf{j}^n(0)$ is defined. A *jump upper semilattice with 0* is a pjusl with 0 where join and jump are total, and a *jump partial ordering with 0* is one where jump is total but join is undefined.

In this section $\mathcal{D}$ represents $\langle \mathbf{D}, \leq_T, \vee, ', 0\rangle$.

5.1. A negative answer. The direct generalization of Theorem 4.17 to jusls with 0 is false.

THEOREM 5.2. *Not every quantifier free 1-type of jusl with 0 is realizable in $\mathcal{D}$.*

PROOF. We shall prove that there are continuum many quantifier free 1-types of jusl with 0 which contain a formula of the form $x \leq \mathsf{j}^n(0)$. But there are only countably many arithmetic Turing degrees. Therefore, not all of these types can be realized in $\mathcal{D}$.

Given a set $A \subseteq \omega$, we construct, $p_A(x)$, a quantifier free 1-type of jusl with 0. Put in $p_A(x)$ all the formulas

$$\mathsf{j}^n(x) \geq_{\mathcal{J}} \mathsf{j}^n(0), \qquad \mathsf{j}^n(x) \mid_{\mathcal{J}} \mathsf{j}^{n+1}(0), \qquad \mathsf{j}^n(x) \leq_{\mathcal{J}} \mathsf{j}^{n+2}(0),$$

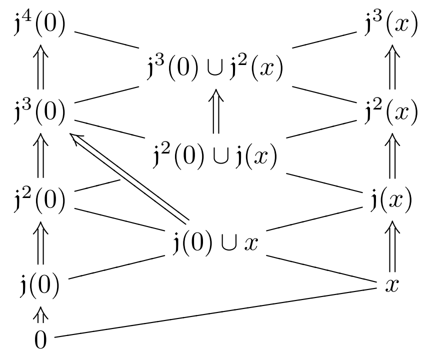
for all $n \in \omega$. Also, for each $n \in A$, add the formula

$$\mathsf{j}\big(\mathsf{j}^n(x) \cup \mathsf{j}^{n+1}(0)\big) = \mathsf{j}^{n+1}(x) \cup \mathsf{j}^{n+2}(0),$$

and for $n \notin A$, the formula

$$\mathsf{j}\big(\mathsf{j}^n(x) \cup \mathsf{j}^{n+1}(0)\big) = \mathsf{j}^{n+3}(0).$$

Of course, we add to $p_A(x)$ all the formulas which can be deduced form the ones already in $p_A(x)$.



In the picture above the reader can see how a realization of $p_A(x)$ would look like, and convince himself that $p_A(x)$ is consistent with the axioms of jusl with 0. (In the picture, the double arrows ($\Rightarrow$) represent the jump operator. In the example drawn, $0 \notin A$ but $1 \in A$.) It is also easy to see that for $A \neq B$, $p_A \neq p_B$. ☐

Remark 5.3. If $p(x)$ is the type of an arithmetic degree $\mathbf{x} \in \mathbf{D}$, then necessarily $p(x) \leq_T 0^\omega$. Because given an index for a set in $\mathbf{x}$, all the quantifier free formulas of jusl with 0 can be decided uniformly in 0^ω.

Since realizing quantifier free n-types of jusl is equivalent to embedding jusl with n generators, we get the following corollary.

Corollary 5.4. *Not every countable jusl with 0 is embeddable into $\mathcal{D}$.*

5.2. A positive answer. Now we consider jpo with 0. The situation here changes because there are only countably many quantifier free 1-types of jpo with 0 containing a formula of the form $x \leq \mathrm{j}^n(0)$. Moreover, all of these types are recursive.

We need a stronger version of Theorem 2.2.

Definition 5.5. Given a jpo $\mathcal{P}$, we say that $H : P \to \omega^\omega$ is *almost a jump hierarchy over $\mathcal{P}$* if for all $x \in P$

- $\mathcal{P} \restriction \mathrm{j}(x) \leq_T H(x)$, where $\mathcal{P} \restriction x$ is the restriction of $\mathcal{P}$ to $\{y \in P : y \leq_P x\}$.
- $\bigoplus_{y \in \mathcal{P} \restriction x} H(y) \leq_T H(x)$;
- $H(x)' \leq_T H(\mathrm{j}(x))$.

Theorem 5.6. *Suppose that $\mathcal{J}$ is a countable jpo that supports an almost jump hierarchy H. Then there is an embedding from $\mathcal{J}$ into $\mathcal{D}$ presented by $R : \mathcal{J} \to \omega^\omega$ such that*

$$\forall x, y \in J\big(H(x) \leq_T R(y) \Longrightarrow H(x) \leq_T H(y)\big). \tag{5.1}$$

Proof. We construct R in the same way as in Theorem 2.2. We have to prove that an almost jump hierarchy is enough to guarantee that R is an embedding, and that we also get (5.1). To prove that R represents an embedding, we have to verify that the proof in section 2 works in the same way as there. We only used that $\mathcal{J} \leq_T H(x)$ for all $x \in \mathcal{J}$ in observation 2.15 and Lemma 2.17. Observe that in both cases we only needed that $\mathcal{J} \restriction \mathrm{j}(x) \leq_T H(x)$. We used that $x <_{\mathcal{J}} y \Longrightarrow H(x)' \leq_T H(y)$ in corollary 2.19, but we only used that $H(x)' \leq_T H(\mathrm{j}(x))$.

Let us prove now that (5.1) holds. Suppose that $H(x) = \{e\}^{R(y)}$. Then, there is some $p \in \mathbb{P}$ such that $p \Vdash \{e\}^{R(y)} = R(x)^{[0]}(= H(x))$. So, for every $q \leq p$ and $m \in \omega$ such that $\{e\}^{R_q(y)}(m)\downarrow$, we have that $\{e\}^{R_q(y)}(m) = H(x)(m)$. We also know that for every m there is some $q \leq p$ such that $\{e\}^{R_q(y)}(m)\downarrow$. Now, given $m \in \omega$, we can find $\bar{q} \in \mathbb{P}_{p,y}$ such that $\{e\}^{R_{\bar{q}}(y)}(m)\downarrow$, recursively in $H(y)$, because $\mathbb{P}_{p,y} \leq_T H(y)$. Then $H(x)(m) = \{e\}^{R_{\bar{q}}(y)}(m)$. This shows that $H(x) \leq_T H(y)$. □

DEFINITION 5.7. Given a jpo with 0 $\mathcal{J}$, the *archimedean part* of $\mathcal{J}$ is

$$J_a = \{x \in J : \exists n \in \omega\big(x \leq_{\mathcal{J}} \mathfrak{j}^n(0)\big)\}.$$

We say that $\mathcal{J}$ is *archimedean* if $J = J_a$. Observe that J_a is closed under jump. So, let $\mathcal{J}_a$ be the restriction to J_a of $\mathcal{J}$ as a jpo.

We say that a type of jpo with 0, $p(x_1, ..., x_n)$ is *archimedean* if for some $m \in \omega$ it contains the formula "$x_1 \leq \mathfrak{j}^m(0)$ & ... & $x_n \leq \mathfrak{j}^m(0)$".

THEOREM 5.8. *Let $\mathcal{J} = \langle J, \leq_{\mathcal{J}}, \mathfrak{j}, 0\rangle$ be a finitely generated jpo with 0 such that every pair $x, y \in J_a$ has a least upper bound. Then, any embedding of $\mathcal{J}_a$ into $\mathcal{D}$ extends to an embedding of $\mathcal{J}$ into $\mathcal{D}$ (not necessarily preserving join but preserving* 0*).*

PROOF. Suppose that we have an embedding of $\mathcal{J}_a$ presented by $R\colon J_a \to \omega^\omega$. We start by defining a particular almost jump hierarchy, K, over $\mathcal{J}$. We need to begin with a couple of observations. First observe that, by corollary 4.14, since $\mathcal{J}$ is finitely generated, it is well founded. So, by Remark 3.3, there is a jump hierarchy, H, over $\mathcal{J}$ such that for all $x \in J$, $H(x) \geq_T R \oplus (\mathcal{J})'$. Second, say that $x \gg 0$ if $\forall n(x \geq_{\mathcal{J}} \mathfrak{j}^n(0))$. Now observe that for every $x \in J$ either $x \gg 0$ or there is a $x_a \in J_a$ such that

$$\forall y \in J_a(y \leq_{\mathcal{J}} x \iff y \leq_{\mathcal{J}} x_a).$$

This is because $\mathcal{J}$ is finitely generated: Let $\bar{a} = \{a_1, ..., a_{n-1}\}$ be a set of generators of $\mathcal{J}$, let $F = \{x_1, ..., x_{n_1}\}$, and suppose that $x \not\gg 0$. Then, there is some m such that $\mathfrak{j}^m(0) \not\leq_T x$. So, each $y \leq_{\mathcal{J}} x$ has to be of the form $\mathfrak{j}^k(a_i)$ for some $i < n$ and $k < m$. Therefore, there are only finitely many $y \in J_a$ with $y \leq_{\mathcal{J}} x$. Let x_a be the least upper bound of $\{y \in J_a : y \leq_{\mathcal{J}} x\}$, which exists by hypothesis.

Now we define $K\colon \mathcal{J} \to \omega^\omega$ as follows

$$K(x) = \begin{cases} R(x_a) & \text{if } x \not\gg 0, \\ H(x) & \text{if } x \gg 0. \end{cases}$$

We claim that K is almost a jump hierarchy over $\mathcal{J}$. For each $x \in J$ we have to check the conditions in Definition 5.5. For $x \not\gg 0$ we have that $\mathcal{J} \restriction \mathfrak{j}(x) \leq_T K(x)$ because $\mathcal{J} \restriction \mathfrak{j}(x)$ is finite; we have that $\bigoplus_{y \in \mathcal{J} \restriction x} K(y) \leq_T K(x)$ because $\mathcal{J} \restriction x$ is finite and for all $y \leq_{\mathcal{J}} x$, $y_a \leq_{\mathcal{J}} x_a$; and we have that $K(x)' \leq_T K(\mathfrak{j}(x))$ because $\mathfrak{j}(x_a) \leq_{\mathcal{J}} (\mathfrak{j}(x))_a$. For $x \gg 0$, we have that $\mathcal{J} \leq_T K(x)$ and that $K(x)' \leq_T K(\mathfrak{j}(x))$ because H is a jump hierarchy over $\mathcal{J}$. To prove that $K(y) \leq_T K(x)$ uniformly in y observe that using $\mathcal{J}'$ we can decide whether $y \gg 0$, and if $y \not\gg 0$ we can find y_a. Then since R and $\bigoplus_{y \leq_{\mathcal{J}} x} H(y)$ are recursive $H(x) = K(x)$, we get that $K(y) \leq_T K(x)$ uniformly in y.

Now, by Theorem 5.6, there is an embedding $\mathcal{J} \to \mathcal{D}$ presented by some $R_1\colon J \to \omega^\omega$ such that

$$\forall x\, y \in J\big(K(x) \leq_T R_1(y) \Longrightarrow K(x) \leq_T K(y)\big). \tag{5.2}$$

Extend R to $\mathcal{J}$ by defining $R(x) = R_1(x)$ for all $x \in J \smallsetminus J_a$. R preserves the jump because it does it for $x \in J_a$ and it does it for $x \in J \smallsetminus J_a$. All we have to prove, to show that R represents an embedding of $\mathcal{J}$ into $\mathcal{D}$, is that for all $x \in J_a$ and $y \in J \smallsetminus J_a$ we have that

$$x \leq_{\mathcal{J}} y \iff R(x) \leq_T R(y)$$

If $y \gg 0$ then $x \leq_{\mathcal{J}} y$ and $R(x) \leq_T R(y)$. So, suppose that $y \not\gg 0$. First assume that $x \leq_{\mathcal{J}} y$. Then $x_a = x \leq_{\mathcal{J}} y_a$. Therefore

$$R(x) = R(y_a) \leq_T K(y_a) \leq_T R_1(y) = R(y).$$

Now suppose that $R(x) \leq_T R(y)$. Since $x \in J_a$, $R(x) = K(x)$, and since $y \notin J_a$, $R(y) = R_1(y)$. So, $K(x) \leq_T R_1(y)$. Then, by (5.2), $K(x) \leq_T K(y)$. Hence, we have that $R(x) \leq_T R(y_a)$. But we know that R restricted to $\mathcal{J}_a$ is an embedding, so $x \leq_{\mathcal{J}} y_a \leq_{\mathcal{J}} y$. □

Corollary 5.9. *Every quantifier free 1-type of jpo with 0 is realized in $\mathcal{D}$.*

Proof. We start by defining the notion of jump trace introduced in [HS91]. A *consistent jump trace* is a pair of sequences $(h_0, h_1, h_2, ...; ..., l_2, l_1, l_0)$ such that for all $k \in \omega$ $h_k \leq h_{k+1} \leq l_{k+1} \leq l_k \leq l_{k+1} + 1$. The *jump trace* of an arithmetic degree $\mathbf{x}$ is $(h_0, h_1, ...; ...l_1, l_0)$ where h_i is the greatest h such that $\mathbf{x}^{(i)} \geq_T 0^{(i+h)}$, and l_i is the least l such that $\mathbf{x}^{(i)} \leq_T 0^{(i+l)}$ is in $p(x)$. Given $p(x)$, an archimedean type of jpo with 0 we can associate to it the jump trace $(h_0, h_1, ...; ...l_1, l_0)$ where h_i is the greatest h such that "$\mathfrak{j}^i(x) \geq \mathfrak{j}^{i+h}(0)$" is in $p(x)$, and l_i is the least l such that "$\mathfrak{j}^i(x) \leq \mathfrak{j}^{i+l}(0)$" is in $p(x)$. It is easy to see that an arithmetic degree $\mathbf{x}$ realizes $p(x)$ if and only if $\mathbf{x}$ and $p(x)$ have the same jump trace. Hinman proved in [Hin99], finishing the cases left by Hinman and Slaman in [HS91], that every consistent jump trace is realizable in $\mathcal{D}$. Hence every archimedean quantifier free 1-type of jpo with 0 is realizable in $\mathcal{D}$.

Now let $p(x)$ be a quantifier free 1-type of jpo with 0 and suppose that no formula of the form "$x \leq \mathfrak{j}^m(0)$" is in $p(x)$. Consider a jpo with 0, $\mathcal{J}$, with one generator a, such that $\mathcal{J} \models p(a)$. By our assumption on $p(x)$, $a \notin J_a$, and hence $J_a = \{0, \mathfrak{j}(0), \mathfrak{j}^2(0), ...\}$. Obviously, $\mathcal{J}_a$ embeds into $\mathcal{D}$, and every pair of elements in $\mathcal{J}_a$ has a least upper bound. So, by Theorem 5.8, the embedding of $\mathcal{J}_a$ into $\mathcal{D}$ extends to $\mathcal{J}$. Therefore, $p(x)$ is realizable in $\mathcal{D}$. □

Lemma 5.10. *Every finitely generated archimedean jpo with 0, $\mathcal{P} = \langle P, \leq_{\mathcal{P}}, \mathfrak{j}, 0\rangle$, can be embedded into a finitely generated archimedean jpo with 0, $\mathcal{J}$, such that every pair of elements has a least upper bound.*

Proof. The idea is to consider the usl with 0 generated by $\mathcal{P}$ and define the jump operator on it by imposing that $\mathfrak{j}(x \cup y) = \mathfrak{j}(x) \cup \mathfrak{j}(y)$. Let $J' = \{F \subset P :$ F finite & $F \neq \emptyset\}$ and define an order on J' as follows:

$$F \leq' G \qquad \iff \qquad \forall x \in F \;\; \exists y \in G (x \leq_{\mathcal{P}} y)$$

Observe that $\leq'$ is transitive and reflexive. Say that F is equivalent to G, $F \equiv G$, if $F \leq' G$ & $G \leq' F$, and write $[F]$ for the equivalence class of F. Let $J = J'/\equiv$, define $[F] \leq [G] \iff F \leq' G$, and $[F] \vee [G] = [F \cup G]$. It is easy to show that $\mathcal{J}$ is an usl with 0 and that the map that sends $x \in P$ into $[\{x\}]$ is an embedding of $\mathcal{P}$ into $\mathcal{J}$. Define a jump operation on $\mathcal{J}$ as follows:

$$\mathfrak{j}([F]) = [\{\mathfrak{j}(x) : x \in F\} \cup \{0\}]$$

One can easily check that $\mathfrak{j}$ is well defined, that it is monotone and strictly increasing and that $\mathcal{J}$ is archimedean.

Now we need to prove that $\mathcal{J}$ is finitely generated as a jpo with 0. Let $\{a_1, ..., a_n\}$ be a set of generators of $\mathcal{P}$. Let m be such that all the generators of $\mathcal{P}$ are below $\mathfrak{j}^m(0)$. We claim that the set

$$A = \{[F] : F \subseteq \{\mathfrak{j}^i(a_j) : i = 0, ..., m-1; \quad j = 1, ..., n\}\}$$

generates $\mathcal{J}$. Take any $[G] \in \mathcal{J}$. G is equivalent to some $G_1 = \{x_1, ..., x_k\}$ such that $\forall i \neq j (x_i \not\leq_{\mathcal{P}} x_j)$. Each x_i is of the form $\mathfrak{j}^{r_i}(a_{s_i})$ for some r_i and for some generator a_{s_i}. Let $r = \min\{r_1, ..., r_k\}$, suppose, without lost of generality, that $r = r_1$. Then

$$[G] = \mathfrak{j}^r([F]) \qquad \text{where } F = \{\mathfrak{j}^{r_i - r}(a_{s_i}) : i = 1, ..., k\}.$$

We have to that for all $i = 1, ..., k$, $r_i - r < m$. Suppose that $r_i - r \geq m$, then $\mathfrak{j}^{r_i - r}(a_{s_i}) \geq_P \mathfrak{j}^m(0) \geq_{\mathcal{P}} a_{s_1}$. Therefore $x_i = \mathfrak{j}^{r_i}(a_{s_i}) \geq \mathfrak{j}^r(a_{s_1}) = x_1$, contradicting our assumption on $G_1 = \{x_1, ..., x_k\}$. □

Corollary 5.11. *If every finitely generated archimedean jpo with 0 can be embedded into $\mathcal{D}$, then every finitely generated jpo with 0 can be embedded into $\mathcal{D}$. Equivalently: If every archimedean quantifier free type of jpo with 0 is realizable in $\mathcal{D}$, then every quantifier free type of jpo with 0 is realizable in $\mathcal{D}$.*

Proof. Let $\mathcal{P}$ be a finitely generated jpo with 0. Let $\bar{\mathcal{J}}$ be an extension of $\mathcal{P}_a$ as in the previous lemma. Let $\mathcal{J}$ be the jpo with 0 obtained by amalgamating $\mathcal{P}$ and $\bar{\mathcal{J}}$ as in Lemma 4.6. Note that $\mathcal{J}$ is still finitely generated, and that its archimedean part is $\bar{\mathcal{J}}$, in which every pair of elements has a least upper bound. By hypothesis $\mathcal{J}_a = \bar{\mathcal{J}}$ can be embedded into $\mathcal{D}$. Then, by Theorem 5.6, $\mathcal{J}$ can be embedded into $\mathcal{D}$. Hence $\mathcal{P}$ can be embedded too. □

§6. Uncountable jump upper semilattices.

So far we have studied countable pjusls. Now, given κ, with $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$, we address the following question: Is every jusl with the size κ and the c.p.p. embeddable in $\mathcal{D}$? In the first subsection we answer this question negatively for $\kappa = 2^{\aleph_0}$. In the second subsection we answer this question positively for κ such that MA(κ) holds.

6.1. A negative answer.

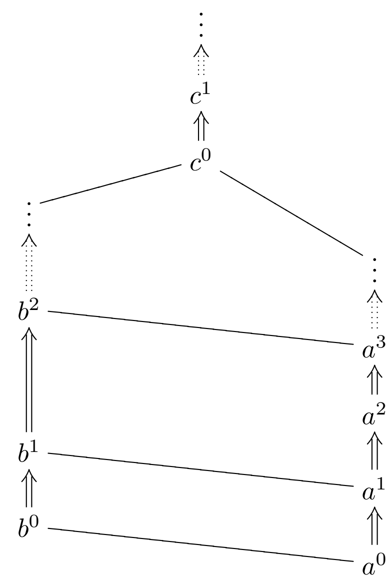
We construct a jpo of size $2^{\aleph_0}$ which cannot be embedded into the degrees.

Definition 6.1. Given a strictly increasing function $f : \omega \to \omega$, we define a jpo $\mathcal{P}_f = \langle P_f, \leq, \mathfrak{j} \rangle$ as follows:

- $P_f = \{a^i : i \in \omega\} \cup \{b^i : i \in \omega\} \cup \{c^i : i \in \omega\}$.
- $\mathfrak{j}(a^i) = a^{i+1}$, $\mathfrak{j}(b^i) = b^{i+1}$ and $\mathfrak{j}(c^i) = c^{i+1}$ for all $i \in \omega$.
- $a^i < a^j$ iff $i < j$, $b^i < b^j$ iff $i < j$, and $c^i < c^j$ iff $i < j$.

- $a^i < c^j$ and $b^i < c^j$ for all i, j.
- $a^i \leq b^j$ iff $f(j) \geq i$.
- for all $i, j \in \omega$, $b^i \not\leq a^j$, $c^i \not\leq a^j$ and $c^i \not\leq b^j$.

In the figure below we draw an example where $f(0) = 0$, $f(1) = 1$, $f(2) = 3$,... (The double arrows ($\Rightarrow$) represent the jump operator.)



It is easy to see that, for every strictly increasing f, $\mathcal{P}_f$ is a jpo.

Lemma 6.2. *Let $f\colon \omega \to \omega$ be strictly increasing, and let ψ be an embedding of $\mathcal{P}_f$ into $\mathcal{D}$. Then $\psi(c^3) \geq_T f$.*

Proof. Let A be a member of $\psi(a^0)$, B be a member of $\psi(b^0)$ and C be a member of $\psi(c^0)$. Since for all $i \in \omega$, $A^{(i)}$ and $B^{(i)}$ are recursive in C, there are functions g and h, recursive in $C^{(3)}$ (actually recursive in $C^{(2)}$ too), such that

$$A^{(i)} = \{g(i)\}^C \qquad \text{and } B^{(i)} = \{h(i)\}^C.$$

Therefore, we can decide whether $B^{(j)} \geq_T A^{(i)}$ recursively in $C^{(3)}$, uniformly in i and j. So, we can compute f from $C^{(3)}$. □

Definition 6.3. Let d be a new symbol and $\mathcal{J}$ be the jpo with generator d (i.e. $J = \{d, \mathrm{j}(d), \mathrm{j}^2(d), ...\}$), and let $\mathcal{F}$ be the set of all strictly increasing functions from ω into itself. Define

$$\mathcal{P} = \mathcal{J} \oplus \bigoplus_{f \in \mathcal{F}} \mathcal{P}_f.$$

In other words: the domain of $\mathcal{P}$ is the disjoint union of $\mathcal{J}$ and all the $\mathcal{P}_f$ with $f \in \mathcal{F}$; the jump operation is defined in the obvious way; and the $\leq$ relation in $\mathcal{P}$ is the disjoint union of the $\leq$ relations of each jpo.

Proposition 6.4. *$\mathcal{P}$ cannot be embedded into $\mathcal{D}$.*

Proof. Suppose that there is an embedding $\psi\colon \mathcal{P} \to \mathcal{D}$. In the degree $\mathbf{d} = \psi(d)$ there is some $f \in \mathcal{F}$. Let $\mathbf{c}^3_f$ be the image under ψ of the element c^3 of $\mathcal{P}_f$ (call that element c^3_f). Then, by the previous lemma, $\mathbf{d} = \deg(f) \leq \mathbf{c}^3_f$. This contradicts the fact that ψ is an embedding since d and c^3_f are incomparable. □

6.2. A positive answer. Now, we prove that if MA(κ) holds, then every jusl with the c.p.p. and size κ can be embedded into $\mathcal{D}$. The idea is the if we have a pjusl of size κ, with the c.p.p. and supporting an almost jump hierarchy, we can carry out the forcing construction of Section 2 as long as we can get generic enough filters. MA(κ) give us the existence of such generic filters.

The hard part is to prove that every pjusl with the c.p.p. extends to another one, also with the c.p.p., which supports an almost jump hierarchy (ajh) and has the same cardinality. We start by proving some facts we will use about end extensions and amalgamations of pjusls. (We say that a partial order $\mathcal{P}$ is an *end extension* of $\mathcal{Q}$ if $\mathcal{Q} \subseteq \mathcal{P}$ and $\mathcal{Q}$ is closed downward in $\mathcal{P}$.)

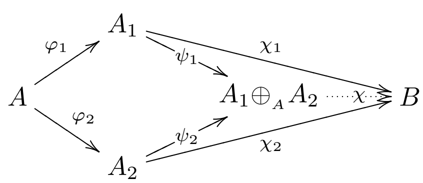
In this section, the jump operation of every pjusl is total.

Definition 6.5. Given pjusls A, A_1 and A_2, and embeddings $\varphi_1\colon A \to A_1$ and $\varphi_2\colon A \to A_2$, let $A_1 \oplus_{A,\varphi_1,\varphi_2} A_2$ be the structure defined in Lemma 4.6. We write $A_1 \oplus_A A_2$ if φ_1 and φ_2 are clear from the context, and we write $A_1 \oplus A_2$ when $A = \emptyset$.

In Lemma 4.6 we also constructed two embeddings, $\psi_1\colon A_1 \to A_1 \oplus_A A_2$ and $\psi_1\colon A_1 \to A_1 \oplus_A A_2$, such that $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$. Observe that if φ_1 and φ_2 are inclusions, we can think of ψ_1 and ψ_2 as inclusions too.

Lemma 6.6. *Let A, A_1, A_2, φ_1 and φ_2 be as in the definition above. Then:*

1. *Given a pjusl B and two homomorphisms (of pjusl) $\chi_1\colon A_1 \to B$ and $\chi_2\colon A_2 \to B$ such that $\chi_1 \circ \varphi_1 = \chi_2 \circ \varphi_2$, there is a unique homomorphism $\chi : A_1 \oplus_A A_2 \to B$ such that the following diagram commutes.*



2. *If A_1 is an end extension of A, then $A_1 \oplus_A A_2$ is an end extension of A_2.*
3. *If A_1 is an end extension of $A_1 \setminus A$, then $A_1 \oplus_A A_2$ is an end extension of $A_1 \setminus A$.*
4. *If A_1 and A_2, both have the c.p.p., then so does $A_1 \oplus_A A_2$.*

Lemma 6.7. *If $\langle A_\xi : \xi < \alpha\rangle$ is a chain of pjusl with the c.p.p. such that for all $\beta < \gamma < \alpha$, A_γ is an end extension of A_β, then $A = \bigcup_{\xi<\alpha} A_\xi$ is a pjusl which is an end extension of each A_ξ and has the c.p.p.*

The proofs of these lemmas are straightforward.

Lemma 6.8. *Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two countable pjusls, such that $\mathcal{J}_2$ is an end extension of $\mathcal{J}_1$. Let H and K be ajhs over $\mathcal{J}_1$ and $\mathcal{J}_2 \setminus \mathcal{J}_1$ respectively, such that $\forall x \in J_2 \setminus J_1 (\mathcal{J}_1 \oplus H \leq_T K(x))$. Define $R\colon \mathcal{J}_2 \to \omega^\omega$ by $R(x) = H(x)$ if $x \in J_1$ and $R(x) = K(x)$ if $x \in J_2 \setminus \mathcal{J}_1$. Then R is an ajh over $\mathcal{J}_2$.*

Proof. Just check the conditions of Definition 5.5. □

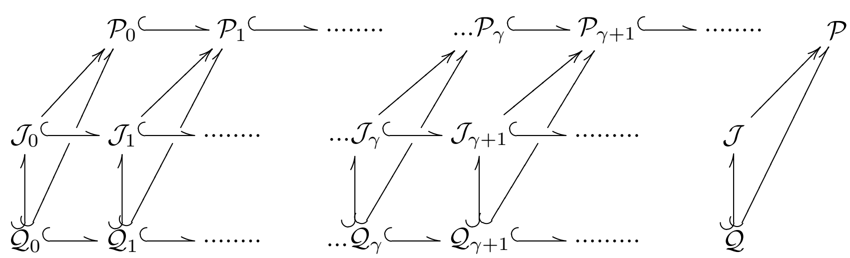
Lemma 6.9. *Let $\mathcal{Q}$ and $\mathcal{J}$ be two countable pjusls, such that $\mathcal{J}$ is an end extension of $\mathcal{Q}$. Let H be an almost Jump Hierarchy over $\mathcal{Q}$. Then, there is a pjusl $\mathcal{P}$ extending $\mathcal{J}$ which is an end extension of $\mathcal{Q}$ and supports an almost jump hierarchy extending H.*

Proof. Let $\bar{\mathcal{P}}$ be a pjusl extending $\mathcal{J}$ and supporting an almost jump hierarchy K such that $\forall x \in \bar{\mathcal{P}}(K(x) \geq_T H \oplus \mathcal{J})$. (A relativized version of Proposition 4.16 would give us such a $\bar{\mathcal{P}}$.) Let $\mathcal{P} = \bar{\mathcal{P}} \oplus_{\mathcal{J} \smallsetminus \mathcal{Q}} J$. Note that $\mathcal{J} \smallsetminus \mathcal{Q}$ is a pjusl and is closed under jump because $\mathcal{J}$ is an end extension of $\mathcal{Q}$. Also observe that, since $\mathcal{J}$ is an end extension of $\mathcal{Q} = \mathcal{J} \smallsetminus (\mathcal{J} \smallsetminus \mathcal{Q})$, $\mathcal{P}$ is an end extension of $\mathcal{Q}$. Now define $R \colon \mathcal{P} \to \omega^\omega$ by $R(x) = H(x)$ if $x \in Q$ and $R(x) = K(x)$ if $x \in \bar{\mathcal{P}}$. By lemma 6.8, R is an ajh over $\mathcal{P}$ extending H. □

Lemma 6.10. *Let $\mathcal{Q}$ and $\mathcal{J}$ be two pjusls, such that $\mathcal{J}$ is an end extension of $\mathcal{Q}$ and let $\kappa = |J|$. Let H be an almost Jump Hierarchy over $\mathcal{Q}$. Then, there is a pjusl $\mathcal{P}$ extending $\mathcal{J}$ which is an end extension of $\mathcal{Q}$, has size κ, and supports an ajh extending H.*

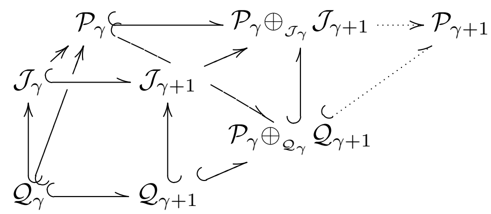
Proof. We use induction on κ. When $\kappa = \aleph_0$, the result is given by the previous lemma. Now suppose that $\kappa > \aleph_0$ and that the lemma is true for all cardinals smaller that κ. Let $\{a_\xi : \xi < \kappa\}$ be a well ordering of the elements of $\mathcal{J}$, let $\mathcal{J}_\gamma$ be the downward closure of the pjusl generated by $\{a_\xi : \xi < \gamma\}$, and let $\mathcal{Q}_\gamma = \mathcal{Q} \cap \mathcal{J}_\gamma$. Note that $|\mathcal{J}_\gamma| = |\gamma| + \aleph_0$. Now, we construct a sequence $\{\mathcal{P}_\gamma\}_{\gamma \leq \kappa}$ as in the figure below (where $A \hookrightarrow B$ indicates that B is an end extension of A). We do it by induction on γ, and we want the sequence to have the following properties.

- $\mathcal{P}_\gamma$ supports an ajh, K_γ.
- if $\beta < \gamma$ then $\mathcal{P}_\gamma$ is an end extension of $\mathcal{P}_\beta$, and $K_\beta \subseteq K_\gamma$.
- $\mathcal{P}_\gamma$ extends $\mathcal{J}_\gamma$.
- $\mathcal{P}_\gamma$ is an end extension of Q_γ and K_γ extends $H \restriction \mathcal{Q}_\gamma$.
- $|\mathcal{P}_\gamma| \leq |\gamma|$.



Suppose that $\mathcal{J}_0 = \mathcal{Q}_0 = \emptyset$ and let $\mathcal{P}_0$ be empty too. Now assume we have defined $\mathcal{P}_\gamma$ and we want to define $\mathcal{P}_{\gamma+1}$. (We do it as in the diagram below.) Let $H_\gamma = \mathcal{K}_\gamma \cup H \restriction Q_{\gamma+1}$; it is an ajh over $\mathcal{P}_\gamma \oplus_{\mathcal{Q}_\gamma} \mathcal{Q}_{\gamma+1}$. Let $\mathcal{P}_{\gamma+1}$ be an extension of $\mathcal{P}_\gamma \oplus_{\mathcal{J}_\gamma} \mathcal{J}_{\gamma+1}$, such that $\mathcal{P}_{\gamma+1}$ is an end extension of $\mathcal{P}_\gamma \oplus_{\mathcal{Q}_\gamma} \mathcal{Q}_{\gamma+1}$ and supports an ajh, $\mathcal{K}_\gamma$, extending H_γ. We know such a $\mathcal{P}_{\gamma+1}$ exists because $|\mathcal{P}_\gamma \oplus_{\mathcal{J}_\gamma} \mathcal{J}_{\gamma+1}| \leq |\gamma| < \kappa$. Moreover we can get $\mathcal{P}_{\gamma+1}$ of size $\leq |\gamma|$. Note that since both $\mathcal{P}_\gamma$ and

$\mathcal{Q}_{\gamma+1}$ are end extensions of $\mathcal{Q}_\gamma$, $\mathcal{P}_\gamma \oplus_{\mathcal{Q}_\gamma} \mathcal{Q}_{\gamma+1}$ is an end extension of both $\mathcal{P}_\gamma$ and $\mathcal{Q}_{\gamma+1}$. Now, since $\mathcal{P}_{\gamma+1}$ is an end extension of $\mathcal{P}_\gamma \oplus_{\mathcal{Q}_\gamma} \mathcal{Q}_{\gamma+1}$, it is also an end extension of both $\mathcal{P}_\gamma$ and $\mathcal{Q}_{\gamma+1}$.



When γ is a limit ordinal let $\mathcal{P}_\gamma = \bigcup_{\xi<\gamma} \mathcal{P}_\xi$ and $K_\gamma = \bigcup_{\xi<\gamma} K_\xi$. It is easy to check that $\{\mathcal{P}_\gamma\}_{\gamma\leq\kappa}$ has the properties mentioned above and that $\mathcal{P} = \mathcal{P}_\kappa$ is as wanted. □

PROPOSITION 6.11. *Every pjusl with the c.p.p. can be extended to one of the same cardinality which also has the c.p.p. and supports an ajh.*

PROOF. Apply the previous lemma with $\mathcal{Q} = \emptyset$. □

Now we use Martin's Axiom to prove that some uncountable jusls can be embedded into $\mathcal{D}$.

DEFINITION 6.12. MA(κ) is the statement: Whenever $\langle I\!P, \leq\rangle$ is a non-empty c.c.c. partial order, and $\mathcal{F}$ is a family of $\leq \kappa$ dense subsets of $I\!P$, then there is a filter G in $I\!P$ such that $\forall D \in \mathcal{F}(G \cap D \neq \emptyset)$. We say that a p.o. has the *countable chain condition* (c.c.c.) if every antichain is at most countable.

It is consistent with ZFC that $2^{\aleph_0} > \aleph_1$ and MA(λ) for all $\lambda < 2^{\aleph_0}$ (see [Jec03, Theorem 16.13]).

PROPOSITION 6.13. *If MA(κ) holds, then every jusl with the c.p.p. and of size $\leq \kappa$ can be embedded into $\mathcal{D}$.*

PROOF. Consider a jusl $\mathcal{Q}$ with the c.p.p. and of size $\leq \kappa$. By Proposition 6.11, there is a pjusl $\mathcal{J}$, extending $\mathcal{Q}$, which supports an ajh H, has cardinality $\leq \kappa$ and has the c.p.p. We claim that the construction done in Section 2 works for $\mathcal{J}$ too. Therefore, we would get that $\mathcal{J}$, and hence $\mathcal{Q}$, can be embedded into $\mathcal{D}$. Note that we do not have a jump hierarchy here, but an almost jump hierarchy. As mentioned in Theorem 5.6, this is not a problem.

Let $I\!P$ be the partial order defined in Subsection 2.1. $I\!P$ is a set of pairs of finite partial functions from $J \times \omega$ to ω. Actually we can view $I\!P$ as a set of finite partial functions from $J \times \omega \times 2$ to ω. Such a partial ordering always has the c.c.c. (see [Kun80, Lemma VII.5.4]). Now consider the set of all first order formulas in the language with signature $\{0, S, +, \cdot, <\} \cup \{R_p(x)(\cdot), Sk_p(x)(\cdot) : p \in I\!P, x \in \mathcal{J}\}$. For every such a formula we can define what it means that $p \in I\!P$ forces it (see [SW]). We know that given a formula φ, $\{p : p \Vdash \varphi \vee p \Vdash \neg\varphi\}$ is dense in $I\!P$, and, since $I\!P$ and $\mathcal{J}$ have cardinality $\leq \kappa$, there are at most κ such formulas. Hence, because of MA(κ), there is a filter G in $I\!P$ such that every such formula is decided by some element of G. This generic filter G satisfies all the properties that we needed in Section 2. Therefore, in particular, it gives us an embedding from $\mathcal{J}$ into $\mathcal{D}$. □

COROLLARY 6.14. *Whether every jpo (or jusl) with the c.p.p. and size $\aleph_1$ is embeddable into $\mathcal{D}$ is independent of ZFC.*

PROOF. On the one hand, we get from 6.4 that, if CH holds, not every jpo of size $\aleph_1 = 2^{\aleph_0}$ with the c.p.p. is embeddable in $\mathcal{D}$. On the other hand, if MA($\aleph_1$) holds, we just proved that every jusl with the c.p.p. and size $\aleph_1$ is embeddable into $\mathcal{D}$. □

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