EQUIVALENCE BETWEEN FRAÏSSÉ'S CONJECTURE AND JULLIEN'S THEOREM.

ANTONIO MONTALBÁN

June 15, 2004

Abstract. We say that a linear ordering $\mathcal{L}$ is *extendible* if every partial ordering that does not embed $\mathcal{L}$ can be extended to a linear ordering which does not embed $\mathcal{L}$ either. Jullien's theorem is a complete classification of the countable extendible linear orderings. Fraïssé's conjecture, which is actually a theorem, is the statement that says that the class of countable linear ordering, quasiordered by the relation of embeddablity, contains no infinite descending chain and no infinite antichain. In this paper we study the strength of these two theorems from the viewpoint of Reverse Mathematics and Effective Mathematics. As a result of our analysis we get that they are equivalent over the basic system of $\mathsf{RCA}_0+\Sigma^1_1\text{-}\mathsf{IND}$.

We also prove that Fraïssé's conjecture is equivalent, over RCA_0, to two other interesting statements. One that says that the class of well founded labeled trees, with labels from $\{+,-\}$, and with a very natural order relation, is well quasiordered. The other statement says that every linear ordering which does not contain a copy of the rationals is equimorphic to a finite sum of indecomposable linear orderings.

While studying the proof theoretic strength of Jullien's theorem, we prove the extendibility of many linear orderings, including ω^2 and η, using just $\mathsf{ATR}_0+\Sigma^1_1\text{-}\mathsf{IND}$. Moreover, for all these linear orderings, $\mathcal{L}$, we prove that any partial ordering, $\mathcal{P}$, which does not embed $\mathcal{L}$ has a linearization, hyperarithmetic (or equivalently Δ^1_1) in $\mathcal{P}\oplus\mathcal{L}$, which does not embed $\mathcal{L}$.

1. Introduction

We compare the strength of two known theorems about linear orderings. We will conclude that, in some sense that we specify below, these two theorems are equally hard to prove.

The two theorems. On the one hand, we have Fraïssé's conjecture. A binary relation $\leq_P$ on a set P is a *quasiordering* if it is reflexive and transitive. A quasiordering is a *well quasiordering* if, for every sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of P, there exists $i<j$ such that $x_i \leq_P x_j$. An equivalent definition of well quasiordering, that might be easier to visualize, is that $\leq_P$ contains no infinite descending chains and no infinite antichain. The proof of the equivalence follows from Ramsey's theorem. FRA is the statement that says that the countable linear orderings form a well quasiordering under the relation of embeddablity. Roland Fraïssé conjectured in [Fra48] that there are no sequences of countable linear orderings which are strictly descending under embeddablity. Although this statement is slightly

This research is going to be part of my Ph.D. thesis [Mon05]. I want to thank my thesis adviser, Richard A. Shore, for introducing me to the problem and for many helpful and inspiring discussions.

different from FRA, FRA became known as Fraïssé's conjecture. Moreover, FRA is still known as Fraïssé's conjecture even though it is not a conjecture anymore. Richard Laver proved FRA in [Lav71] using Nash-Williams complicated notion of better quasiordering [NW68].

On the other hand, we have Jullien's Theorem and the study of the extendibility linear orderings. A *linearization* of a partial ordering $\mathcal{P} = \langle P, \leq_P \rangle$ is a linear ordering $\langle P, \leq_L \rangle$ such that $\forall x, y \in P(x \leq_P \Rightarrow x \leq_L y \rangle)$. A linear ordering $\mathcal{L}$ is *extendible** if every countable partial ordering, $\mathcal{P}$, which does not embed $\mathcal{L}$ has a linearization which does not embed $\mathcal{L}$ either. For example, the extendibility of ω^* (the linear ordering of the negative integers) is a well known result and it can be translated as every well founded partial ordering has a well ordered linearization. (We give a proof of this in Lemma 6.2.) But for instance, **2**, the linear ordering with two elements, is not extendible. Other linear orderings which are not extendible are the ones of the form $\langle \rightarrow, \leftarrow \rangle$. We say that $\mathcal{L}$ is of the form $\langle \rightarrow, \leftarrow \rangle$ if $\mathcal{L}$ can be written as a sum of two linear orderings, $\mathcal{A}$ and $\mathcal{B}$, such that $\mathcal{A}$ embeds in every final segment of itself and $\mathcal{B}$ embeds in every initial segment of itself; for example $\mathcal{L} = \omega + \omega^*$. The extendibility of η, the order type of the rational numbers, was proved by Bonnet and Pouzet in [BP69] (see also [BP82, p. 140]). Linear orderings which do not contain a copy of η are called *scattered*. A characterization of exactly which linear orderings are extendible has been given by Jullien in his Ph.D. thesis [Jul69]. There, he proved that every scattered linear ordering has a unique minimal decomposition, and then he gave a characterization of the extendible linear orderings which depends on the minimal decomposition of the linear ordering (see Definition 3.8 and Statement 5.8). Here, we will study Jullien's result and also an equivalent formulation that is simpler to state because it does not use minimal decompositions. This new equivalent formulation, that we call JUL, says that a linear ordering is not extendible if and only if contains a linear ordering of the form either **2** or $\langle \rightarrow, \leftarrow \rangle$ in an essential way (see Statement 5.2).

Reverse Mathematics. What we would like to know is exactly which set existence axioms are needed to prove these two theorems. The questions of what axioms are necessary to do mathematics is of great importance in Foundations of Mathematics and is the main question behind Friedman and Simpson's program of Reverse Mathematics. Old known examples along this line of investigations are Euclid's question of whether the fifth postulate was necessary to do geometry and the question of the necessity of the Axiom of Choice to do mathematics. To analyze this question formally it is necessary to fix a logic system. Reverse Mathematics deals with subsystems of Z_2, the system of second-order arithmetic. Second-order Arithmetic, even though it is a lot weaker than set theory, is rich enough to be able to express an important fragment of classical mathematics. This fragment includes number theory, calculus, countable algebra, real and complex analysis, differential equations and combinatorics among others. Almost all of mathematics that can be modeled with, or coded by, countable objects can be done in Z_2.

* This property is sometimes called *weakly extendability* and extendibility refers to the same property but considering all partial orderings $\mathcal{P}$, and not only the countable ones. A characterization of these linear orderings has been given by Bonnet [BP82]. Since we are only interested in countable objects, we omit the word "weakly". Other names given to this property in the literature are *enforceable* and *Szpilrajn*.

It happens often that the analysis of theorems from the viewpoint of reverse math gives a deeper understanding of the theorems and sometimes leads to new proofs. This is definitely the case in this paper.

The idea of Reverse Mathematics is as follows. We start by fixing a basic system of axioms. The most commonly used system is RCA_0 which closely related to Computable Mathematics. When this program started, RCA, which is slightly stronger than RCA_0, was often used as the basic system. In RCA, as in RCA_0, the only sets we can assume exist are the ones that we can describe via an effective algorithm. Now, given a theorem of "ordinary" mathematics, the question is what axioms do we need to add to the basic system to prove this theorem. Moreover, we want the least set of axioms needed. It is often the case in Reverse Mathematics that we can prove that a certain set of axioms is needed to prove a theorem by proving the axioms from the theorem using some basic system. Because of this idea this program is called Reverse Mathematics. When we have that a theorem can be proved from a certain system of axioms and that the axioms can be proved from the theorem using for example RCA, we say that the theorem and the system are *equivalent over* RCA. Many different system of axioms have been defined and studied. But a very interesting fact is that most of the theorems that have been analyzed, have been proved equivalent over RCA_0 to one of five systems. These five systems are RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0, listed in increasing order of strength. The basic reference for this subject is [Sim99].

The language of second order arithmetic is the usual language of first order arithmetic (which contains non-logical symbols 0, 1, $+$, $\times$ and $\leq$) augmented with set variables and a membership relation $\in$. (We use the letters $x, y, z, n, m, ...$ for number variables and capital letters $X, Y, Z, A, ...$ for set variables.) The axioms of Z_2, are divided in three groups. First we have the *Basic axioms* which say that the natural numbers form an ordered semiring. Then we have the *Induction axioms.* Given a formula $\varphi(x)$ of second-order arithmetic we have the axiom:

$$(\mathsf{IND}(\varphi)) \qquad \varphi(0) \;\&\; \forall x(\varphi(x) \Rightarrow \varphi(x+1)) \Rightarrow \forall x\varphi(x).$$

Last, we have the *Comprehension axioms.* These axioms are *set existence axioms* in the sense that they say that sets with certain properties exist. Again, we have one for each formula $\varphi(x)$:

$$(\mathsf{CA}(\varphi)) \qquad \exists X\forall x(x \in X \Leftrightarrow \varphi(x)).$$

The formula φ above may have first or second order, free variable other than x. In that case, $(\mathsf{IND}(\varphi))$ and $(\mathsf{CA}(\varphi))$ are the universal closure of the formulas shown above. Subsystems of Z_2 are obtained by restricting the induction and comprehension axioms to certain classes of formulas. The basic system RCA_0 consist of the basic axioms, and the schemes of Σ^0_1-induction and Δ^0_1-comprehension. Σ^0_1-induction is the scheme of axioms that contains a sentence $(\mathsf{IND}(\varphi))$ for each Σ^0_1 formula $\varphi(x)$. The *Recursive Comprehension Axiom scheme* or Δ^0_1-comprehension consist of the axioms of the form

$$\forall x(\varphi(x) \Leftrightarrow \neg\psi(x)) \Rightarrow \exists X\forall x(x \in X \Leftrightarrow \varphi(x)).$$

where φ and ψ are Σ^0_1 formulas. (A formula ψ is Σ^0_0 if it contains no set quantifiers and all the first order quantifiers are bounded, that is, of the form either $(\forall y < t)$ or $(\exists y < t)$. A formula φ is Σ^0_1 if it is of the form $\exists z\psi(z)$, where ψ is a Σ^0_0 formula.) Another important system is ACA_0. Its axioms are the ones of RCA_0 plus the

Arithmetic Comprehension Axiom scheme, which consist of the sentences ($\mathsf{CA}(\varphi)$) for arithmetic formulas $\varphi(x)$. (A formula is *arithmetic* if it contains no second order quantifiers.) The scheme of arithmetic comprehension is equivalent to the sentence that says that for every set X, there exists a set X' which is the Turing jump of X. For other classes, Γ, of formulas, like Π^1_1 for example, the system Γ-CA_0 is defined analogously. A system that will be important in this paper is ATR_0. It consist of RCA_0 and the axiom scheme of *Arithmetic Transfinite Recursion*. The scheme of Arithmetic Transfinite Recursion is a little technical so we omit the details. What it says is that arithmetic comprehension can be iterated along any ordinal, which is equivalent to say that the Turing jump can be iterated along any ordinal. For example, ATR_0 is equivalent to the fact that any two ordinals are comparable.

All the systems we have described have restricted induction. The subindex 0 in the notation of a system means that the induction scheme it contains is Σ^0_1-induction. If we drop the subindex 0, and for example get RCA or ATR, is because we are adding the Full induction scheme to the system. The *Full induction scheme* consists of the sentences ($\mathsf{IND}(\varphi)$), for all formulas $\varphi(x)$. A subindex $*$, as in ATR_*, indicates that the system contains the scheme of Σ^1_1-induction. (Σ^1_1-induction, also called Σ^1_1-IND, is defined analogously to Σ^0_1-induction. A formula φ is Σ^1_1 if it is of the form $\exists X \psi(X)$, where ψ is an arithmetic formula.)

Fraïsé's conjecture. The theory of well quasiorderings has been of interest to people studying reverse math because it contains results that seem to be very difficult to prove in comparison with results from other areas of mathematics. Most of the proof seem to require Π^1_2-CA_0, which is more that what is usually needed. However, none of these theorems have been proved to be equivalent to Π^1_2-CA_0 and for most of them the exact proof theoretic strength is unknown. A very interesting example is Kruskal's theorem [Kru60] which says that the class of finite trees is well quasiordered under embeddablity (preserving greatest lower bounds). Harvey Friedman proved that Kruskal's theorem can not be proved in ATR_0. (See [Sim85] for a proof of Friedman's result and [RW93] for an analysis of the exact proof theoretic strength of Kruskal's theorem.) The reader can find a survey on the theory of well quasiorderings studied from the viewpoint of reverse mathematics in [Mar].

The exact proof theoretic strength of FRA is also unknown. It is known that Laver's proof of FRA can be carried out in Π^1_2-CA_0, and that since FRA is a true Π^1_2 statement, it cannot imply Π^1_1-CA_0. (Because every true Π^1_2 sentence holds in every β-model, but Π^1_1-CA_0 does not.) Shore [Sho93] proved that the fact that the class of well orderings is well quasiordered under embeddablity implies ATR_0, getting as a corollary that FRA implies ATR_0. But we still do not know whether FRA could be proved using just ATR_0 (not even Π^1_1-CA_0), as has been conjectured by Peter Clote [Clo90], Stephen Simpson [Sim99, Remark X.3.31] and Alberto Marcone [Mar].

Along with FRA, we study two other statements equivalent to it over RCA_0. One, that we call $\mathsf{WQO(ST)}$, says that the class of signed trees is well quasiordered. A signed tree is a well founded tree which has each node labeled with either a $+$ or a $-$. Given signed trees T and $\check{T}$, we say that $T \preccurlyeq \check{T}$ if there is a homomorphism from T to $\check{T}$ (see Definition 2.1). A useful property of signed trees is that if there exists a homomorphism between two recursive signed trees, then there is one that is hyperarithmetic (Lemma 2.5 says even more than this). This helps us reduce the quantifier complexity of certain formulas talking about them when working in

ATR_0. It might also be useful when trying to prove FRA in ATR_0. We are interested in signed trees because they can be used to represent certain linear orderings that we will call *h-indecomposable*. We will show that, under certain assumptions, every indecomposable linear ordering is equimorphic to an h-indecomposable one. We are also interested in signed trees because they give us a better understanding of the embeddablity relation on linear orderings. For example, in a forthcoming paper [Mon] we use signed trees to prove that every hyperarithmetic linear ordering is equimorphic to a recursive one.

The other statement we prove equivalent to FRA is the *Finite decomposability of linear orderings*, that we call FINDEC. A version of FINDEC was proved by Laver in [Lav71]. It says that every scattered linear ordering can be decomposed, up to equimorphism, as a sum of h-indecomposable linear orderings. (A partial ordering is *scattered* if it does not contain a copy of the rational numbers. Two linear orderings are equimorphic if each one can be embedded into the other.) The representation of the scattered linear orderings that FINDEC gives us will allow us to prove properties about them as, for example, extendibility. We will also look at minimal decompositions of scattered linear orderings. A *minimal decomposition* is a finite decomposition of minimal length. The interesting feature of minimal decomposition is that they are unique up to equimorphism. We will prove that the existence of minimal decompositions for every scattered linear ordering is also equivalent to FRA.

Jullien's Theorem. In the case of extendibility of linear orderings, people have been interested not only in its reverse mathematical strength, but also in the effective content of certain theorems. For example, Szpilrajn proved in [Szp30] that every partial ordering has a linearization. This can be done in an effective way; that is, for every partial ordering we can effectively construct a linearization of it (see [Dow98, Observation 6.1]). The effectiveness of the extendibility of ω^* has also been studied: Rosenstein and Kierstead proved that every recursive well founded partial ordering has a recursive well founded linearization; and Rosenstein and Statman proved that there is a recursive partial ordering without recursive descending sequences which has no recursive linearization without recursive descending sequences. (For proofs of these results and other related ones see [Ros84] and see [Ros82] for more background.) The proof theoretic strength of the fact that ω^* is extendible was studied by Rod Downey, Denis Hirschfeldt, Steffen Lempp and Reed Solomon in [DHLS03]. They showed that the extendibility of ω^* can be proved in ACA_0, that it implies WKL_0, and that it is not implied by WKL_0. It is not known whether it is equivalent to ACA_0, or it is strictly in between WKL_0 and ACA_0. In that same paper they studied the extendibility of ζ, the order type of the integers, and of η, the order type of the rationals. They prove that the extendibility of ζ is equivalent to ATR_0 over RCA_0. For η, they adapted Bonnet and Pouzet's proof of its extendibility to work in Π^1_2-CA_0 and then they give a modification of their proof, due to Howard Becker, that uses only Π^1_1-CA_0. Joseph Miller [Mil] proved that the extendibility of η implies WKL_0 and that over Σ^1_1-AC_0, it implies ATR_0. We prove in this paper that the extendibility of η is provable in ATR_*, which is strictly weaker than Π^1_1-CA_0, using a completely different proof. Our proof is based on a general analysis of the extendibility of h-indecomposable linear orderings and on the fact that if a partial ordering does not embed η, there is some h-indecomposable linear ordering that is does not embed either.

Rod Downey and R. B. Remmel asked about the effective content of the Bonnet-Jullien result that here we call Jullien's theorem in [DR00, Question 4.1] and also in [Dow98, Question 6.1]. In [DR00] they observe that Jullien's proof requires Π^1_2-CA_0, and they mention that it would be remarkable if Jullien's theorem was equivalent to Π^1_2-CA_0. It will follow from our results that this is not the case. (Because it is implied by RCA_*+FRA which does not even imply Π^1_1-CA_0.)

As we said above, Jullien's theorem, as stated in his thesis, says that a scattered linear ordering is extendible if and only if it has a minimal decomposition of a certain kind. The first problem that we have here is that the existence of minimal decompositions is proof theoretically too strong (it implies FINDEC, which implies FRA). Therefore, the statement that we call JUL(min-dec), and asserts that a linear ordering which has a minimal decomposition is extendible if and only if a certain property of the decomposition holds, does not completely characterize the extendible linear orderings. However, we do study the proof theoretic strength of JUL(min-dec), and we prove that it is equivalent to ATR_* over RCA_*. This proof is divided in to parts. In one, we prove that every h-indecomposable linear ordering is extendible. Moreover, we prove that for all h-indecomposable linear orderings, $\mathcal{L}$, any partial ordering, $\mathcal{P}$, which does not embed $\mathcal{L}$ has a linearization, hyperarithmetic in $\mathcal{P} \oplus \mathcal{L}$, which does not embed $\mathcal{L}$. In the other part, we use this result to prove that every linear ordering, $\mathcal{L}$, which is a finite sum of h-indecomposable ones satisfying a certain property is extendible. We also get that the linearizations can be taken to be hyperarithmetic in $\mathcal{L}$ and the partial ordering. The fact that we are getting hyperarithmetic linearizations not only is interesting in its own right from the viewpoint of effective mathematics, but also it is useful to reduce the complexity of some formulas we need to prove by induction. We will use the fact that existential quantification over the hyperarithmetic sets is, in certain cases, equivalent to universal second order quantification. This will allow us to transform some complicated formulas into Π^1_1 equivalents and then prove them by Σ^1_1-IND. The extendibility of η will follow from the extendibility of h-indecomposable linear orderings and the fact that if a partial ordering does not embed η, there is some h-indecomposable linear ordering which it does not embed either.

Because of the problem about the minimal decomposition we mentioned earlier we study the equivalent formulation, JUL, of Jullien's theorem. We will show that one of the directions of JUL can be proved in RCA_0; it is the other direction that is proof theoretically strong. It will also not be hard to show that JUL follows from JUL(min-dec) and the existence of minimal decompositions for every scattered linear ordering. Using this, we show that JUL follows from FRA and Σ^1_1-IND. We will also prove that JUL implies FRA over RCA_0, getting that JUL and FRA are equivalent over RCA_*.

We have to note that we are not proving the equivalence of FRA and JUL over RCA_0. Instead we prove it over RCA_*, which in addition to RCA_0 has Σ^1_1-IND. RCA_* is still a very weak system and, as RCA_0 and RCA, is closely related to to Computable mathematics. From our work, one can still get that the amount of set existence axioms needed to prove JUL and FRA is the same.

Simpson claimed in [Sim99, pag. 176] that, over RCA_0, Friedman's system, ATR_0, is the weakest set of axioms which permits the development of a decent theory of countable ordinals. Similarly, we should conclude from our work that, over RCA_*, FRA (which could still be equivalent to ATR_*) is the weakest set of axioms which permits the development of a decent theory of countable linear orderings modulo equimorphisms.

1.1. **Basic Definitions.** We use $I\!N$ for the set of all the natural numbers and ω for the linear ordering $\omega = \langle I\!N, \leq_{I\!N}\rangle$. Some authors use ω for the standard first order model of the natural numbers. Since we are not dealing with models at all, this will not cause confusion.

Even though our language only let us talk about natural numbers, we can encode pairs and finite sequences of natural numbers as natural numbers. We have a recursive pairing function $\langle\cdot,\cdot\rangle$, and recursive projection functions $(\cdot)_0$ and $(\cdot)_1$ such that $(\langle x,y\rangle)_0 = x$ and $(\langle x,y\rangle)_1 = y$. The same for triplets of elements, $\langle x,y,z\rangle$, and strings $\langle x_0,...,x_{n-1}\rangle$ of any finite length. Given a set X, we denote by Seq_X the set of strings of elements of X. We use Seq for $\mathrm{Seq}_{I\!N}$ and Seq_2 for $\mathrm{Seq}_{\{0,1\}}$, the set of binary stings. For a string $\sigma = \langle x_0,...,x_{n-1}\rangle$ we define $|\sigma| = n$, $\sigma(i) = x_i$, $\mathrm{last}(\sigma) = x_{n-1}$, $\sigma^- = \langle x_0,...,x_{n-2}\rangle$, $\sigma^\frown x = \langle x_0,...,x_{n-1},x\rangle$, $\sigma^\frown\langle y_0,...,y_{m-1}\rangle = \langle x_0,...,x_{n-1},y_0,...,y_{m-1}\rangle$ and $\sigma\restriction m = \langle x_0,...,x_{m-1}\rangle$.

Orderings. A binary relation $\leq_P$ on a set P is a *quasiordering* if it is reflexive and transitive. It is a *partial ordering* if it is also antisymmetric, and a *linear ordering* if it is also total (i.e: $\forall x,y \in P(y \leq_P x \vee x \leq_P y)$). If a partial ordering is called $\mathcal{P}$, we will usually use the letter P for its domain and $\leq_P$ for its relation. An *embedding* from a partial ordering $\mathcal{P} = \langle P, \leq_P\rangle$ to another partial ordering $\mathcal{Q} = \langle Q, \leq_Q\rangle$ is a one-to-one map $f\colon P \to Q$ such that $\forall x,y \in P(x <_P y \Leftrightarrow f(x) <_Q f(y))$. If this is the case, we write $f\colon \mathcal{P} \hookrightarrow \mathcal{Q}$. When such an f exists, we say that $\mathcal{P}$ *embeds* in $\mathcal{Q}$, and write $\mathcal{P} \preccurlyeq \mathcal{Q}$. Two linear orderings $\mathcal{L}_1$ and $\mathcal{L}_2$ are *equimorphic* if $\mathcal{L}_1 \preccurlyeq \mathcal{L}_2$ and $\mathcal{L}_2 \preccurlyeq \mathcal{L}_1$. We write $\mathcal{L}_1 \sim \mathcal{L}_2$ when $\mathcal{L}_1$ and $\mathcal{L}_2$ are equimorphic. An *equimorphism* between $\mathcal{L}_1$ and $\mathcal{L}_2$ is a pair $\langle f_1, f_2\rangle$, where $f_1\colon \mathcal{L}_1 \hookrightarrow \mathcal{L}_2$ and $f_2\colon \mathcal{L}_2 \hookrightarrow \mathcal{L}_1$. If the embeddings f_1 and f_2 are inverses of each other, we have an *isomorphism*, we say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are *isomorphic* and we write $\mathcal{L}_1 \cong \mathcal{L}_2$. A linearization of a partial ordering $\langle P, \leq_P\rangle$ is a relation $\leq_Q$ on P such that $\langle P, \leq_Q\rangle$ is a linear ordering and $\leq_Q$ extends $\leq_P$ in the sense that $\forall x,y \in P(x \leq_P y \Rightarrow x \leq_Q y)$.

Some examples of linear orderings are: $\mathbf{1}$, the linear ordering with one element; $\mathbf{m}$, the linear ordering with m many elements; ω, the order type of the natural numbers; ζ, the order type of the integers; η, the order type of the rationals; and ω_1^{CK}, the first non-recursive ordinal. A partial ordering which does not embed η is said to be *scattered*.

We have some operations on the class of orderings. The *reverse* partial ordering of $\mathcal{P} = \langle P, \leq_P\rangle$ is $\mathcal{P}^* = \langle P, \geq_P\rangle$. The *product*, $\mathcal{P} \times \mathcal{Q}$, of two partial orderings $\mathcal{P}$ and $\mathcal{Q}$ is obtained by substituting a copy of $\mathcal{P}$ for each element of $\mathcal{Q}$. That is: $\mathcal{P} \times \mathcal{Q} = \langle P \times Q, \leq_{P\times Q}\rangle$ where $\langle x,y\rangle \leq_{P\times Q} \langle x',y'\rangle$ iff $y <_Q y'$ or $y = y'$ and $x \leq_P x'$. The *sum*, $\sum_{i\in\mathcal{P}} \mathcal{P}_i$, of a set of partial orderings $\{\mathcal{P}_i\}_{i\in P}$ indexed by another partial ordering $\mathcal{P}$, is constructed by substituting a copy of $\mathcal{P}_i$ for each element $i \in P$. So, for example, $\mathcal{P} \times \mathcal{Q} = \sum_{i\in\mathcal{Q}} \mathcal{P}$. When $\mathcal{P} = \mathbf{m}$, we sometimes write $\mathcal{P}_0 + ... + \mathcal{P}_{m-1}$, $\sum_{i<m} \mathcal{P}_i$ or $\sum_{i=0}^{m-1} \mathcal{P}_i$ instead of $\sum_{i\in\mathbf{m}} \mathcal{P}_i$. When $\mathcal{P} = \omega$, we sometimes write $\sum_{i=k}^{\infty} \mathcal{P}_i$ or $\sum_{i\in\omega, i\geq k} \mathcal{P}_i$ instead of $\sum_{i\in\omega} \mathcal{P}_{i+k}$. The *direct sum*, $\bigoplus_{i\in I} \mathcal{P}_i$, of a set of partial orderings $\{P_i\}_{i\in I}$ indexed by a set I, is constructed by taking the disjoint union of the $\mathcal{P}_i$ and letting elements from different P_i's be incomparable. So $\bigoplus_{i\in I} \mathcal{P}_i = \sum_{i\in\mathcal{I}} \mathcal{P}_i$, where $\mathcal{I}$ is the partial ordering with domain I where all the elements are incomparable.

Given a linear ordering $\mathcal{L} = \langle L, \leq_L \rangle$, we can order Seq_L in various ways. The first ordering we have is the one given by inclusion. For two strings, σ and τ, we use the word *incompatible* when they are incomparable under inclusion, and write $\sigma|\tau$. The most common linear ordering on Seq_L is the *lexicographic ordering*, $\leq_{\mathrm{Seq}_L}$: Given $\sigma_0, \sigma_1 \in \mathrm{Seq}_L$, we let $\sigma_0 \leq_{\mathrm{Seq}_L} \sigma_1$ iff either $\sigma_0 \subseteq \sigma_1$ or $\sigma_0|\sigma_1$ and $x_0 \leq_L x_1$, where x_0 and x_1 are such that for some τ, $\tau^\frown x_0 \subseteq \sigma_0$, $\tau^\frown x_1 \subseteq \sigma_1$, and $x_0 \neq x_1$. On Seq_2 we also have the *Left-to-right ordering*, $\leq_{LR}$. It coincides with the lexicographic ordering on incompatible stings. When $\sigma \subset \tau$ we let $\sigma \leq_{LR} \tau$ if $\tau(|\sigma|) = 1$ and $\sigma \geq_{LR} \tau$ if $\tau(|\sigma|) = 0$. Observe that $\langle \mathrm{Seq}_2, \leq_{LR} \rangle$ has order type η.

Given a partial ordering $\mathcal{P} = \langle P, \leq_P \rangle$, and $x \in P$, we let $P_{(<x)} = \{y \in P : y <_P x\}$ and $\mathcal{P}_{(<x)} = \langle P_{(<x)}, \leq_P \rangle$. Analogously we define $\mathcal{P}_{(>x)}$, $\mathcal{P}_{(\leq x)}$, and $\mathcal{P}_{(\geq x)}$. We let $(x, y)_P$ be the interval $\{z : x <_P z <_P y\}$.

A linear ordering, $\mathcal{L}$, is *indecomposable* if whenever $\mathcal{L} = \mathcal{A} + \mathcal{B}$, either $\mathcal{L} \preccurlyeq \mathcal{B}$ or $\mathcal{L} \preccurlyeq \mathcal{A}$. $\mathcal{L}$ is *indecomposable to the right (left)* if whenever $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and $\mathcal{B} \neq \emptyset$ $(\mathcal{A} \neq \emptyset)$, $\mathcal{L} \preccurlyeq \mathcal{B}$ $(\mathcal{L} \preccurlyeq \mathcal{A})$. Sometimes, instead of saying that $\mathcal{L}$ is indecomposable to the right (left), we say that $\mathcal{L}$ is $\rightarrow$ (is $\leftarrow$).

Lemma 1.1. *(RCA_0) If $\mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A}$, then $\eta \preccurlyeq \mathcal{A}$.*

Proof. Assume $\mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A}$. Observe that then $\mathcal{A} + \mathbf{1} + \mathcal{A} \preccurlyeq \mathcal{A} + \mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A}$. So we have two embeddings $f_0, f_1 \colon A \hookrightarrow A$ and an $a \in \mathcal{A}$ such that $\forall x, y \in A(f_0(x) <_A a <_A f_1(y))$. Now, given $\sigma \in \mathrm{Seq}_2$ define

$$f(\sigma) = f_{\sigma(0)}(f_{\sigma(1)}(...(f_{\sigma(|\sigma|-1)}(a))...)).$$

f is an embedding of $\langle \mathrm{Seq}_2, \leq_{LR} \rangle \cong \eta$ into $\mathcal{A}$. □

Lemma 1.2. *(RCA_0) If $\mathcal{A}$ is scattered, indecomposable to the right, and different from $\mathbf{1}$, then $\mathbf{1} + \mathcal{A} \sim \mathcal{A}$ but $\mathcal{A} + \mathbf{1} \not\preccurlyeq \mathcal{A}$.*

Proof. For the first part decompose $\mathcal{A}$ as $\mathcal{B} + \mathcal{C}$ with $\mathcal{B}$ and $\mathcal{C}$ non-empty. Then $\mathbf{1} \preccurlyeq \mathcal{B}$ and $\mathcal{A} \preccurlyeq \mathcal{C}$.

For the second part, if $\mathcal{A} + \mathbf{1} \preccurlyeq \mathcal{A}$, we have that for some $a \in \mathcal{A}$, $\mathcal{A} \preccurlyeq \mathcal{A}_{(<a)}$. Since $\mathcal{A} \preccurlyeq \mathcal{A}_{(\geq a)}$, we have that $\mathcal{A} + \mathcal{A} \preccurlyeq A$. By the previous lemma, this contradicts the assumption that $\mathcal{A}$ is scattered. □

2. Signed trees and h-indecomposable linear orderings

In this section we introduce signed trees and h-indecomposable linear orderings. An h-indecomposable (or *hereditarily indecomposable*) linear ordering is an indecomposable linear ordering that is built up recursively from simpler h-indecomposable linear orderings (Definition 2.6). We are interested in them because, since we have a nice way of representing them, it is easier to prove properties about them. It can be proved (in classical mathematics) that every indecomposable scattered linear ordering is equimorphic to an h-indecomposable one (Lemma 3.3). Therefore, since we are only interested in the class of linear orderings up to equimorphism, we are not loosing generality by only considering the h-indecomposable linear orderings. To represent h-indecomposable linear ordering we use signed trees. Signed trees are easy to deal with (see for example Lemma 2.5) and they encode the whole structure of the h-indecomposable linear orderings.

2.1. Signed trees.

Definition 2.1. *A* signed tree *is pair $\langle T, s_T\rangle$, where T is a well founded subtree of* Seq *and s_T is a map, called a* sign function*, from T to $\{+,-\}$. We will usually write T instead of $\langle T, s_T\rangle$. A* homomorphism *from a signed tree T to another signed tree $\check{T}$ is map $f\colon T \to \check{T}$ such that*

- *for all $\sigma \subset \tau \in T$ we have that $f(\sigma) \subset f(\tau)$ and*
- *for all $\sigma \in T$, $s_{\check{T}}(f(\sigma)) = s_T(\sigma)$.*

In the class of signed trees, we define a binary relation $\preccurlyeq$. We let $T \preccurlyeq \check{T}$ if there exists a homomorphism $f\colon T \to \check{T}$.

We will also consider the empty tree with the empty sign function $\langle \emptyset, \emptyset\rangle$ as a signed tree. We will denote it by $\emptyset$.

Remark 2.2. *For f to be a homomorphism, we do not require that $\sigma|\tau$ implies $f(\sigma)|f(\tau)$.*

Notation 2.3. *For $\sigma \in T$, we let $T_\sigma = \{\tau : \sigma^\frown\tau \in T\}$ and $s_{T_\sigma}(\tau) = s_T(\sigma^\frown\tau)$.*

Statement 2.4. *Let WQO(ST) be the statement that says that the class of signed trees is well quasiordered under $\preccurlyeq$: For every sequence $\langle T_i\rangle_{i\in\mathbb{N}}$ of signed trees there are $i < j$ such that $T_i \preccurlyeq T_j$.*

It will follow from Proposition 2.13 that WQO(ST) follows from Fraïsé's conjecture, and therefore it is provable in classical mathematics. We will prove in the next section that FRA and WQO(ST) are actually equivalent over RCA_0. WQO(ST) seems to be a statement that is easier to deal with than Fraïsé's conjecture, and it might be useful for the study of the latter one.

The following lemma is an important property about signed trees that we will use later.

Lemma 2.5. *(ATR_0) Given recursive signed trees T and $\check{T}$ we can decide whether $T \preccurlyeq \check{T}$ recursively in $0^{2\alpha+2}$ where α is the rank of T. Moreover, if $T \preccurlyeq \check{T}$, then we can find a homomorphism recursively in $0^{2\alpha+2}$.*

Proof. $T \preccurlyeq \check{T}$ if and only if there is a $\sigma \in \check{T}$ such that $s_T(\emptyset) = s_{\check{T}}(\sigma)$ and for each n there is an m such that $T_{\langle n\rangle} \preccurlyeq \check{T}_{\sigma^\frown m}$. Then, by effective transfinite recursion we can construct a $\Sigma^0_{2\alpha+2}$-computable formula which says $T \preccurlyeq \check{T}$. (See [AK00, Chapter 7] for a definition of Σ^0_α-computable formulas.) More specifically, given $\tau \in T$, $\tau' \in \check{T}$, define a formula $\varphi_{\tau,\tau'}$ by effective transfinite recursion as follows:

$$\varphi_{\tau,\tau'} \equiv \exists \sigma \in \check{T}(\tau' \subseteq \sigma \;\&\; s_T(\tau) = s_{\check{T}}(\sigma) \;\&\; \forall n(\tau^\frown n \in T \Rightarrow \exists m(\varphi_{\tau^\frown n, \sigma^\frown m}))).$$

By transfinite induction we can prove that $\varphi_{\tau,\tau'}$ is a $\Sigma^0_{2\,\mathrm{rk}(T_\tau)+2}$-computable formula. Then, $0^{2\alpha+2}$ can compute the truth value of these formulas. We claim that $T \preccurlyeq \check{T}$ if and only if $\varphi_{\emptyset,\emptyset}$ holds. It $f\colon T \to \check{T}$ is a homomorphism, then we can prove by transfinite induction that for every $\tau \in T$, $\varphi_{\tau,f(\tau)}$ holds, and then that $\varphi_{\emptyset,\emptyset}$ holds too. On the other hand, we can prove, also by transfinite induction, that if $\varphi_{\tau,\tau'}$ holds, there is a homomorphism $f_\tau\colon T_\tau \to \check{T}_{\tau'}$ recursive in $0^{2\,\mathrm{rk}(T_\tau)+2}$. To define the homomorphism we have to search for a $\sigma \in \check{T}_{\tau'}$, and then for each n find an m_n and a homomorphism $f_n\colon T_{\tau^\frown n} \hookrightarrow \check{T}_{\sigma^\frown m_n}$; $0^{2\,\mathrm{rk}(T_\tau)+2}$ can do this uniformly. Then let $f_\tau(\emptyset) = \sigma$ and $f_\tau(n^\frown\pi) = \sigma^\frown m_n^\frown f_n(\pi)$. $\square$

2.2. **H-indecomposable linear orderings.** We associate to each signed tree T, a linear ordering $\mathrm{lin}(T)$. The idea is the following: If $T = \{\emptyset\}$, then we let $\mathrm{lin}(T) = \omega$ or $\mathrm{lin}(T) = \omega^*$ depending on whether $s_T(\emptyset) = +$ or $s_T(\emptyset) = -$. Now suppose $T \supsetneq \{\emptyset\}$. For $i \in \mathbb{N}$, let T_i be the tree $\{\sigma : i^\frown\sigma \in T\}$, and consider the signed function over T_i defined by $s_{T_i}(\sigma) = s_T(i^\frown\sigma)$. If $s_T(\emptyset) = +$, we want $\mathrm{lin}(T)$ to be an ω sum of copies of T_0, T_1,..., where each T_i appears infinitely often in the sum. So, we let

$$\mathrm{lin}(T) = \sum_{n\in\omega} \mathrm{lin}(T_{(n)_0}).$$

If $s_T(\emptyset) = -$, we let

$$\mathrm{lin}(T) = \sum_{n\in\omega^*} \mathrm{lin}(T_{(n)_0}).$$

Now we give the formal definition of $\mathrm{lin}(T)$. It is not hard two see that the two definitions coincide.

Definition 2.6. *To each signed tree T we assign a linear ordering $\mathrm{lin}(T) = \langle L, \leq_T\rangle$. Given $\sigma \in \mathrm{Seq}$, let $(\sigma)_0 = \langle(\sigma(0))_0, (\sigma(1))_0, ..., (\sigma(|\sigma|-1))_0\rangle$. Let $\widehat{T} = \{\sigma \in \mathrm{Seq} : (\sigma)_0 \in T\}$. Let L be the set of strings $\sigma^\frown m \in \mathrm{Seq}$ such that σ is an end node of $\widehat{T}$ and $m \in \mathbb{N}$. Let $\sigma_1^\frown m_1$ and $\sigma_2^\frown m_2$ be distinct elements in L, let $\tau \in \widehat{T}$ and $n_1 \neq n_2 \in \mathbb{N}$ be such that $\tau^\frown n_1 \subseteq \sigma_1^\frown m_1$ and $\tau^\frown n_2 \subseteq \sigma_2^\frown m_2$. We define*

$$\sigma_1^\frown m_1 <_T \sigma_2^\frown m_2 \Leftrightarrow \begin{cases} n_1 < n_2 \ \&\ s_T((\tau)_0) = + \text{ or} \\ n_1 > n_2 \ \&\ s_T((\tau)_0) = -. \end{cases}$$

lin *of the empty signed tree is defined to be* $\mathbf{1}$.

We say that a linear ordering, $\mathcal{L}$, is h-indecomposable *if it is of the form* $\mathrm{lin}(T)$ *for some signed tree T. $\mathcal{L}$ is* h-indecomposable to the right *if $s_T(\emptyset) = +$ and* h-indecomposable to the left *otherwise.*

Remark 2.7. *One should observe that the definition of* $\mathrm{lin}(T)$ *depends on the pairing function used, which is something that, usually, one would like to avoid. But, in this paper, we are only interested in linear orderings up to equimorphisms. It is not hard to see that if we use another paring function, as long as it satisfies that*

$$\forall i \exists^\infty n (i = (n)_0),$$

we will get an equimorphic linear ordering.

Example 2.8. *We show how the function* lin *behaves on small signed trees. We represent the signed trees with a picture, where the root is on top and on every node we put a $+$ or $-$ depending on the value of s_T on it.*

$$\mathrm{lin}(+) = \omega; \qquad \mathrm{lin}\left(\begin{array}{c} - \\ | \\ - \\ | \\ - \end{array}\right) = ... + (... + \omega^* + \omega^*) + (... + \omega^* + \omega^*);$$

$$\mathrm{lin}\left(\begin{array}{c} + \\ | \\ - \end{array}\right) = \omega^* + \omega^* + \omega^* + ...; \qquad \mathrm{lin}\left(\begin{array}{ccc} & + & \\ \swarrow & & \searrow \\ - & & + \end{array}\right) \sim \omega + \omega^* + \omega + \omega^* ...$$

In the rest of this section we will prove that h-indecomposable linear orderings are indecomposable and scattered, and that the quasi-ordering $\preccurlyeq$ on signed trees coincides with the quasi-ordering $\preccurlyeq$ on h-indecomposable linear orderings. In the last subsection of this section we will prove that $\mathsf{WQO(ST)}$ implies ATR_0. In a first reading of the paper, the reader could assume these results and move on to the next section.

Lemma 2.9. *(RCA_0) Every h-indecomposable linear ordering, $\mathcal{L}$, is indecomposable. Moreover, if $\mathcal{L}$ is h-indecomposable to the right (left), for every $x \in L$ we can find an embedding $f\colon \mathcal{L} \hookrightarrow \mathcal{L}_{(>x)}$, ($f\colon \mathcal{L} \hookrightarrow \mathcal{L}_{(<x)}$), uniformly recursively in x and $\mathcal{L}$.*

Proof. **1** is both, h-indecomposable and indecomposable. So suppose that $\mathcal{L}$ is h-indecomposable to the right. Think of the domain of $\mathcal{L}$ as $\{\langle m, y\rangle : y \in L_m\}$, where, if $\mathcal{L} = \mathrm{lin}(T)$, then $\mathcal{L}_m = \mathrm{lin}(T_{(m)_0})$. Say $x = \langle \bar{m}, y\rangle$, $y \in L_{\bar{m}}$. Consider an increasing function $h\colon I\!N \to I\!N$ such that $\forall n(h(n) > \bar{m}\ \&\ (h(n))_0 = (n)_0)$. (For example $h(0) = \langle 0, \bar{m}+1\rangle$ and $h(n) = \langle (n)_0, h(n-1)\rangle$.) Define $f(\langle m, y\rangle) = \langle h(m), y\rangle$. It is not hard to see that $f\colon \mathcal{L} \hookrightarrow \mathcal{L}_{(>x)}$. □

A version of the converse of this lemma will be proved in 3.3 using stronger assumptions.

Lemma 2.10. *(RCA_0) Every h-indecomposable linear ordering is scattered.*

Proof. Suppose that we have an embedding $f : \mathbb{Q} \hookrightarrow \mathcal{L}$, where $\mathcal{L} = \mathrm{lin}(T)$ is h-indecomposable. Given $\sigma \in T$, let $\mathcal{L}_\sigma = \mathrm{lin}(T_\sigma)$. By recursion on n, we define a_n and $b_n \in \mathbb{Q}$ and $\sigma_n \in T$, such that $a_n <_{\mathbb{Q}} b_n$, and $f(a_n)$ and $f(b_n)$ belong to the same copy of $\mathcal{L}_{\sigma_n}$ in $\mathcal{L}$. Let $\sigma_0 = \emptyset$ and a_0 and b_0 be any two different elements of $\mathbb{Q}$. Suppose we have already defined a_n and $b_n \in \mathbb{Q}$ and $\sigma_n \in T$. So, we have that

$$f((a_n, b_n)_{\mathbb{Q}}) \subseteq \mathcal{L}_{\sigma_n} = \sum_{m\in\omega(\text{or } \omega^*)} \mathcal{L}_{\sigma_n^\frown (m)_0}.$$

Since $(a_n, b_n)_{\mathbb{Q}}$ does not embed in either ω or ω^*, there have to be some $m \in I\!N$, and some a_{n+1} and $b_{n+1} \in \mathbb{Q}$, with $a_n \leq_{\mathbb{Q}} a_{n+1} <_{\mathbb{Q}} b_{n+1} \leq_{\mathbb{Q}} b_n$, such that $f(a_{n+1}), f(b_{n+1}) \in \mathcal{L}_{\sigma_n^\frown (m)_0}$. Note that we can find m, a_{n+1} and b_{n+1} recursively. Let $\sigma_{n+1} = \sigma_n^\frown (m)_0$. We have just defined partial recursively sequences $\langle \sigma_n\rangle_n$, $\langle a_n\rangle_n$ and $\langle b_n\rangle_n$, and proved by induction that $f(a_n)$ and $f(b_n)$ belong to the same copy of $\mathcal{L}_{\sigma_n}$ and that for every n, σ_n, a_n and b_n are defined. We can also show by induction that $\forall n < m(\sigma_n \subsetneq \sigma_m)$. Therefore, we have constructed a an infinite path in T, contradicting the fact that it is well founded. □

Before proving that the quasi-ordering $\preccurlyeq$ on signed trees coincides with the ordering $\preccurlyeq$ on h-indecomposable linear orderings we need to prove the following lemma.

Lemma 2.11. *(RCA_0) If $\mathcal{L}$ is h-indecomposable to the right and $\mathcal{L} \preccurlyeq \sum_{i\in\alpha^*} \mathcal{A}_i$, where α is well ordered, then for some $i \in \alpha$, $\mathcal{L} \preccurlyeq \mathcal{A}_i$.*

(ACA_0) Moreover, given recursive indices for $\mathcal{L}$, $\langle \mathcal{A}_i : i \in \alpha\rangle$, and the embedding $f\colon \mathcal{L} \hookrightarrow \sum_{i\in\alpha^*} \mathcal{A}_i$ we can find an i and a recursive index for an embedding $g\colon \mathcal{L} \hookrightarrow \mathcal{A}_i$, uniformly recursively in $0'$.

Proof. Consider $f\colon \mathcal{L} \hookrightarrow \sum_{i\in\alpha^*} \mathcal{A}_i$. Write $\mathcal{L}$ as $\sum_{m\in\omega} \mathcal{L}_m$, and for each m let x_m be a member of $\mathcal{L}_m$ (say the least one in the order of the natural numbers). Note that the sequence $\langle x_m\rangle_{m\in I\!N}$ is co-final in $\mathcal{L}$. For each m, let $a_m \in \alpha^*$ be such that $f(x_m) \in \mathcal{A}_{a_m}$. The sequence $\langle a_m\rangle_{m\in I\!N}$ is decreasing in α (increasing in α^*). Since α is well ordered, there is some m_0 such that $\forall m \geq m_0(f(x_m) \in \mathcal{A}_{a_{m_0}})$. Let $i = a_{m_0}$. (Observe that if $0'$ exists, it can find i.) Therefore f maps $\sum_{j=m_0+1}^{\infty} \mathcal{L}_j$

into $\mathcal{A}_i$. Then, we can construct g by composing f with an embedding of $\mathcal{L}$ into $\sum_{j=m+1}^{\infty}\mathcal{L}_j$, that we have by Lemma 2.9. □

Corollary 2.12. *(RCA_0) If $\mathcal{L}$ is h-indecomposable to the right and $\mathcal{L}+\mathbf{1}\preccurlyeq\sum_{i\in\omega}\mathcal{A}_i$, then for some $i\in\omega$, $\mathcal{L}\preccurlyeq\mathcal{A}_i$.*

(ACA_0) Moreover, given recursive indices for $\mathcal{L}$, $\langle\mathcal{A}_i : i\in\omega\rangle$, and the embedding $f\colon\mathcal{L}+\mathbf{1}\hookrightarrow\sum_{i\in\omega}\mathcal{A}_i$ we can find an i and a recursive index for an embedding $g\colon\mathcal{L}\hookrightarrow\mathcal{A}_i$, uniformly recursively in $0'$.

Proof. If we have an embedding of $\mathcal{L}+\mathbf{1}$ into $\sum_{i\in\omega}\mathcal{A}_i$, we have an embedding of $\mathcal{L}$ into $\sum_{i<n}\mathcal{A}_i$ for some n. Since the linear ordering $\mathbf{n}\cong\mathbf{n}^*$ is well ordered, the corollary follows from the previous lemma. □

Proposition 2.13. *(ACA_0) Let T and $\check{T}$ be signed trees. Then*

$$T\preccurlyeq\check{T}\Leftrightarrow\operatorname{lin}(T)\preccurlyeq\operatorname{lin}(\check{T}).$$

Proof. If either T or $\check{T}$ is empty, then the result is trivial. So suppose neither is empty. First assume that f is a homomorphism witnessing $T\preccurlyeq\check{T}$. Without lost of generality, we can assume that T, $\check{T}$ and f are recursive. Because if they are not, we can relativize the proof. We use effective transfinite recursion to construct an embedding $g\colon\operatorname{lin}(T)\to\operatorname{lin}(\check{T})$. Since for each n, $T_{\langle n\rangle}$ has rank less than T, we can assume that for each n, we have uniformly defined an embedding $g_n\colon\operatorname{lin}(T_{\langle n\rangle})\to\operatorname{lin}(\check{T}_{f(\langle n\rangle)})$. For each n, let $a_n\in\mathbb{N}$ be such that $f(\emptyset)^\frown a_n\subseteq f(\langle n\rangle)$. We can easily modify each g_n and assume that $g_n\colon\operatorname{lin}(T_{\langle n\rangle})\to\operatorname{lin}(\check{T}_{f(\emptyset)^\frown a_n})$. Let $h\colon\mathbb{N}\to\mathbb{N}$ be an increasing function such that $\forall n((h(n))_0=a_{(n)_0}$. (For example, let $h(n+1)=\langle a_{(n)_0},h(n)\rangle$.) We know that $s_T(\emptyset)=s_{\check{T}}(f(\emptyset))$. Assume, without lost of generality, that $s_T(\emptyset)=+$. Now, use the embeddings g_n to construct an embedding

$$\operatorname{lin}(T)=\sum_{n\in\omega}\operatorname{lin}(T_{\langle(n)_0\rangle})\preccurlyeq\sum_{n\in\omega}\operatorname{lin}(\check{T}_{f(\emptyset)^\frown(h(n))_0}),$$

and then use the obvious embeddings

$$\sum_{n\in\omega}\operatorname{lin}(\check{T}_{f(\emptyset)^\frown(h(n))_0})\preccurlyeq\sum_{n\in\omega}\operatorname{lin}(\check{T}_{f(\emptyset)^\frown(n)_0})=\operatorname{lin}(\check{T}_{f(\emptyset)})\preccurlyeq\operatorname{lin}(\check{T}).$$

For the other direction, consider $g\colon\operatorname{lin}(T)\hookrightarrow\operatorname{lin}(\check{T})$. Again, we can assume that T, $\check{T}$ and g are recursive. We will define $\sigma\in\check{T}$ such that $s_{\check{T}}(\sigma)=s_T(\emptyset)$ and assign to each $n\in\mathbb{N}$ an $m_n\in\mathbb{N}$ and a recursive index for an embedding $g_n\colon\operatorname{lin}(T_{\langle n\rangle})\hookrightarrow\operatorname{lin}(\check{T}_{\sigma^\frown m_n})$. We do it uniformly recursively in $0'$ so that we can use $0'$-effective transfinite recursion to define f as follows: ¿From the embeddings g_n, we can get homomorphisms $f_n\colon T_{\langle n\rangle}\to\check{T}_{\sigma^\frown m_n}$. Then, define $f(\emptyset)=\sigma$ and $f(\langle n\rangle^\frown\tau)=\sigma^\frown m_n^\frown f_n(\tau)$.

We start by defining σ and $\bar{g}\colon\operatorname{lin}(T)\hookrightarrow\operatorname{lin}(\check{T}_\sigma)$. For this purpose, we define a sequence $\bar{\sigma}_0$, $\bar{g}_0$, $\bar{\sigma}_1$, $\bar{g}_1$,..., $\bar{\sigma}_n$, $\bar{g}_n$ by recursion. Let $\bar{\sigma}_0=\emptyset$, and $\bar{g}_0=g$. Suppose now, we have already defined $\bar{\sigma}_j$ and $\bar{g}_j$. If $s_T(\emptyset)=s_{\check{T}}(\bar{\sigma}_j)$, let $n=j$, $\sigma=\bar{\sigma}_j$ and $\bar{g}=\bar{g}_j$. Otherwise, suppose that $s_T(\emptyset)=+$ and $s_{\check{T}}(\bar{\sigma}_j)=-$. (The other case is analogous.) Then, by Lemma 2.11, we can find $i\in\mathbb{N}$ and $\bar{g}_{j+1}\colon\mathcal{L}\hookrightarrow\mathcal{L}_{\bar{\sigma}_j^\frown(i)_0}$. Let $\bar{\sigma}_{j+1}=\bar{\sigma}_j^\frown(i)_0\in T$. Since T is well founded, this process cannot go for ever. So, at some point we have to find a j with $s_T(\emptyset)=s_{\check{T}}(\sigma_j)$ and define σ and $\bar{g}$.

Suppose that $s_T(\emptyset) = s_{\check{T}}(\sigma) = +$. (The other case is analogous.) For every $n \in I\!N$ we have

$$\mathrm{lin}(T_{\langle n\rangle}) + \mathbf{1} \preccurlyeq \mathrm{lin}(T) \preccurlyeq \mathrm{lin}(\check{T}_\sigma) = \sum_{m\in\omega} \mathrm{lin}(\check{T}_{f(\sigma)^\frown(m)_0}).$$

So, by Corollary 2.12, for some m_n, we have a recursive index for an embedding, g_n, of $\mathrm{lin}(T_{\langle n\rangle})$ into $\mathrm{lin}(\check{T}_{f(\sigma)^\frown(m_n)_0})$. $0'$ can find these uniformly. □

2.3. **WQO(ST) implies ATR$_0$.** Shore proved in [Sho93] that the fact that the class of well orderings is well quasiordered under embeddablity implies ATR_0. We will use Shore's result to prove the following proposition.

Proposition 2.14. *(RCA_0) $\mathsf{WQO(ST)}$ implies ATR_0.*

The Proposition will follow from the following three lemmas.

Lemma 2.15. *(ACA_0) $\mathsf{WQO(ST)}$ implies ATR_0.*

Proof. We work in ACA_0 and assume $\mathsf{WQO(ST)}$. We will prove that for every sequence $\langle\alpha_i\rangle_{i\in I\!N}$ of ordinals, there are $i < j$ such that α_i embeds in α_j. By Shore's result, this implies ATR_0. For each i we construct a tree T_i as follows: Let T_i be the tree of descending sequences $\langle a_0, ..., a_n\rangle$ with entries in α_i. Consider T_i as a signed tree using the constant function equal to $+$ as the sign function s_{T_i}. By $\mathsf{WQO(ST)}$, there are $i < j$ such that $T_i \preccurlyeq T_j$. We claim that this implies that α_i embeds in α_j. Let f be a homomorphism $T_i \to T_j$. We define $g : \alpha_i \to \alpha_j$ as follows: Given $a \in \alpha_i$, let

$$g(a) = \min\{b \in \alpha_j : \exists\sigma \in T_i(a = \mathrm{last}(\sigma)\ \&\ b = \mathrm{last}(f(\sigma)))\}$$

where $\mathrm{last}(\tau)$ is the last entry of τ. Note that ACA_0 can prove the existence of g. We have to show that $a_0 < a_1 \in \alpha_i$ implies $g(a_0) < g(a_1)$. Let $\sigma \in T_i$ be such that $\mathrm{last}(\sigma) = a_1$ and $\mathrm{last}(f(\sigma)) = g(a_1)$. Consider $\tau = \sigma^\frown a_0 \in T_i$ and let $b_0 = \mathrm{last}(f(\tau))$. Necessarily $f(\tau) \supset f(\sigma)$, and hence, b_0 is smaller than $\mathrm{last}(f(\sigma)) = g(a_1)$. So $g(a_0) \le b_0 < g(a_1)$. □

Now we have to prove that $\mathsf{WQO(ST)}$ implies ACA_0 over RCA_0. We first prove that $\mathsf{WQO(ST)}$ implies ACA_0 over RCA_2, and then prove that $\mathsf{WQO(ST)}$ implies RCA_2. RCA_2 is the system that consist of RCA_0 together with the axiom scheme of Σ^0_2-induction.

Lemma 2.16. *(RCA_2) $\mathsf{WQO(ST)}$ implies ACA_0.*

Proof. We will prove that $\mathsf{WQO(ST)}$ implies that $K = 0'$ exists. Then, by relativizing the proof, as usual, we can get that for all set X, X' exists, and hence ACA_0.

Let T be the tree of sequences $\langle s_0, ..., s_n\rangle \in \mathrm{Seq}$ such that $\{s_0 < ... < s_n\}$ is the set of stages that look true at stage s_n for the enumeration of K. We say that a stage t *looks true at stage s for the enumeration of* K if for all u between t and s, $k_u \ge k_t$, where $\{k_u\}_{u\in I\!N}$ is an enumeration of K. t is a *true* stage if it looks true at every $s \ge t$. Note that if T has a path, it is unique and is the set of the true stages of the enumeration of K. So, from that path we would be able to compute $0'$. Also note that using Σ^0_2-induction we can prove that for every m there is an s_m which is a true stage for the enumeration of K and for which there are m many true stages before s_m. (Σ^0_2-induction is needed because a statement that says that there exits a stage that is a true stage which satisfies some recursive predicate is Σ^0_2.) Assume, toward a contradiction, that $0'$ does not exists as a set. Then we would have that T is well founded. For each $n \in I\!N$, let T_n be the signed tree $\langle T, s_{T_n}\rangle$ where

$$s_{T_n}(\sigma) = \begin{cases} + & \text{if } \sigma \in T \ \&\ |\sigma| \neq n \\ - & \text{if } \sigma \in T \ \&\ |\sigma| = n. \end{cases}$$

Now use $\mathsf{WQO(ST)}$ to get $n < m$ such that $T_n \preccurlyeq T_m$. Let f be an homomorphism from T_n into T_m. Let s be a true stage such that there are $n-1$ many true stages before s. Let σ be the corresponding tuple $\in T$. (i.e.: $\sigma = \langle s_0, ..., s_{n-1}\rangle$, where $\{s_0 < ... < s_{n-1} = s\}$ is the set of stages that look true at s.) Since $s_{T_n}(\sigma) = -$, we have to have that $s_{T_m}(f(\sigma)) = -$, and hence $|f(\sigma)| = m > n$.

We claim that $f(\sigma) \supset \sigma$. Let t be the last element of $f(\sigma)$. If $t > s$, then, since s is a true stage, we would have that $\sigma \subseteq f(\sigma)$. Then $\sigma \subset f(\sigma)$ because $|\sigma| < |f(\sigma)|$. Suppose then, that $t < s$, and σ is incomparable with $f(\sigma)$. There are at most $s-t-1$ many $\tau \in T$ extending $f(\sigma)$. Consider the $s+(s-t)$th true stage and the corresponding sequence in T. We can construct a sequence $\{\sigma_i\}_{i<s-t}$ of nodes of T, such that

$$\sigma \subset \sigma_1 \subset \sigma_2 \subset ... \subset \sigma_{s-t-1}.$$

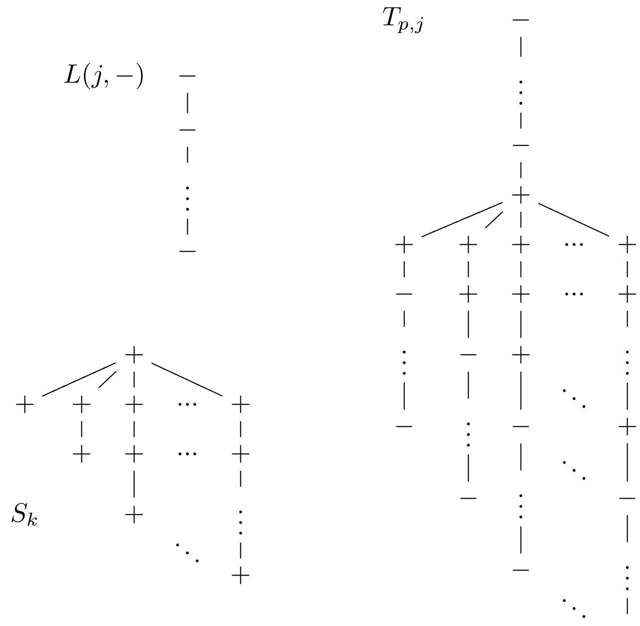
Then, for every $i < j < s-t$ we have to have that $f(\sigma) \subset f(\sigma_i) \subset f(\sigma_j)$. But there are not $s-t$ different nodes on T above $f(\sigma)$. This contradiction to the Pigeon-Hole Principle proves our claim.

Now we can prove by induction that for every n, $f^n(\sigma) \subset f^{n+1}(\sigma)$. Therefore, using f, we can compute the infinite path of T and hence $0'$ too. □

Lemma 2.17. *(RCA_0) $\mathsf{WQO(ST)}$ implies RCA_2.*

Proof. Let $\psi(x) = \exists u \forall v \phi(x, u, v)$ be a Σ^0_2 formula. To verify the instance of the induction scheme for ψ, it suffices to prove that, for each $n \in I\!N$, there exists a set $Z = \{x < n : \psi(x)\}$. Because we can then employ the induction axiom with Z as a parameter, and get induction for ψ up to any n. For each $j < n$ there is a $u_j \leq \omega$ such that, if $\psi(j)$, u_j is the first witness for $\exists u \forall v \phi(j, u, v)$, and if $\neg\psi(j)$, $u_j = \omega$. (Each u_j exists by bounded Σ^0_1-comprehension. See [Sim99, Definition II.3.8 and Theorem II.3.9] for the technical definition and proof of this principle in RCA_0. Note that we are not claiming the existence of the tuple $\langle u_j : j < n\rangle$.)

We will construct a sequence $\langle T_i\rangle_{i\in I\!N}$ of signed trees and then apply $\mathsf{WQO(ST)}$ to it. Each T_i will have n branches $T_{i,j}$, $j = 0, ..., n-1$. Given $k \in I\!N$ and $* \in \{+, -\}$, let $L(k, *)$ bet the signed tree which is linearly ordered, has size k and all its nodes have sign $*$. Given $l \in \omega+1$, let S_l be the signed tree which has a root signed $+$ and for each $i < l$ there is a branch of the form $L(i, +)$. To construct $T_{p,j}$ attach a copy of $S_{u_j - p}$ after the end node of $L(j, -)$ and then attach a copy of $L(n-j, -)$ after each end node of $S_{u_j - p}$. (If $p > u_j$ let $u_j - p = 0$ and if $u_j = \omega$, let $u_j - p = \omega$.) See pictures of $L(k, *)$, $S_{u_j - p}$ and $T_{p,j}$ below. It is not hard to see how to construct $T_{p,j}$ recursively.



By $\mathsf{WQO(ST)}$, there exists $p < q$ such that $T_p \preccurlyeq T_q$. Then, for every $j_0 < n$ there is a $j_1 < n$ such that $T_{p,j_0} \preccurlyeq T_{q,j_1}$. We claim that necessarily $j_0 = j_1$. Every path though T_{p,j_0} consists of $n - j_0$ nodes signed $-$, then some nodes signed $+$ and then j_0 nodes signed $-$. Every path though T_{p,j_1} consists of $n - j_1$ nodes signed $-$, then some nodes signed $+$ and then j_1 nodes signed $-$. The n nodes signed $-$ in a path though T_{p,j_0} have to be mapped into the n nodes signed $-$ in a path though T_{p,j_1}, and the nodes signed $+$ have to be mapped to nodes signed $+$. Therefore, it has to be the case that $j_0 = j_1$. We have also proved that necessarily $S_{u_{j_0}-p} \preccurlyeq S_{u_{j_1}-q}$

The second observation is that if $\psi(j)$ and $T_{p,j} \preccurlyeq T_{q,j}$, then $u_j \leq p$. This is because, to have that $S_{u_j-p} \preccurlyeq S_{u_j-q}$, we need to have that $u_j - p \leq u_j - q = 0$.

So we have that $\psi(j) \Leftrightarrow (\exists u \leq p)\forall v \psi(j,u,v)$. Therefore, Z can be proved to exists in RCA_0 by bounded Σ^0_1 comprehension. □

3. Finite decomposability

Definition 3.1. *A* finite decomposition *of a linear ordering, $\mathcal{L}$, is a finite tuple signed trees $\langle T_0, ..., T_n \rangle$, such that*

$$\mathcal{L} \sim \sum_{i=0}^{n} \mathrm{lin}(T_i).$$

If $\mathcal{F}_i = \mathrm{lin}(T_i)$, we may abuse notation and say that the tuple of h-indecomposable linear orderings $\langle \mathcal{F}_0, ..., \mathcal{F}_n \rangle$ is a finite decomposition of $\mathcal{L}$. In this case we will implicitly assume that the sequence $\langle T_0, ..., T_n \rangle$ is also given.

Statement 3.2. *Let FINDEC be the statement that says that every scattered linear ordering has a finite decomposition.*

FINDEC gives us a nice representation of scattered linear orderings up to equimorphism. This representation will be very useful in the proof of Jullien's Theorem. A proof of FINDEC can be extracted from [Jul69] using Fraïssé's conjecture and Π^1_1-DC_0 (which is equivalent to Σ^1_2-DC_0 and to Δ^1_2-CA_0 plus Σ^1_2-induction [Sim99, Theorem VII.6.9.2], and strictly stronger than, for example, Π^1_1-CA plus Σ^1_2-induction). In this section we prove that FINDEC is equivalent to WQO(ST)and in the next section that it is equivalent to Fraïssé's conjecture. We also analyze finite decompositions of minimal length. We show in ATR_0that, if finite decompositions exists, then minimal decompositions also exists and are unique modulo equimorphisms.

The next lemma uses FINDEC to show that h-indecomposability is the same as indecomposability, modulo equimorphism.

Lemma 3.3. *(RCA_0) FINDEC implies that every scattered indecomposable linear ordering is equimorphic to an h-indecomposable linear ordering.*

Proof. Let $\mathcal{L}$ be scattered and indecomposable, say to the right. By FINDEC, $\mathcal{L} \sim \sum_{i=0}^n \mathcal{F}_i$, where each $\mathcal{F}_i$ is h-indecomposable. Since $\mathcal{L}$ is indecomposable to the right, $\mathcal{L} \preccurlyeq \mathcal{F}_n$. Obviously $\mathcal{F}_n \preccurlyeq \mathcal{L}$, therefore $\mathcal{L} \sim \mathcal{F}_n$ which is h-indecomposable. □

3.1. **FINDEC and WQO(ST).** We prove that FINDEC is equivalent to WQO(ST) over RCA_0.

Lemma 3.4. *(RCA_0) WQO(ST) implies FINDEC.*

Proof. Clote proved that ATR_0 implies that every scattered linear ordering, $\mathcal{L}$, can be embedded in $\mathbb{Z}^\alpha$ for some ordinal α [Clo89, Theorem 16]. ($\mathbb{Z}^\alpha$ is defined by effective transfinite induction as follows: $\mathbb{Z}^0 = \mathbf{1}$, $\mathbb{Z}^{\alpha+1} = \mathbb{Z}^\alpha \times \mathbb{Z}$ and $\mathbb{Z}^{\lim_m \alpha_m} = \sum_{m\in\omega} \mathbb{Z}^{\alpha_m} \times \omega^* + \sum_{m\in\omega^*} \mathbb{Z}^{\alpha_m} \times \omega$.) The least α such that $\mathcal{L}$ embeds in a finite sum of $\mathbb{Z}^\alpha$s is the *rank* of $\mathcal{L}$. The rank can be defined in ATR_0 as in [Clo89]. Recall that WQO(ST) implies ATR_0, so we can use ATR_0 here. By arithmetic transfinite induction we prove that for every ordinal α the following holds: Every recursive linear ordering $\mathcal{L}$ of rank α, for which there is a recursive embedding $\mathcal{L} \preccurlyeq \mathbb{Z}^\alpha \times \mathbf{m}$ for some $m \in \mathbb{N}$, is equimorphic to a finite sum, $\sum_{i=0}^n \mathcal{F}_i$, of h-indecomposable linear orderings such that

- each $\mathcal{F}_i$ has of rank $\leq \alpha$,
- each $\mathcal{F}_i$ is recursive in $0^{2(\alpha+1)^2}$, and
- the equimorphism is recursive in $0^{2(\alpha+1)^2}$.

To do this using only arithmetic transfinite induction (which we have in ATR_0, even in ACA_0; see [Sim99, Lemma V.2.1]) we need to fix a big ordinal α_0 and prove that the statement above holds for every $\alpha < \alpha_0$ by induction on α. Note that ATR_0 implies that $0^{2\alpha_0^2}$ exists as a set. This is why the sentence that we are proving by transfinite induction is just arithmetic. To get finite decomposability for every scattered linear ordering $\mathcal{L}$, the proof has to work for every ordinal α_0 and relative to every set X.

We can write $\mathcal{L}$ as a finite sum of ω or ω^* sums of linear orderings of rank less than α. Clearly, it suffices to consider the case that $\mathcal{L}$ is equal to one of these sums, $\sum_{i\in\omega} \mathcal{L}_i$. By inductive hypothesis, for each i there is a equimorphism recursive in $0^{2\alpha^2}$ between $\mathcal{L}_i$ and a finite sum of h-indecomposable linear orderings of rank $< \alpha$ recursive in $0^{2\alpha^2}$. So now, we have that $\mathcal{L} \sim \sum_{i\in\omega} G_i$, where each $G_i = \mathrm{lin}(T_i)$ is h-indecomposable. Recursively in $0^{2\alpha^2+2}$ we can find these equimorphisms uniformly, and hence the equimorphism $\mathcal{L} \sim \sum_{i\in\omega} G_i$. By Lemma 2.5, recursively uniformly in $0^{2\alpha^2+2\alpha}$, we can tell, for each i and j, whether $G_i \preccurlyeq G_j$ or not. Moreover, if $G_i \preccurlyeq G_j$, we can find the embedding. By WQO(ST) and Proposition 2.13, it cannot happen that

$$\forall k \exists i, j \geq k \forall l > j (G_i \not\preccurlyeq G_l).$$

Otherwise we could define a subsequence $\langle G_{k_i} \rangle_{i \in I\!N}$ such that $\forall i < j (G_i \not\preccurlyeq G_j)$. Let k_0 be such that $\forall i, j \geq k_0 \exists l > j (G_i \preccurlyeq G_l)$. Let $T = \{i^\frown\sigma : \sigma \in T_{i+k_0}\}$, $s_T(\emptyset) = +$ and $s_T(i^\frown\sigma) = s_{T_{i+k_0}}(\sigma)$. We claim that

$$L \sim \sum_{i=0}^{k_0-1} G_i + \mathrm{lin}(T).$$

We have to construct an equimorphism between

$$\sum_{i=k_0}^{\infty} G_i \quad \text{and} \quad \mathrm{lin}(T) = \sum_{m \in \omega} G_{(m)_0+k_0}.$$

The equimorphism can be easily constructed given the pairs $\langle i, j \rangle$ such that $G_i \preccurlyeq G_j$ and the embeddings $f_{ij} \colon G_i \hookrightarrow G_j$, which we have recursively in $0^{2\alpha^2+2\alpha}$. Note that $2(\alpha+1)^2 \geq 2\alpha(\alpha+1) = 2\alpha^2 + 2\alpha$. Then $\langle T_0, ..., T_{k_0-1}, T \rangle$ is a finite decomposition of $\mathcal{L}$. □

The proof of the other direction is divided in two steps.

Lemma 3.5. *(ACA_0) FINDEC implies WQO(ST).*

Proof. Suppose WQO(ST) is false. Then, using Proposition 2.13, there is a sequence $\langle \mathcal{L}_i \rangle_{i \in I\!N}$ of h-indecomposable linear orderings such that for all $i < j$, $\mathcal{L}_i \not\preccurlyeq \mathcal{L}_j$. By taking an infinite subsequence, we can assume that all the $\mathcal{L}_i$ are h-indecomposable in the same direction. Let us assume they are all h-indecomposable to the right. Let $\mathcal{L} = \sum_{i \in \omega} \mathcal{L}_i$. We claim that $\mathcal{L}$ is scattered but it can not be decomposed as a finite sum of h-indecomposables and therefore that FINDEC does not hold. By Lemma 2.10, each $\mathcal{L}_i$ is scattered, so $\mathcal{L}$ is scattered too. Suppose, toward a contradiction, that $\mathcal{L} \sim \sum_{j=0}^{n} \mathcal{F}_j$, where each $\mathcal{F}_j$ is h-indecomposable.

First we show that for some $k \in I\!N$, $\mathcal{F}_n \sim \sum_{i=k}^{\infty} \mathcal{L}_i$. Let f and g be an embeddings, $f \colon \mathcal{L} \hookrightarrow \sum_{j=0}^{n} \mathcal{F}_j$ and $g \colon \sum_{j=0}^{n} \mathcal{F}_j \hookrightarrow \mathcal{L}$. Let $h \colon \mathcal{L} \hookrightarrow \mathcal{L}$ be the composition of g and f. We claim that for every $k \in I\!N$, the image of $\sum_{i=k}^{\infty} \mathcal{L}_i$ under h is included in $\sum_{i=k}^{\infty} \mathcal{L}_i$. The proof of the claim is a straightforward induction using that each $\mathcal{L}_i$ is indecomposable to the right and hence cannot be embedded into a proper initial segment of it. Now, let k_0 be such that $f^{-1}(F_n) = \mathcal{G}_{k_0} + \sum_{i=k_0+1}^{\infty} \mathcal{L}_i$, where $\mathcal{G}_{k_0}$ is a non-empty final segment of $\mathcal{L}_{k_0}$, and let k_1 be the greatest k such that $g(F_n) \subseteq \sum_{i=k}^{\infty} \mathcal{L}_i$. Since then $h(G_{k_0} + \sum_{i=k_0+1}^{\infty} \mathcal{L}_i) \cap \mathcal{L}_{k_1} \neq \emptyset$, by the claim above, $k_0 \leq k_1$. Therefore

$$\sum_{i=k_0}^{\infty} \mathcal{L}_i \preccurlyeq \mathcal{G}_{k_0} + \sum_{i=k_0+1}^{\infty} \mathcal{L}_i \preccurlyeq \mathcal{F}_n \preccurlyeq \sum_{i=k_1}^{\infty} \mathcal{L}_i \preccurlyeq \sum_{i=k_0}^{\infty} \mathcal{L}_i.$$

So, $\mathcal{F}_n \sim \sum_{i=k_0}^{\infty} \mathcal{L}_i$. Let $k = k_0$.

Since $\mathcal{F}_n$ is indecomposable, either $\mathcal{F}_n \preccurlyeq \mathcal{L}_k$ or $\mathcal{F}_n \preccurlyeq \sum_{i=k+1}^{\infty} \mathcal{L}_i$. The former case is not possible because we would have that $\mathcal{L}_k + 1 \preccurlyeq \mathcal{L}_k$, which contradicts Lemma 1.2. In the latter case we would have that $\mathcal{L}_k + 1 \preccurlyeq \sum_{i=k+1}^{\infty} \mathcal{L}_i$. Then, by Corollary 2.12, $\mathcal{L}_k \preccurlyeq \mathcal{L}_m$ for some $m \geq k+1$, contradicting our initial assumption. □

Lemma 3.6. *(RCA_0) FINDEC implies ACA_0.*

Proof. We will prove that FINDEC implies that $K = 0'$ exists. Then, by relativizing the proof, as usual, we can get that for all set X, X' exists, and hence ACA_0.

Let $\{k_0, k_1, ...\}$ be a recursive enumeration of K. For each $s \in \mathbb{N}$ let $K_s = \{k_0, ..., k_s\}$ and $\sigma_s = K_s \upharpoonright k_s + 1$. Consider the following ordering of $\mathbb{N}$.

$$s <_B t \Leftrightarrow \sigma_s <_{KB} \sigma_t,$$

where $<_{KB}$ is the Kleene-Brouwer ordering of Seq_2. ($\sigma <_{KB} \tau$ iff $\sigma \supseteq \tau$ or $\sigma|\tau$ and $\sigma \leq_{\mathrm{Seq}_\omega} \tau$.) Let $\mathcal{B} = \langle \mathbb{N}, \leq_B \rangle$. For each s we have that either for some $t > s$, $k_t < k_s$, in which case we have that $\forall t' \geq t (s <_B t')$, or that for every $t > s$, $k_t > k_s$ (in other words, s is a *true stage*), in which case we have that $\forall t' > s (t' <_B s)$. In the former case we say that s is in the *left side* of $\mathcal{B}$, and in the latter case that s is in the *right side*. Just for the sake of giving some intuition about the shape of $\mathcal{B}$, we observe that ACA_0 proves that $\mathcal{B}$ has order type $\omega + \omega^*$. RCA_0 cannot prove this fact. Furthermore, if we had an order preserving map from ω^* into $\mathcal{B}$, then we could compute infinitely many true stages and hence K.

FINDEC implies that $\mathcal{B}$ is equimorphic to a finite sum of h-indecomposable linear orderings. Since $\mathcal{B}$ is infinite, at least one of the summands has to be infinite. Because of the fact that every element has finitely many elements either to the right or to the left, we are left with three possible decomposition of $\mathcal{B}$:

$$\mathbf{1} + \mathbf{1} + ...\mathbf{1} + \omega + \mathbf{1} + ... + \mathbf{1};$$
$$\mathbf{1} + \mathbf{1} + ...\mathbf{1} + \omega * + \mathbf{1} + ... + \mathbf{1};$$
$$\mathbf{1} + \mathbf{1} + ...\mathbf{1} + \omega + \omega * + \mathbf{1} + ... + \mathbf{1}.$$

We can eliminate the first possibility by proving that there is no embedding $\mathcal{B} \preccurlyeq \mathbf{1} + \mathbf{1} + ...\mathbf{1} + \omega + \mathbf{1} + ... + \mathbf{1}$. To do this all we have to show is that every element in the right side has to be mapped to one of the $\mathbf{1}$s at the left of the copy of ω, and then that there are infinitely many elements in the right side of $\mathcal{B}$, or in other words, infinitely many true stages. (The second possibility can be eliminated too. But we do not need to do it.) Therefore, we have a map from ω^* to $\mathcal{B}$ as we needed to compute K. □

Corollary 3.7. *$\mathsf{WQO(ST)}$ and FINDEC are equivalent over RCA_0.*

Proof. Use the previous three lemmas. □

3.2. **Minimal decomposition.** Finite decompositions of a linear ordering are not unique. For example, $\langle \omega^2 \rangle$ and $\langle \omega, \mathbf{1}, \omega^2 \rangle$ are two finite decompositions of ω^2. This is why we are interested in considering minimal finite decompositions of linear orderings.

Jullien proved that every scattered linear ordering has a minimal decomposition, and, in a certain sense, a unique one [Jul69]. His definitions of finite and minimal decompositions were, although essentially the same, a bit different from ours. Because of this, our proof of uniqueness is simpler than his. The existence of minimal decompositions follows easily from the existence of finite decompositions and Σ^1_2-induction. To prove it using just ATR_0, a little work is required.

Definition 3.8. *A* minimal decomposition *of a linear ordering is a finite decomposition of minimal length.*

Lemma 3.9. *(RCA_2) If $\langle \mathcal{F}_0, ..., \mathcal{F}_n\rangle$ and $\langle \check{\mathcal{F}}_0, ..., \check{\mathcal{F}}_m\rangle$ are finite decomposition of $\mathcal{L}$, then there exists a set $X \subseteq \{0, ..., m\}$ of size at most $n+1$ such that $\sum_{i\in X} \check{\mathcal{F}}_i \sim \mathcal{L}$. Moreover, there exists an embedding*

$$g\colon \sum_{i=0}^{n} \mathcal{F}_i \hookrightarrow \sum_{i\in X} \check{\mathcal{F}}_i.$$

such that for each $i \leq n$, there is a $j \in X$, such that the image of $\mathcal{F}_i$ under g is contained in $\check{\mathcal{F}}_j$.

Proof. Let f be an embedding $f\colon \sum_{i=0}^{n} \mathcal{F}_i \hookrightarrow \sum_{i=0}^{m} \check{\mathcal{F}}_i$. As in the proof of Lemma 2.11, for each $i \leq n$, there is an $x_i \in F_i$ and a $j_i \leq m$ such that $\check{\mathcal{F}}_{j_i}$ contains the the image under f of $\mathcal{F}_{i(\geq x_i)}$ if $\mathcal{F}_i$ is $\rightarrow$, and of $\mathcal{F}_{i(\leq x_i)}$ if $\mathcal{F}_i$ is $\leftarrow$. (If $\mathcal{F}_i$ is $\mathbf{1}$, let x_i be the the only element of F_i.) The sequence $\langle x_0, ..., x_n\rangle$ exists by Σ^0_2-induction. Let $X = \{j_i : i \leq n\}$. Now, using Lemma 2.9 we can construct embeddings $g_i\colon \mathcal{F}_i \hookrightarrow \check{\mathcal{F}}_{j_i}$, uniformly in i, such that the image of g_i is contained in the image of $\mathcal{F}_i$ under f. Then, putting all the g_is together, we can construct $g\colon \sum_{i=0}^{n} \mathcal{F}_i \hookrightarrow \sum_{i\in X} \check{\mathcal{F}}_i$. So, we have that

$$\mathcal{L} \preccurlyeq \sum_{i=0}^{n} \mathcal{F}_i \preccurlyeq \sum_{i\in X} \check{\mathcal{F}}_i \preccurlyeq \mathcal{L}.$$

□

Proposition 3.10. *(ATR_0) If a linear ordering $\mathcal{L}$ has a finite decomposition, then it has a minimal decomposition. Moreover, this minimal decomposition is unique up to equimorphism.*

Proof. The uniqueness of the minimal decomposition follows from the previous lemma: If $\langle \mathcal{F}_0, ..., \mathcal{F}_n\rangle$ and $\langle \check{\mathcal{F}}_0, ..., \check{\mathcal{F}}_n\rangle$ are minimal decomposition of $\mathcal{L}$, then the X given by the previous lemma has to be the whole set $\{0, ..., n\}$. Then, necessarily $j_i = i$ for all $i \leq n$, and hence $\mathcal{F}_i \preccurlyeq \check{\mathcal{F}}_i$. Analogously we get $\check{\mathcal{F}}_i \preccurlyeq \mathcal{F}_i$ for each i, and therefore $\mathcal{F}_i \sim \check{\mathcal{F}}_i$.

Now, let $\langle \mathcal{F}_0, ..., \mathcal{F}_n\rangle$ be a finite decomposition of $\mathcal{L}$. We will prove that $\mathcal{L}$ has a minimal decomposition. We consider the least m such that there is a subset X of $\{0, ..., n\}$ of size $m+1$ such that

$$\sum_{i=0}^{n} \mathcal{F}_i \preccurlyeq \sum_{i\in X} \mathcal{F}_i.$$

The existence of such an m requires induction. We will prove that the formula $\sum_{i=0}^{n} \mathcal{F}_i \preccurlyeq \sum_{i\in X} \mathcal{F}_i$ is Σ^0_1 over some parameters that, using ATR_0, we can prove exist. Let $\langle T_0, ..., T_n\rangle$ be a sequence of signed trees such that $\mathrm{lin}(T_i) = \mathcal{F}_i$. Let α be the maximum of the ranks of the T_is plus 1. We claim that we can decide whether $\sum_{i=0}^{n} \mathcal{F}_i \preccurlyeq \sum_{i\in X} \mathcal{F}_i$ recursively in $Z^{(2\alpha+2)}$, where Z is some set that computes $\langle T_0, ..., T_n\rangle$. Let $\{j_0 < ... < j_m\} = X$. If $\sum_{i=0}^{n} \mathcal{F}_i \preccurlyeq \sum_{i\in X} \mathcal{F}_i$, then, by the previous lemma, there exists an embedding g such that for each $i \leq n$, there is a $j \in X$, such that the image under g of each $\mathcal{F}_i$ is contained in $\mathcal{F}_j$. So $\sum_{i=0}^{n} \mathcal{F}_i \preccurlyeq \sum_{i\in X} \mathcal{F}_i$ is equivalent to

$$\bigvee_{0=i_0\leq ...\leq i_m\leq n} \left(\bigwedge_{k\leq m} \mathcal{F}_{i_k} + \mathcal{F}_{i_k+1} + ... + \mathcal{F}_{i_{k+1}-1} \preccurlyeq \mathcal{F}_{j_k} \right).$$

Observe that in general, if $\mathcal{C}$ is $\rightarrow$, then $\mathcal{A}+\mathcal{B} \preccurlyeq \mathcal{C}$ if and only if $\mathcal{A}+\mathbf{1} \preccurlyeq \mathcal{C}$ and $\mathcal{B} \preccurlyeq \mathcal{C}$. Also observe that $\mathcal{A}+\mathbf{1} \preccurlyeq \mathcal{C}$ if and only if $\mathcal{A}\times\omega \preccurlyeq \mathcal{C}$. So, now, the question "$\mathcal{F}_{i_k}+\mathcal{F}_{i_k+1}+...+\mathcal{F}_{i_{k+1}-1} \preccurlyeq \mathcal{F}_{j_k}$?", supposing $\mathcal{F}_{j_k}$ is $\rightarrow$, becomes a conjunction of formulas of the forms $\mathcal{F}_i \preccurlyeq F_j$ and $\mathcal{F}_i\times\omega \preccurlyeq F_j$. Since, by Lemma 2.5, $Z^{(2\alpha+2)}$ can answer all these questions, it can tell whether $\sum_{i=0}^n \mathcal{F}_i \preccurlyeq \sum_{i\in X}\mathcal{F}_i$. This proves our claim.

Now, by Σ^0_1-induction, there is an m and an X as required above. We claim that $\langle \mathcal{F}_i : i\in X\rangle$ is a minimal decomposition of $\mathcal{L}$. Suppose, toward a contradiction, that $\langle \check{\mathcal{F}}_0,...,\check{\mathcal{F}}_l\rangle$ is a finite decomposition of $\mathcal{L}$ of length $l+1<m+1$. But then, by the Lemma above, there is some $Y\subset X$ such that $\sum_{i\in Y}\mathcal{F}_i$ is equimorphic to $\mathcal{L}$, contradicting the minimality of m. ☐

Since FINDEC implies ATR_0, we obtain the following equivalence.

Corollary 3.11. *The following are equivalent over RCA_0:*
(1) *FINDEC.*
(2) *Every scattered linear ordering has a minimal decomposition.*

4. Fraïssé's conjecture

Statement 4.1. *Fraïssé's conjecture, FRA, is the statement that says that the class of linear orderings is well quasiordered under embeddablity.*

As we said in the introduction, the exact proof theoretic strength of FRA is unknown. All we know it that it is provable in Π^1_2-CA_0, that it implies ATR_0 (Shore [Sho93]) but that it does not imply Π^1_1-CA_0. We prove in this section that it is equivalent to the two statements studied above.

Theorem 4.2. *The following are equivalent over RCA_0:*
(1) *$\mathsf{WQO(ST)}$*
(2) *FINDEC*
(3) *FRA*

Proof. We have already proved that $\mathsf{WQO(ST)}$ and FINDEC are equivalent. Obviously FRA implies that the class of h-indecomposable linear orderings is well quasiordered. It follow from Proposition 2.13, and the fact that FRA implies ACA_0, that FRA implies $\mathsf{WQO(ST)}$.

Now we show that $\mathsf{WQO(ST)}$ implies FRA. Recall that $\mathsf{WQO(ST)}$ implies ATR_0, so we can use ATR_0 here. Consider a sequence $\langle \mathcal{L}_i : i\in \mathbb{N}\rangle$ of linear orderings. For some set X and ordinal α, we have that these linear orderings are all recursive in X and have rank less than α. (The rank of a scattered linear ordering is defined at the beginning of the proof of Lemma 3.4.) By relativization, assume X is recursive.

The idea is like the one in the proof of [Fra00, 7.5.4], but we have to be a little bit more careful. We prove that, for every ordinal α, the set of recursive linear orderings of rank less that α is well quasiordered. We use Higman's theorem which is provable in ACA_0; see, for example, [Mar]. Higman's theorem says that if $\mathcal{P}$ is well quasiordered, then $\langle \mathrm{Seq}_P, \preccurlyeq_P\rangle$ is well quasiordered too, where $\sigma \preccurlyeq_P \tau$ if there is a strictly increasing $f:\{0,...,|\sigma|-1\}\rightarrow\{0,...,|\tau|-1\}$ such that $\forall i<|\sigma|(\sigma(i)\leq_P \tau(f(i)))$.

Let $\mathcal{H}_\alpha$ be the set of h-indecomposable linear orderings of rank less than α, which are recursive in $0^{2\alpha^2}$. It follows from $\mathsf{WQO(ST)}$ that $\mathcal{H}_\alpha$ is well quasiordered, and then, by Higman's theorem, that $\langle \mathrm{Seq}_{\mathcal{H}_\alpha}, \preccurlyeq_{\mathcal{H}_\alpha}\rangle$ is well quasiordered too. For each i, let $S_i = \langle \mathcal{F}_0, ..., \mathcal{F}_n\rangle \in \mathrm{Seq}_{\mathcal{H}_\alpha}$ be such that $\mathcal{L}_i = \sum_{j=0}^n \mathcal{F}_j$. ($S_i$ exists by Lemma 3.4 and its proof.) Then, for some $i < j$, $S_i \preccurlyeq_{\mathcal{H}_\alpha} S_j$. Hence $\mathcal{L}_i \preccurlyeq \mathcal{L}_j$. □

The following lemma gives us another statement equivalent to FRA. We will use it later.

Lemma 4.3. *The following are equivalent over* ACA_0*:*

(1) *FRA.*
(2) *There is no infinite strictly descending sequence of linear orderings which are h-indecomposable to the right.*

Proof. Clearly FRA implies (2). Let us prove that (2) implies WQO(ST), and hence FRA. Suppose, toward a contradiction, that $\langle T_i\rangle_{i\in\mathbb{N}}$ is a sequence of signed trees such that for all $i < j$, $T_i \not\preccurlyeq T_j$. For each n, define a signed tree $S_n = \{i^\frown\sigma : \sigma \in T_i, i \geq n\}$ and $s_{S_n}(\emptyset) = +$ and $s_{S_n}(i^\frown\sigma) = s_{T_i}(\sigma)$. We claim that for all $n < m$, $S_n \succ S_m$. Take $n < m$. Clearly $S_n \succcurlyeq S_m$. If $S_n \preccurlyeq S_m$, then for some $j \geq m$, $T_n \preccurlyeq T_j$, contradicting our assumption on $\langle T_i\rangle_{i\in\mathbb{N}}$. Therefore $\langle \mathrm{lin}(S_n)\rangle_{n\in\mathbb{N}}$ is a strictly descending sequence of linear orderings h-indecomposable to the right. □

5. Jullien's theorem

In his doctoral dissertation [Jul69] Jullien characterized all the extendible linear orderings. We want to analyze the proof theoretic strength of Jullien's theorem. The first problem we have is that, as formulated in [Jul69], Jullien's theorem does not make sense if FINDEC does not hold. We formulate Jullien's theorem in two different ways which do not need FINDEC to make sense.

Definition 5.1. *A segment $\mathcal{B}$ of a linear ordering $\mathcal{L} = \mathcal{A} + \mathcal{B} + \mathcal{C}$ is* essential *if whenever we have $\mathcal{L} \preccurlyeq \mathcal{A} + \mathcal{B}' + \mathcal{C}$ for some linear ordering $\mathcal{B}'$, it has to be the case that $\mathcal{B} \preccurlyeq \mathcal{B}'$.*

Statement 5.2. *JUL is the statement: A scattered linear ordering $\mathcal{L}$ is extendible if and only if it does not have an essential segment $\mathcal{B}$ of either of the following forms:*

- *$\mathcal{B} = \mathcal{R} + \mathcal{Q}$ where R is indecomposable to the right and $\mathcal{Q}$ is indecomposable to the left, or*
- *$\mathcal{B} = \mathbf{2}$.*

This version of Jullien's theorem is different from the ones that appear in the literature. We find it more natural than Jullien's formulation and it does not require the notion of minimal decompositions. We will describe Jullien's formulation of his theorem in subsection 5.3. The fact that the two formulations are equivalent follows from an analysis of the essential segments of a linear ordering with a given minimal decomposition. See, for example, Lemma 5.9 below.

Notation 5.3. *We say that a linear ordering $\mathcal{B}$ has the form $\langle \rightarrow, \leftarrow\rangle$ if $\mathcal{B} = \mathcal{R} + \mathcal{Q}$ where $\mathcal{R}$ is indecomposable to the right and $\mathcal{Q}$ is indecomposable to the left.*

5.1. Proof of the easy direction.

We start by proving, using just RCA_0, that if $\mathcal{L}$ has an essential segment of the form either $\mathbf{2}$ or $\langle \rightarrow, \leftarrow\rangle$, then it is not extendible.

Lemma 5.4. *(RCA_0) If $\mathcal{L}$ has an essential segment which is not extendible, then $\mathcal{L}$ is not extendible.*

Proof. Write $\mathcal{L}$ as $\mathcal{A}+\mathcal{B}+\mathcal{C}$ where $\mathcal{B}$ is an essential, not extendible segment of $\mathcal{L}$. There is some partial ordering $\mathcal{P}$ such that $\mathcal{B} \not\preccurlyeq \mathcal{P}$, but $\mathcal{B}$ embeds in any linearization of $\mathcal{P}$. Let $\mathcal{Q} = \mathcal{A}+\mathcal{P}+\mathcal{C}$. First note that $\mathcal{L} \not\preccurlyeq \mathcal{Q}$: This is because any embedding $\mathcal{L} \preccurlyeq \mathcal{Q}$, induces an embedding of $\mathcal{L}$ into $\mathcal{A}+\mathcal{B}'+\mathcal{C}$, where $\mathcal{B}'$ is a chain in $\mathcal{P}$, and hence $\mathcal{B} \not\preccurlyeq \mathcal{B}'$, contradicting the essentially of $\mathcal{B}$. On the other hand, $\mathcal{L}$ embeds in any linearization of $\mathcal{Q}$, because a linearization of $\mathcal{Q}$ is of the form $\mathcal{A}+\mathcal{D}+\mathcal{C}$, where $\mathcal{D}$ is a linearization of $\mathcal{P}$, and $\mathcal{B}$ embeds in any linearization of $\mathcal{P}$. □

The proof of the following lemma is exactly the one in [Jul69, Lemma V.2.2]. Since Jullien's thesis [Jul69] was never published, we include the proof here.

Lemma 5.5. *(RCA_0) The following linear orderings are not extendible.*

- $\mathbf{2}$,
- *any linear ordering of the form $\langle \rightarrow, \leftarrow \rangle$.*

Proof. To see that $\mathbf{2}$ is not extendible consider the poset which consist of two incomparable elements.

For the other case, let $\mathcal{A} = \mathcal{B}+\mathcal{C}$ be such that $\mathcal{B}$ is $\rightarrow$ and $\mathcal{C}$ is $\leftarrow$. We will define a partial ordering $\mathcal{P}$ such that $\mathcal{A} \not\preccurlyeq \mathcal{P}$, but $\mathcal{A}$ embeds in every linearization of $\mathcal{P}$.

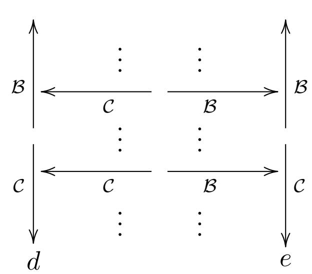
First, suppose that $\mathcal{B} \not\preccurlyeq \mathcal{C}$ and $\mathcal{C} \not\preccurlyeq \mathcal{B}$. Let $\mathcal{D} = \mathcal{C}+\mathcal{B}$ and $\{d, e\}$ be two elements not in D. We first define a set P:

$$P = (\{d\} \cup D \cup \{e\}) \times D.$$

Now we define and ordering $\leq_P$ on P.

$$\langle w, x\rangle \leq_P \langle y, z\rangle \Leftrightarrow \begin{cases} w = d \;\&\; x \leq_D z, \text{ or} \\ y = e \;\&\; x \leq_D z, \text{ or} \\ w \leq_D y \;\&\; x = z. \end{cases}$$

See picture of $\mathcal{P}$ below. (In the picture, an element of $\mathcal{P}$ is greater than another if it is above, or to the right, of it.)



We claim that $\mathcal{A} \not\preccurlyeq \mathcal{P}$, but $\mathcal{A}$ embeds in every linearization of $\mathcal{P}$. Every maximal chain in $\mathcal{P}$ is either of the form $\mathcal{C}+\mathcal{B}$, of the form $\mathcal{C}_1+\mathcal{C}+\mathcal{B}+\mathcal{C}_0+\mathcal{B}$ where $\mathcal{C}_1+\mathcal{C}_0 = \mathcal{C}$, or of the form $\mathcal{C}+\mathcal{B}_0+\mathcal{C}+\mathcal{B}+\mathcal{B}_1$ where $\mathcal{B}_0+\mathcal{B}_1 = \mathcal{B}$. In any case, any chain of $\mathcal{P}$ can be embedded into a linear ordering of the form $\mathcal{C}+\mathcal{B}_0+\mathcal{C}+\mathcal{B}+\mathcal{C}_0+\mathcal{B}$, where $\mathcal{B}_0$ is a proper initial segment of $\mathcal{B}$ and $\mathcal{C}_0$ a proper final segment of $\mathcal{C}$. ¿From these six summands, $\mathcal{B}$ only embeds in the ones isomorphic to $\mathcal{B}$ and $\mathcal{C}$ in the ones isomorphic to $\mathcal{C}$. Therefore, if we had an embedding of $\mathcal{A}$ into $\mathcal{C}+\mathcal{B}_0+\mathcal{C}+\mathcal{B}+\mathcal{C}_0+\mathcal{B}$, we should have that a final segment of $\mathcal{B}$ is mapped into one of the copies of $\mathcal{B}$ and that an initial segment of $\mathcal{C}$ into one of the copies of $\mathcal{C}$, which is impossible. Now let $\mathcal{Q} = \langle P, \leq_Q\rangle$ be a linearization of $\mathcal{P}$. If for every $x, y \in D$, $\langle d, x\rangle \leq_Q \langle e, y\rangle$, then

$\{d\} \times B \cup \{e\} \times C$ is a subset of $\mathcal{Q}$ of type $\mathcal{A}$. Otherwise, there exists $x, y \in D$ such that $\langle d, x\rangle \geq_Q \langle e, y\rangle$, then $B \times \{y\} \cup C \times \{x\}$ is a subset of $\mathcal{Q}$ of type $\mathcal{A}$. In any case, $\mathcal{A} \preccurlyeq \mathcal{Q}$.

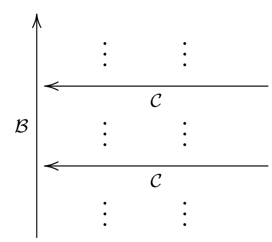
The second case is that $\mathcal{B} \preccurlyeq \mathcal{C}$ but $\mathcal{C} \not\preccurlyeq \mathcal{B}$.

$$P = (\{d\} \cup C) \times B,$$

where d is a new element. Now we define and ordering $\leq_P$ on P.

$$\langle w, x\rangle \leq_P \langle y, z\rangle \Leftrightarrow \begin{cases} w = d \;\&\; x \leq_D z, \text{ or} \\ w \leq_C y \;\&\; x = z. \end{cases}$$

See picture of $\mathcal{P}$ below. (In the picture, again, an element of $\mathcal{P}$ is grater than another if it is above of to the right of it.)



We claim that $\mathcal{A} \not\preccurlyeq \mathcal{P}$ but $\mathcal{A}$ embeds in every linearization of $\mathcal{P}$. Every chain in $\mathcal{P}$ is can be embedded in $\mathcal{B}_0 + \mathcal{C}$, where $\mathcal{B}_0$ is an initial segment of $\mathcal{B}$. If f is an embedding $\mathcal{A} \hookrightarrow \mathcal{B}_0 + \mathcal{C}$, then, since $\mathcal{B} \not\preccurlyeq \mathcal{B}_0$, there is some $x \in B$ such that $f(x) \in C$. But then, we have an embedding of $\mathbf{1} + \mathcal{C}$ into $\mathcal{C}$ contradicting Lemma 1.2. So $\mathcal{A} = \mathcal{B} + \mathcal{C} \not\preccurlyeq \mathcal{B}_0 + \mathcal{C}$. Now let $\mathcal{Q} = \langle P, \leq_Q\rangle$ be a linearization of $\mathcal{P}$. If for every $x, z \in B$ and $y \in C$, $\langle d, x\rangle \leq_Q \langle y, z\rangle$, then $\{d\} \times B \cup C \times \{x\}$ for some $x \in B$ is a subset of $\mathcal{Q}$ of type $\mathcal{A}$. Otherwise, there exists $x, z \in B$ and $y \in C$ such that $\langle d, x\rangle \geq_Q \langle y, z\rangle$. Then $\mathcal{B} + \mathcal{C}$ embeds into $\langle C_{(<y)} \times \{z\} \cup C \times \{x\}, \leq_Q\rangle$. In any case, $\mathcal{A} \preccurlyeq \mathcal{Q}$.

The case where $\mathcal{B} \not\preccurlyeq \mathcal{C}$ and $\mathcal{C} \preccurlyeq \mathcal{B}$ is analogous. It cannot be the case that $\mathcal{B} \preccurlyeq \mathcal{C}$ and $\mathcal{C} \preccurlyeq \mathcal{B}$, because we would have $\mathcal{B} + \mathbf{1} \preccurlyeq \mathcal{C} + \mathbf{1} \preccurlyeq \mathcal{C} \preccurlyeq \mathcal{B}$, contradicting Lemma 1.2. □

Corollary 5.6. *The implication from left to right in* JUL *is provable in* RCA_0*.*

Proof. Immediate from the previous two lemmas. □

5.2. **Consequences of JUL.** Now we show that FRA is necessary to prove the right to left direction of JUL.

Lemma 5.7. *(*RCA_0*) JUL implies FRA.*

Proof. First we prove that JUL implies ATR_0. For this observe that ζ, the linear ordering of the integers does not have essential intervals of the form $\mathbf{2}$, or $\langle\rightarrow, \leftarrow\rangle$. Then, by JUL, ζ has to be extendible. Downey, Hirschfeldt, Lempp and Solomon proved in [DHLS03] that the extendibility of ζ implies ATR_0.

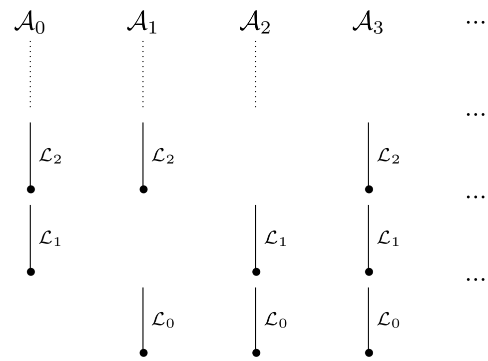
Suppose that FRA does not hold. Then, by Lemma 4.3, there is a sequence $\langle \mathcal{L}_i\rangle_{i\in\mathbb{N}}$ of linear orderings which are h-indecomposable to the right such that for all $i < j$, $\mathcal{L}_i \succ \mathcal{L}_j$. Assume that each $\mathcal{L}_i$ has a first element 0_{L_i}; otherwise add a first element to $\mathcal{L}_i$. Let

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \cdots + \mathcal{L}_n + \cdots ,$$

and, for each $n \in I\!N$, define

$$\mathcal{A}_n = \mathcal{L}_0 + \cdots + \mathcal{L}_{n-1} + \mathcal{L}_{n+1} + \cdots .$$

Let $\mathcal{P} = \bigoplus_{n \in I\!N} \mathcal{A}_n$. (See the diagram of $\mathcal{P}$ below. In the picture, an element of $\mathcal{P}$ is grater than another if it is above it.)



We think of the domain of $\mathcal{L}$ as $\{\langle i,x\rangle : x \in L_i, i \in I\!N\}$, the domain of $\mathcal{A}_n$ as $\{\langle i,x\rangle : x \in L_i, i \in I\!N, i \neq n\}$ and the domain of $\mathcal{P}$ as $\{\langle n,i,x\rangle : x \in L_i, i,n \in I\!N, i \neq n\}$.

The first claim is that $\mathcal{L} \not\preccurlyeq \mathcal{P}$. Suppose that there is an embedding $f\colon \mathcal{L} \hookrightarrow \mathcal{P}$. Then, for some n, f is an embedding $\mathcal{L} \preccurlyeq \{n\} \times \mathcal{A}_n$. Think of f as en embedding into $\mathcal{A}_n$. We prove, by induction on $i < n$, that for every $x \in L_i$, $\langle i,x\rangle <_{\mathcal{A}_n} f(\langle i+1, 0_{L_{i+1}}\rangle)$. Suppose it is true for $i-1$, but that $f(\langle i+1, 0_{L_{i+1}}\rangle) \leq_{\mathcal{A}_n} \langle i,x\rangle$ for some $x \in \mathcal{L}_i$. So,

$$f(\{i\} \times L_i \cup \{\langle i+1, 0_{L_{i+1}}\rangle\}) \subseteq \{i\} \times L_i.$$

But, since $\mathcal{L}_i$ is h-indecomposable to the right, $\mathcal{L}_i + \mathbf{1} \not\preccurlyeq \mathcal{L}_i$. Contradiction. This implies that

$$f(\{n\} \times L_n \cup \{\langle n+1, 0_{L_{n+1}}\rangle\}) \subseteq \sum_{j \geq n+1} L_j,$$

which, by Corollary 2.12, implies that for some $j > n$, $\mathcal{L}_n \preccurlyeq \mathcal{L}_j$, contradicting our assumptions.

The second claim is that $\mathcal{L}$ embeds in every linearization of $\mathcal{P}$. Let $\leq_Q$ be a linearization of $\leq_P$ and $\mathcal{Q} = \langle P, \leq_Q\rangle$. We consider three possible cases. First, suppose that for every $n > 0$ and every $x \in L_{n-1}$, $\langle n, n-1, x\rangle \leq_Q \langle n+1, n, 0_{L_n}\rangle$. Then, $f(\langle i,x\rangle) = \langle i+1, i, x\rangle$ is an embedding of $\mathcal{L}$ into $\mathcal{Q}$. Second, if for some n, for every $y \in L_n$, $\langle n+1, n, y\rangle \leq_Q \langle n, n+1, 0_{L_{n+1}}\rangle$, then

$$f(\langle i,y\rangle) = \begin{cases} \langle n+1, i, y\rangle & \text{if } i \leq n \\ \langle n, i, y\rangle & \text{if } i > n \end{cases}$$

is an embedding of $\mathcal{L}$ into $\mathcal{Q}$. Last, suppose that neither of the above is the case. Then, for some $n > 0$ and $x \in L_{n-1}$, $\langle n, n-1, x\rangle \geq_Q \langle n+1, n, 0_{L_n}\rangle$, and for some

$y \in L_n$ $\langle n+1, n, y\rangle \geq_Q \langle n, n+1, 0_{L_{n+1}}\rangle$. Therefore, for all $z \in L_{n-1}$, $z \geq_{L_{n-1}} x$,

$$\langle n+1, n, 0_{L_n}\rangle \leq_Q \langle n, n-1, z\rangle \leq_Q \langle n+1, n, y\rangle$$

Let h_n be an embedding of $\mathcal{L}_n$ into $\mathcal{L}_{n-1(>x)}$ and h_{n+1} be an embedding of $\mathcal{L}_{n+1}$ into $\mathcal{L}_{n(>y)}$. Now, define $f\colon \mathcal{L} \to \mathcal{Q}$ as follows

$$f(\langle i, z\rangle) = \begin{cases} \langle n+1, i, z\rangle & \text{if } i < n \\ \langle n, n-1, h_n(z)\rangle & \text{if } i = n \\ \langle n+1, n, h_{n+1}(z)\rangle & \text{if } i = n+1 \\ \langle n+1, i, z\rangle & \text{if } i > n+1. \end{cases}$$

The reader can check that f is an embedding of $\mathcal{L}$ into $\mathcal{Q}$.

The third claim, needed to get a contradiction to JUL, is that $\mathcal{L}$ does not have an essential segment which is either $\mathbf{2}$, or of the form $\langle \rightarrow, \leftarrow\rangle$. If $\mathcal{A}$ is segment of $\mathcal{L}$ of order type $\mathbf{2}$, then $\mathcal{A} \subset \{i\} \times L_i$ for some i. But, since for all $x \in L_i$,

$$\mathcal{L} \sim \mathcal{L}_0 + \cdots + \mathcal{L}_{i-1} + \mathcal{L}_{i(>x)} + \mathcal{L}_{i+1} + \cdots,$$

$\mathcal{A}$ cannot be essential. Now suppose that $\mathcal{L} = \mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D}$, where $\mathcal{B}$ is indecomposable to the right and $\mathcal{C}$ is indecomposable to the left and $\mathcal{B} + \mathcal{C}$ is an essential segment. Let i be least such that $C \cap L_i \neq \emptyset$. $\mathcal{C}$ cannot contain a final segment of $\mathcal{L}_i$, because otherwise $\mathcal{L}_i + \mathbf{1} \preccurlyeq \mathcal{C} + \mathbf{1} \preccurlyeq \mathcal{C} \preccurlyeq \mathcal{L}_i$. So $\mathcal{C}$ is contained in a proper initial segment of $\mathcal{L}_i$. Let j be maximal such that $B \cap L_j \neq \emptyset$. j could be either i or $i-1$. $\mathcal{B}$ cannot contain a final segment of $\mathcal{L}_{j-1}$, because if it did we would have $\mathcal{L}_{j-1} \preccurlyeq \mathcal{B} \preccurlyeq \mathcal{L}_j$. So $\mathcal{B} \subseteq \mathcal{L}_j$. If $j = i$, then $\mathcal{B} + \mathcal{C}$ is contained in a proper initial segment of $\mathcal{L}_i$, and therefore $\mathcal{L} \preccurlyeq \mathcal{A} + \mathcal{D}$. If $j = i-1$, $\mathcal{B}$ is a final segment of $\mathcal{L}_j$, and hence $\mathcal{B} \sim \mathcal{L}_j$ and $\mathcal{C}$ is a proper initial segment of $\mathcal{L}_i$. So, we have that $\mathcal{L} \preccurlyeq \mathcal{A} + \mathcal{B} + \mathcal{D}$ Then, since $\mathcal{B} + \mathcal{C}$ is essential, $\mathcal{B} + \mathcal{C} \preccurlyeq \mathcal{B}$, and therefore $\mathcal{L}_j + \mathbf{1} \preccurlyeq \mathcal{B} + \mathcal{C} \preccurlyeq \mathcal{B} \preccurlyeq \mathcal{L}_j$. This contradicts Lemma 1.2. □

5.3. **Minimal decomposition and the proof of Jullien's theorem.** Our next goal is to prove JUL in the system RCA_*+FRA.

What Jullien did in [Jul69] is to prove that every scattered linear ordering has a unique minimal decomposition, and then characterize the extendible linear orderings by putting conditions on their minimal decompositions:

Statement 5.8. *JUL(min-dec) is the statement that says that if $\langle \mathcal{F}_0, ..., \mathcal{F}_n\rangle$ is a minimal decomposition of $\mathcal{L}$, then $\mathcal{L}$ is extendible if and only if there is no i such that either $\mathcal{F}_i = \mathcal{F}_{i+1} = \mathbf{1}$ or $\mathcal{F}_i$ is indecomposable to the right and $\mathcal{F}_{i+1}$ is indecomposable to the left.*

The problem with this statement is that, without knowing that minimal decompositions always exists, JUL(min-dec) is not enough to classify all the extendible linear orderings, as Jullien did. So, from the viewpoint of reverse math, this is not a satisfactory formulation of Jullien's classification of the extendible linear orderings. We could say that Jullien's theorem, as stated in [Jul69], is the conjunction of JUL(min-dec) and the sentence that says that every scattered linear ordering has a minimal decomposition (which is equivalent to FRA; see Corollary 3.11 and Theorem 4.2).

We will prove that JUL(min-dec) is equivalent to ATR_* over RCA_*. Then, use this result to prove that that FRA implies JUL.

Lemma 5.9. *(RCA_0) If $\langle \mathcal{F}_i : i \leq n\rangle$ is a minimal decomposition of $\mathcal{L}$, and either $\mathcal{F}_i = \mathcal{F}_{i+1} = \mathbf{1}$ or $\mathcal{F}_i$ is h-indecomposable to the right and $\mathcal{F}_{i+1}$ is h-indecomposable to the left, then $\mathcal{F}_i + \mathcal{F}_{i+1}$ is an essential segment of $\mathcal{L}$.*

Proof. Suppose, toward a contradiction, that f is an embedding,

$$f\colon \mathcal{L} \hookrightarrow (\mathcal{F}_0 + \cdots + \mathcal{F}_{i-1}) + \mathcal{A} + (\mathcal{F}_{i+2} + \cdots + \mathcal{F}_n),$$

and $\mathcal{F}_i + \mathcal{F}_{i+1} \not\preccurlyeq \mathcal{A}$. If $\mathcal{F}_i = \mathcal{F}_{i+1} = 1$, then $\mathcal{A}$ has to be either $\emptyset$ or $\mathbf{1}$, so $\mathcal{F}_0 + \cdots + \mathcal{F}_{i-1} + \mathcal{A} + \mathcal{F}_{i+2} + \cdots + \mathcal{F}_n$ is a decomposition of $\mathcal{L}$ with less less than $n+1$ terms. This contradicts the minimality of the decomposition of $\mathcal{L}$. Now suppose that $\mathcal{F}_i$ is h-indecomposable to the right and $\mathcal{F}_{i+1}$ is h-indecomposable to the left. If there exist $x \in F_i$ and $y \in F_{i+1}$ such that both $f(x)$ and $f(y)$ belong to A, then

$$\mathcal{F}_i + \mathcal{F}_{i+1} \preccurlyeq \mathcal{F}_{i(>x)} + \mathcal{F}_{i+1(<y)} \preccurlyeq \mathcal{A}.$$

So, either $\forall x \in \mathcal{F}_i(f(x) \notin \mathcal{A})$ or $\forall x \in \mathcal{F}_{i+1}(f(x) \notin \mathcal{A})$. Suppose the former is the case. The other case is analogous. If, there is some $x \in \mathcal{F}_i$ such that $f(x) \in \mathcal{F}_{i+2}+\cdots+\mathcal{F}_n$, then, since $\mathcal{F}_i \sim \mathcal{F}_{i(>x)}$, we have that $\mathcal{F}_i+\cdots+\mathcal{F}_n \preccurlyeq \mathcal{F}_{i+2}+\cdots+\mathcal{F}_n$. Hence

$$\mathcal{L} \sim \mathcal{F}_1 + \cdots + \mathcal{F}_{i-1} + \mathcal{F}_{i+2} + \cdots \mathcal{F}_n,$$

contradicting the minimality of $\langle \mathcal{F}_i : i \leq n\rangle$. So, for every $x \in F_i$, $f(x) \in F_0 + \cdots + F_{i-1}$. Then

$$\mathcal{L} \sim \mathcal{F}_1 + \cdots + \mathcal{F}_{i-1} + \mathcal{F}_{i+1} + \cdots \mathcal{F}_n,$$

contradicting, again, the minimality of $\langle \mathcal{F}_i : i \leq n\rangle$. □

Corollary 5.10. *The direction from left to right of JUL(min-dec) is provable in RCA_0.*

Proof. Use the previous lemma and Corollary 5.6. □

Now we want to prove the other direction of JUL(min-dec) using ATR_*. We will use that ATR_* proves that $\mathcal{L}$ and $\mathbf{1} + \mathcal{L} + \mathbf{1}$ are extendible when $\mathcal{L}$ is h-indecomposable, and not $\mathbf{1}$, which we will prove in the next section. Moreover, in the next section, in Proposition 6.18, we will prove that every partial ordering, $\mathcal{P}$, which does not embed $\mathbf{1} + \mathcal{L} + \mathbf{1}$, has a linearization which is hyperarithmetic in $\mathcal{L}$ and $\mathcal{P}$, and does not embed $\mathbf{1} + \mathcal{L} + \mathbf{1}$. We could get JUL(min-dec), using ATR and the results of the next section, using a proof similar to Jullien's. But, since we want to use ATR_*, we have to make some modifications.

One important fact that we use to lower the complexity of certain formulas is the following.

Lemma 5.11. [Sim99, Theorem VIII.3.20] *For any Σ^1_1 formula $\varphi(X, Y)$, we can find a Σ^1_1 formula $\varphi'(X)$ such that ATR_0 proves*

$$\varphi'(X) \Leftrightarrow \forall Y(Y \text{ hyperarithmetic in } X \Rightarrow \varphi(X, Y)).$$

The plan of the proof is as follows. First, we prove that every scattered linear ordering of the right form has a finite decomposition of a certain kind:

Lemma 5.12. *(ATR_0) Let $\langle \mathcal{F}_0, ..., \mathcal{F}_n\rangle$ be a minimal decomposition of a linear ordering $\mathcal{L}$, such that $\mathcal{F}_i + \mathcal{F}_{i+1}$ is neither $\mathbf{2}$ nor $\langle \rightarrow, \leftarrow\rangle$ for any $i < n$. Then, $\mathcal{L}$*

has a finite decomposition of one of the following forms:

$$\langle \mathbf{1}, \check{\mathcal{F}}_0, \mathbf{1}, \check{\mathcal{F}}_1, \mathbf{1}, ..., \mathbf{1}, \check{\mathcal{F}}_m, \mathbf{1}\rangle,$$
$$\langle \check{\mathcal{F}}_0, \mathbf{1}, \check{\mathcal{F}}_1, \mathbf{1}, ..., \mathbf{1}, \check{\mathcal{F}}_m, \mathbf{1}\rangle,$$
$$\langle \mathbf{1}, \check{\mathcal{F}}_0, \mathbf{1}, \check{\mathcal{F}}_1, \mathbf{1}, ..., \mathbf{1}, \check{\mathcal{F}}_m\rangle, \qquad \text{or}$$
$$\langle \check{\mathcal{F}}_0, \mathbf{1}, \check{\mathcal{F}}_1, \mathbf{1}, ..., \mathbf{1}, \check{\mathcal{F}}_m\rangle,$$

where each $\check{\mathcal{F}}_i$ is h-indecomposable, either to the left or to the right, but not $\mathbf{1}$.

Next, we use this decomposition of $\mathcal{L}$ to reduce the problem of the extendibility of $\mathcal{L}$ to the extendibility of $\mathbf{1} + \check{\mathcal{F}}_i + \mathbf{1}$ for each i:

Lemma 5.13. *(ATR_*) Suppose that $\mathcal{L}$ has a finite decomposition of the form $\langle \mathbf{1}, \mathcal{F}_0, \mathbf{1}, \mathcal{F}_1, \mathbf{1}, ..., \mathcal{F}_m, \mathbf{1}\rangle$, where each $\mathcal{F}_i$ is h-indecomposable but not $\mathbf{1}$. Consider a partial ordering $\mathcal{P} = \langle P, \leq_P\rangle$ such that $\mathcal{L} \not\preccurlyeq \mathcal{P}$. Then there exists a partition $\langle P_i : i \leq m\rangle$ of P such that*

- *if $x \in P_i$, $y \in P_j$ and $x \leq_P y$, then $i \leq j$, and*
- *for all $i \leq m$, $\mathbf{1} + \mathcal{F}_i + \mathbf{1} \not\preccurlyeq \mathcal{P}_i$, where $\mathcal{P}_i = \langle P_i, \leq_P\rangle$.*

Then, we will use the results in the next section to linearize each $\mathcal{P}_i$ and get a linear ordering which does not embed $\mathbf{1} + \mathcal{F}_i + \mathbf{1}$. We will show that Σ^1_1-IND is enough to get all these linearization simultaneously and construct a linearization of $\mathcal{P}$ which does not embed $\mathcal{L}$.

Proof of Lemma 5.12. All we need to observe is that if $\mathcal{F}_i$ is $\leftarrow$, then $\mathcal{F}_i \sim \mathcal{F}_i + \mathbf{1}$, and if $\mathcal{F}_i$ is $\rightarrow$, then $\mathcal{F}_i \sim \mathbf{1} + \mathcal{F}_i$. Therefore, if none of the $\mathcal{F}_i + \mathcal{F}_{i+1}$ is of the form $\langle \rightarrow, \leftarrow\rangle$ or $\mathbf{2}$, then $\mathcal{F}_i + \mathcal{F}_{i+1} \sim \mathcal{F}_i + \mathbf{1} + \mathcal{F}_{i+1}$. Apply this to insert $\mathbf{1}$s in the finite decomposition $\langle \mathcal{F}_0, ..., \mathcal{F}_n\rangle$ to get the desired decomposition. □

Proof of Lemma 5.13. We prove by induction on $i \leq m$ that there exists a partition $\langle P_0, ..., P_{i-1}, \bar{P}_i\rangle$ of P, hyperarithmetic in $\mathcal{P}$, such that

- for $j, k < i$, if $x \in P_j$, $y \in P_k$ and $x \leq_P y$, then $j \leq k$,
- for $j < i$, if $x \in P_j$, $y \in \bar{P}_i$ then $y \not\leq_P x$,
- for all $j < i$, $\mathbf{1} + \mathcal{F}_j + \mathbf{1} \not\preccurlyeq \mathcal{P}_j = \langle P_j, \leq_P\rangle$, and
- $\mathbf{1} + \mathcal{F}_i + \mathbf{1} + ... + \mathbf{1} + \mathcal{F}_m + \mathbf{1} \not\preccurlyeq \langle \bar{P}_i, \leq_P\rangle$.

The case $i = m$ will give us the Lemma. By Lemma 5.11, the formula we are proving by induction is equivalent to a Π^1_1 one. (Π^1_1-induction is equivalent to Σ^1_1-IND [Sim99, Lemma VIII.4.9].) The base case $i = 0$ is trivial; just take the trivial partition $\langle P\rangle$. Now suppose we have $\langle P_0, ..., P_{i-1}, \bar{P}_i\rangle$ satisfying the conditions above. Let $\phi_+(x)$ be the Σ^1_1-formula that says that

$$x \in \bar{P}_i \text{ and } \mathbf{1} + \mathcal{F}_i + \mathbf{1} \preccurlyeq \bar{P}_{i(\leq x)}$$

and $\phi_-(x)$ be the Σ^1_1-formula that says that

$$x \in \bar{P}_i \text{ and } \mathbf{1} + \mathcal{F}_{i+1} + ... + \mathcal{F}_m + \mathbf{1} \preccurlyeq \mathcal{P}_{(\geq x)}.$$

Since $\mathbf{1} + \mathcal{F}_i + \mathbf{1} + ... + \mathbf{1} + \mathcal{F}_m + \mathbf{1} \not\preccurlyeq \bar{P}_i$, there is no x such that $\phi_+(x)$ & $\phi_-(x)$. Then, by Σ^1_1-separation (which is equivalent to ATR_0; see [Sim99, Theorem V.5.1]), there is a set $Q \subseteq \bar{P}_i$ such that

$$\forall x(\phi_-(x) \Rightarrow x \in Q \;\&\; \phi_+(x) \Rightarrow x \in \bar{P}_i \smallsetminus Q).$$

Moreover, Q can be taken hyperarithmetic in $\mathcal{P}$. (Let f be a recursive map that assigns to each x a recursive linear ordering such that $\neg\phi_+(x)$ iff $f(x)$ is a well ordering [Sim99, Proof of Lemma VII.3.4]. By the Σ^1_1 bounding principle [Sim99, Lemma V.6.2], there is an ordinal α such that for all x with $\phi_-(x)$, $f(x) \leq \alpha$. Now, let Q be the set of x's such that α has an initial segment isomorphic to $f(x)$. Q is hyperarithmetic (see the proof of [Sim99, Lemma VII.3.19]).) Let P_i be the downward closure of Q in $\bar{P}_i$. (i.e.: $P_i = \{x \in \bar{P}_i : \exists y \in Q(x \leq_P y)\}$.) Since for no $x \in Q$, $\mathbf{1} + \mathcal{F}_i + \mathbf{1} \preccurlyeq \mathcal{P}_{(\leq x)}$, we have that that $\mathbf{1} + \mathcal{F}_i + \mathbf{1} \not\preccurlyeq P_i$. Analogously $\mathbf{1} + \mathcal{F}_{i+1} + ... + \mathcal{F}_m + \mathbf{1} \not\preccurlyeq \bar{P}_i \smallsetminus P_i$. Let $\bar{P}_{i+1} = \bar{P}_i \smallsetminus \mathcal{P}_i$. It is not hard to see that $\langle P_0, ..., P_i, \bar{P}_{i+1}\rangle$ satisfies the conditions above. □

Theorem 5.14. *JUL(min-dec) is equivalent to ATR_* over RCA_*.*

Proof. Assume JUL(min-dec). Since $\omega^* + \omega$ is a minimal decomposition of ζ, we have that ζ is extendible. It is proved in [DHLS03, Theorem 3] that the extendibility of ζ implies ATR_0 over RCA_0.

Let us prove JUL(min-dec) from ATR_*. The direction from left to right was proved in Corollary 5.10. We now prove the other direction. Let $\langle \check{\mathcal{F}}_i : i \leq n\rangle$ be a minimal decomposition of $\mathcal{L}$ such that for no i, $\check{\mathcal{F}}_i = \check{\mathcal{F}}_{i+1} = \mathbf{1}$ or $\check{\mathcal{F}}_i + \check{\mathcal{F}}_{i+1}$ is $\langle \rightarrow, \leftarrow\rangle$. Let $\mathcal{P}$ be a partial ordering which does not embed $\mathcal{L}$. By Lemma 5.12, $\mathcal{L}$ has a finite decomposition of one of four possible forms. Suppose $\mathcal{L}$ has a decomposition of the form $\langle \mathbf{1}, \mathcal{F}_0, \mathbf{1}, \mathcal{F}_1, \mathbf{1}, ..., \mathcal{F}_m, \mathbf{1}\rangle$. The other cases are similar to this one. Then, consider a partition, $\{P_i : i \leq m\}$, of P as in Lemma 5.13.

Now, by induction on $i \leq m$, we prove that there exists a sequence $\langle \mathcal{Q}_0, ..., \mathcal{Q}_i\rangle$, hyperarithmetic in $\mathcal{P}$ and $\mathcal{L}$, such that for each $j \leq i$, $\mathcal{Q}_i = \langle P_i, \leq_Q\rangle$ is a linearization of $\mathcal{P}_i$ which does not embed $\mathbf{1} + \mathcal{F}_i + \mathbf{1}$. The formula we are proving by induction is equivalent to a Π^1_1 one by Lemma 5.11, so we can do this using Σ^1_1-IND. The base case and induction step follow immediately from Proposition 6.18.

Define $\mathcal{Q} = \sum_{i \leq m} Q_i$, and think of the domain of $\mathcal{Q}$ as P. So, $\mathcal{Q}$ is a linearization of $\mathcal{P}$. We claim that $\mathcal{Q}$ does not embed $\mathcal{L}$. Suppose, toward a contradiction, that we have an embedding $f\colon \mathbf{1} + \mathcal{F}_0 + ... + \mathcal{F}_m + \mathbf{1} \hookrightarrow \mathcal{Q}$. Let $x_0 \leq_Q x_1 \leq_Q ... \leq_Q x_{m+1}$ be the image under f of the $\mathbf{1}$s in $\mathbf{1} + \mathcal{F}_0 + ... + \mathcal{F}_m + \mathbf{1}$. For each $i \leq m+1$ let $j_i \leq m$ be such that $x_i \in P_{j_i}$. Note that for every $i \leq m$, $j_i \leq j_{i+1}$. We claim that for some i, $j_i = j_{i+1} = i$. It can be easily proved by induction on i that, if the claim is not true, then for every i, $j_i \geq i$. We then get a contradiction when we let $i = m+1$. So, there exists some i such that $x_i, x_{i+1} \in P_i$. But then, f maps $\mathbf{1} + \mathcal{F}_i + \mathbf{1}$ into $\mathcal{Q}_i$, contradicting the definition of the Q_is. □

Corollary 5.15. *JUL is equivalent to FRA over RCA_*.*

Proof. We have proved, in Lemma 5.7, that JUL implies FRA. Now assume FRA holds, and hence FINDEC too. Recall that FRA implies ATR_0, so from the theorem above, we have JUL(min-dec). Let $\mathcal{L}$ a scattered linear ordering which does not have any essential segment of the form $\mathbf{2}$ or $\langle \rightarrow, \leftarrow\rangle$. Using FINDEC and Proposition 3.10, we get that $\mathcal{L}$, has a minimal decomposition $\langle \mathcal{F}_i : i \leq n\rangle$. ¿From Lemma 5.9, we get that for no i, $\mathcal{F}_i + \mathcal{F}_{i+1}$ is of the form $\mathbf{2}$ or $\langle \rightarrow, \leftarrow\rangle$. Using JUL(min-dec) we get that $\mathcal{L}$ is extendible. □

6. Extendibility of h-indecomposable linear orderings

This section is devoted to proving the following theorem in ATR_*.

Theorem 6.1. *(ATR_*) Every h-indecomposable linear ordering is extendible.*

Every result in this section is going to be proved in ATR_*. So, unless otherwise stated, we will be working in ATR_*.

ATR_* it is not strong enough to prove the existence of ω_1^{CK}. But it can prove the existence of a linear ordering which contains ω_1^{CK}. Let ξ be a recursive linear ordering such that every hyperarithmetic well ordering embeds into ξ as an initial segment. We write $x \in \omega_1^{CK}$ as an abbreviation for $x \in \xi$ and $\xi_{(<x)}$ is well ordered. The existence of such a ξ in ATR_0 follows from [Sim99, Lemma VIII.3.14 and Theorem VIII.3.15].

6.1. **Extendibility of ω^* and $(\omega^2)^*$.** Before we prove the extendibility of an arbitrary h-indecomposable linear ordering, we provide two examples. These examples will illustrate some key ideas used in the general case.

Theorem 6.2. *ω^* is extendible.*

A stronger version of this theorem is proved in [DHLS03]. They prove that ω^* is extendible in ACA_0. Our proof, even though it uses ATR_0, is easier to understand and incorporates an idea that we will generalize later.

Proof. Consider a recursive partial ordering $\mathcal{P}$ which does not embed ω^*, or equivalently, which is well founded. If $\mathcal{P}$ is not recursive, relativize. Consider the rank function, $\mathrm{rk}_{\mathcal{P}}$, on $\mathcal{P}$. Let $\alpha \in \omega_1^{CK}$ be the rank of $\mathcal{P}$. Define a linearization, $\leq_Q$, of $\mathcal{P}$ as follows: let $x \leq_Q y$ iff $\mathrm{rk}_{\mathcal{P}}(x) < \mathrm{rk}_{\mathcal{P}}(y)$ or $\mathrm{rk}_{\mathcal{P}}(x) = \mathrm{rk}_{\mathcal{P}}(y)$ and $x \leq_{\mathbb{N}} y$ (where $\leq_{\mathbb{N}}$ is the ordering of the natural numbers; recall that the domain of $\mathcal{P}$ is a subset of $\mathbb{N}$). Observe now that $\omega^* \not\preceq \langle P, \leq_Q\rangle$. $\square$

Using Proposition 6.7, we get as a corollary of the previous theorem that $\mathbf{1} + \omega^*$ is extendible too. We will use this in the next theorem.

Theorem 6.3. *(ATR_0) $(\omega^2)^*$ is extendible.*

In ATR_0, this is a new result. The key idea is the use of the trees $T_{x,\omega^{2*}}$ defined below. It allows us to prove this theorem in ATR_0, and is going to be very useful in the more general case.

We write ω^{2*} for $(\omega^2)^*$.

Proof. Consider a partial ordering $\mathcal{P}$ which does not embed ω^{2*}. Assume $\mathcal{P}$ is recursive; otherwise relativize. Let $T_{\mathcal{P},\omega^{2*}}$ be the set of all $\sigma = \langle \pi_0, ..., \pi_{n-1}\rangle$ such that:

- for every $i < n$, π_i is a $(n-i)$-tuple from $\mathcal{P}$;
- for every $i < n$ and $j < k < |\pi_i|$, $\pi_i(j) >_P \pi_i(k)$;
- for every $i, i' < n$, $j < |\pi_i|$ and $j' < |\pi_{i'}|$, if $i < i'$ then $\pi_i(j) >_P \pi_{i'}(j')$.

We claim that $T_{\mathcal{P},\omega^{2*}}$ is well founded. Indeed, a path f though $T_{\mathcal{P},\omega^{2*}}$ codes a sequence $\langle f_0, f_1, ...\rangle$ such that each f_i is a descending sequence in $\mathcal{P}$ and for all x, y and $i < j$, $f_i(x) >_P f_j(y)$. Therefore, f codes a embedding $\omega^{2*} \hookrightarrow \mathcal{P}$. Let $\alpha \in \omega_1^{CK}$ be the rank of $T_{\mathcal{P},\omega^{2*}}$. Now, for each $x \in P$ let $T_{x,\omega^{2*}}$ be the subtree of $T_{\mathcal{P},\omega^{2*}}$ which consist of the $\sigma = \langle \pi_0, ..., \pi_{n-1}\rangle$ such that $\forall i < n \forall j < n-i (\pi_i(j) \leq_P x)$. So $T_{x,\omega^{2*}}$ is $T_{\mathcal{P}_{(\leq x)},\omega^{2*}}$ Let r_x be the rank of $T_{x,\omega^{2*}}$, and for each $\gamma < \alpha$, let

$$Q_\gamma = \{x \in P : r_x = \gamma\}.$$

We claim that for each γ, $\mathbf{1} + \omega^* \not\preceq Q_\gamma$. Suppose, toward a contradiction, that there exists an $f \colon \omega^* \hookrightarrow Q_\gamma$ and an $x \in Q_\gamma$ such that for all $n \in \omega^*$, $x \leq_P f(n)$.

Let $y = f(0) \in P$. We will prove that $r_x < r_y$ contradicting the fact that both x and y are in Q_γ. In order to prove this, we use f to construct an embedding, g, of $T_{x,\omega^{2*}}$ into $T_{y,\omega^{2*}}$ such that $g(\emptyset) \supsetneq \emptyset$. Given $\sigma = \langle \pi_0, ..., \pi_{n-1} \rangle \in T_{x,\omega^{2*}}$, let $g(\sigma) = \langle f \restriction n+1, \pi_0, ..., \pi_{n-1} \rangle \in T_{y,\omega^{2*}}$. Clearly g is an embedding and $g(\emptyset) = \langle\langle f(0) \rangle\rangle$. Therefore, the rank of $\langle\langle f(0)\rangle\rangle$ in $T_{y,\omega^{2*}}$ is grater than or equal the rank of $T_{x,\omega^{2*}}$, and hence $r_y > r_x$. This proves our claim.

Now we want to linearize each Q_γ so that $\mathbf{1} + \omega^*$ does not embed in the linearization. To do this consider $\bigoplus_{\gamma<\alpha} Q_\gamma$ and observe that $\mathbf{1} + \omega^*$ does not embed in it. Therefore, it has a linearization that does not embed $\mathbf{1} + \omega^*$. For each γ, let $\leq_{Q_\gamma}$ be the restriction of this linearization to Q_γ. Now define a linearization $\leq_Q$ of $\mathcal{P}$ as follows: let $x \leq_Q y$ iff $r_x < r_y$ or $r_x = r_y$ and $x \leq_{Q_{r_x}} y$. Note that this is the same as defining

$$\langle P, \leq_Q \rangle = \sum_{\gamma<\alpha} \langle Q_\gamma, \leq_{Q_\gamma} \rangle.$$

Observe that there cannot be an embedding of ω^{2*} into $\langle P, \leq_Q \rangle$ because we would have an embedding of ω^{2*} into some $\langle Q_\gamma, \leq_{Q_\gamma} \rangle$ when not even $\mathbf{1} + \omega^*$ embeds in $\langle Q_\gamma, \leq_{Q_\gamma} \rangle$. □

6.2. **Extendibility of $\mathbf{1} + \mathcal{L} + \mathbf{1}$.** Let $\mathcal{L}$ be an h-indecomposable linear ordering. We study here the relation between the extendibility of $\mathcal{L}$ and the extendibility of $\mathbf{1} + \mathcal{L} + \mathbf{1}$. Assume that $\mathcal{L}$ is h-indecomposable to the left. Note that then, $\mathbf{1} + \mathcal{L} + \mathbf{1} \sim \mathbf{1} + \mathcal{L}$.

The general ideas in this subsection come from [Jul69, Lemma V.2.4].

Lemma 6.4. *If $\mathbf{1} + \mathcal{L} + \mathbf{1}$ is extendible, then so is $\mathcal{L}$.*

Proof. Let $\mathcal{P}$ be a partial ordering such that $\mathcal{L} \not\preccurlyeq \mathcal{P}$. Let $\mathcal{Q} = \mathbf{1} + \mathcal{P} + \mathbf{1}$. Then $\mathbf{1} + \mathcal{L} + \mathbf{1} \not\preccurlyeq \mathcal{Q}$, hence $\mathcal{Q}$ has a linearization $\mathcal{R} = \langle Q, \leq_R \rangle$, such that $\mathbf{1} + \mathcal{L} + \mathbf{1} \not\preccurlyeq \mathcal{R}$. The restriction of $\leq_R$ to P is a linearization of $\mathcal{P}$ which does not embed $\mathcal{L}$. □

Now consider a poset $\mathcal{P}$ such that $\mathbf{1}+\mathcal{L}+\mathbf{1} \not\preccurlyeq \mathcal{P}$ and assume that $\mathcal{L}$ is extendible. We will show how to linearize $\mathcal{P}$, so that $\mathbf{1}+\mathcal{L}+\mathbf{1}$ does not embed in the linearization. We will partition $\mathcal{P}$ into infinitely many pieces $\{\mathcal{P}_m : m \in \omega\}$ such that for each m, $\mathcal{L} \not\preccurlyeq \mathcal{P}_m$. The idea is that then we can use the extendibility of $\mathcal{L}$ to linearize each $\mathcal{P}_m$ and get a linearization of $\mathcal{P}$ as the one required.

Definition 6.5. *If $\mathcal{P}$ has a least element a, let $P_0 = P \setminus \{a\}$, $P_1 = \{a\}$ and $P_n = \emptyset$ for $n > 1$. Suppose now that $\mathcal{P}$ has no least element and that we have already defined P_i for $i < n$. Let a_n be the least, in the order of the natural numbers, element of $P \setminus (\bigcup_{i<n} P_i)$. (We are assuming that the domain of P is a subset of the natural numbers.) Now, let $P_n = \{x \in P \setminus (\bigcup_{i<n} P_i) : x >_P a_n\}$.*

Lemma 6.6. *If $\mathbf{1} + \mathcal{L} + \mathbf{1} \not\preccurlyeq \mathcal{P}$, then*

(1) *$\{\mathcal{P}_m\}_{m\in\omega}$ is a partition of $\mathcal{P}$.*
(2) *if $x \leq_P x'$, $x \in P_m$ and $x' \in P_{m'}$, then $m \geq m'$.*
(3) *For each m, $\mathcal{L} \not\preccurlyeq \mathcal{P}_m$.*

Proof. The first two parts follow easily from the definitions. For the last part, note that if $\mathcal{L} \preccurlyeq \mathcal{P}_m$, then $\mathbf{1} + \mathcal{L} + \mathbf{1} \preccurlyeq \mathbf{1} + \mathcal{L} \preccurlyeq \mathcal{P}$. □

Proposition 6.7. *Given an h-indecomposable linear ordering $\mathcal{L}$, $\mathcal{L}$ is extendible if and only if $\mathbf{1} + \mathcal{L} + \mathbf{1}$ is.*

Proof. We have already shown the implication from right to left. Now assume $\mathcal{L}$ is extendible and consider $\mathcal{P}$ such that $\mathbf{1}+\mathcal{L}+\mathbf{1} \not\preccurlyeq \mathcal{P}$. Let $\{\mathcal{P}_m\}_{m\in\mathbb{N}}$ be as defined above. For each m let $\mathcal{Q}_m = \langle P_m, \leq_{Q_m}\rangle$ be a linearization of $\mathcal{P}_m$ which does not embed $\mathcal{L}$. To get all the linearizations $\{\mathcal{Q}_m\}_{m\in\mathbb{N}}$ uniformly, consider $\mathcal{Q} = \bigoplus_{m\in\mathbb{N}} \mathcal{P}_m$. Observe that $\mathcal{L} \not\preccurlyeq \mathcal{Q}$, and linearize $\mathcal{Q}$ so that $\mathcal{L}$ does not embed in the linearization. Observe now that $\sum_{m\in\omega^*}\mathcal{Q}_m$ is a linearization of $\mathcal{P}$ which does not embed $\mathbf{1}+\mathcal{L}$ (by Corollary 2.12, substituting left for right and ω^* for ω). □

Remark 6.8. *Note that the results we have proved so far in this subsection could have been proved using only RCA_0. But ATR_* is enough for our purposes.*

6.3. **Extendibility of $\sum_{m\in\omega^*}\mathcal{L}_m$.** Now suppose we are given a partial ordering $\mathcal{P}$ such that $\mathcal{L} \not\preccurlyeq \mathcal{P}$. Again assume that $\mathcal{L} = \mathrm{lin}(T)$ is h-indecomposable to the left, and also assume that $\mathcal{L} \neq \omega^*$. Let $\mathcal{L}_k = \mathrm{lin}(T_{\langle (k)_0\rangle})$. (Since $\mathcal{L}\neq\omega^*$, $T_{\langle m\rangle}$ exists.) So $\mathcal{L} = \sum_{k\in\omega^*}\mathcal{L}_k$. We will partition $\mathcal{P}$ into $\{\mathcal{P}_{m,\gamma}\}_{m\in\omega,\gamma\in\omega_1^{CK}}$ such that for each m and γ, $\mathbf{1}+\mathcal{L}_m+\mathbf{1} \not\preccurlyeq \mathcal{P}_{m,\gamma}$. Note that if we could uniformly linearize each $\mathcal{P}_{m,\gamma}$ into a linear ordering $\mathcal{Q}_{m,\gamma}$ such that $\mathbf{1}+\mathcal{L}_m+\mathbf{1} \not\preccurlyeq \mathcal{Q}_{m,\gamma}$, then $\sum_{\langle m,\gamma\rangle\in\omega\times\omega_1^{CK}}\mathcal{Q}_{m,\gamma}$ would be a linearization of $\mathcal{P}$ which does not embed $\mathcal{L}$.

We will construct the partition much as in the proof that ω^{2*} is extendible. But the fact that ω^{2*} is an ω^*-sum of terms which are all equal (all terms are ω^*) made that proof easier. In the general case, instead of considering one tree $T_{\mathcal{P},\mathcal{L}}$, we have to consider a tree $T^m_{\mathcal{P},\mathcal{L}}$ for each $m\in\mathbb{N}$. This modification is needed for the proof of Lemma 6.11(4) below.

Definition 6.9. *Given a poset $\mathcal{P}$ and $m\in\mathbb{N}$, define $T^m_{\mathcal{P},\mathcal{L}}$ to be the set of all $\sigma = \langle\pi_0,\dots,\pi_{n-1}\rangle$ such that:*

- *for every $i<n$, π_i is a $(n-i)$-tuple from $\mathcal{P}$;*
- *for every $i<n$ and $j,k<|\pi_i|$, if $j <_{L_{m+i}} k$, then $\pi_i(j) <_P \pi_i(k)$;*
- *for every $i,i'<n$, $j<|\pi_i|$ and $j'<|\pi_{i'}|$, if $i<i'$ then $\pi_i(j) >_P \pi_{i'}(j')$.*

Lemma 6.10. *If $\mathcal{L}\not\preccurlyeq\mathcal{P}$, then for all m, $T^m_{\mathcal{P},\mathcal{L}}$ is well founded.*

Proof. Suppose that $T^m_{\mathcal{P},\mathcal{L}}$ is not well founded. A path f though $T^m_{\mathcal{P},\mathcal{L}}$ codes a sequence $\langle f_0,f_1,f_2,\dots\rangle$ such that each f_i is an embedding of $\mathcal{L}_{m+i}$ into $\mathcal{P}$ and for all x,y, if $i<j$, then $f_i(x) >_P f_j(y)$. So, we have an embedding

$$\sum_{i\in\omega^*, i\geq m}\mathcal{L}_i \preccurlyeq \mathcal{P}.$$

Since $\mathcal{L}$ is h-indecomposable to the left, we have an embedding $\mathcal{L}\preccurlyeq\mathcal{P}$, contradicting the hypothesis. □

When $\mathcal{L}\not\preccurlyeq\mathcal{P}$, we have that for each $x\in P$ and $m\in\omega$, $T^m_{\mathcal{P}_{(\leq x)},\mathcal{L}}$ is well founded, and uniformly recursive in x and m (and $\mathcal{P}$ and $\mathcal{L}$). Let $T^m_{x,\mathcal{L}} = T^m_{\mathcal{P}_{(\leq x)},\mathcal{L}}$ So, each tree $T^m_{x,\mathcal{L}}$ has a rank $r_{x,m}\in\omega_1^{CK}$. For each x, let r_x be the least of $\{r_{x,m}: m\in\mathbb{N}\}$ and m_x be the least m such that $r_{x,m}=r_x$. Define $\mathrm{rk}_{\mathcal{P},\mathcal{L}}(x) = \langle m_x, r_x\rangle$. Given $\gamma\in\xi$, and $m\in\mathbb{N}$, let $P_{m,\gamma} = \{x\in P : \mathrm{rk}_{\mathcal{P},\mathcal{L}}(x) = \langle m,\gamma\rangle\}$.

Lemma 6.11. *Assume that $\mathcal{L}\not\preccurlyeq\mathcal{P}$ and that $\mathcal{P}$ is hyperarithmetic, then*

1. *For $\gamma\in\xi\setminus\omega_1^{CK}$, $P_{m,\gamma}=\emptyset$.*
2. *$\{P_{m,\gamma}\}_{\gamma\in\xi, m\in\mathbb{N}}$ is a partition of P.*

(3) *If* $x \leq_P x'$, $x \in P_{m,\gamma}$ *and* $x' \in P_{m'\gamma'}$, *then* $\langle m, \gamma\rangle \leq_{\omega\times\xi} \langle m', \gamma'\rangle$. *i.e.* $\gamma <_\xi \gamma'$ *or* $\gamma = \gamma'$ *and* $m \leq m'$.
(4) $\mathbf{1} + \mathcal{L}_m + \mathbf{1} \not\preccurlyeq \mathcal{P}_{m,\gamma}$.

Proof. Part (1) is because the trees $T^m_{\mathcal{P},\mathcal{L}}$ are well founded and hyperarithmetic. Part (2) is clear because for all $x \in P$, $\mathrm{rk}_{\mathcal{P},\mathcal{L}}(x) \in I\!N \times \xi$. To prove part (3) we show that if $x \leq_P y$, then $\mathrm{rk}_{\mathcal{P},\mathcal{L}}(x) \leq_{\omega\times\xi} \mathrm{rk}_{\mathcal{P},\mathcal{L}}(y)$: Since for each m, $T^m_{x,\mathcal{L}} \subseteq T^m_{y,\mathcal{L}}$, we have that $r_{x,m} \leq_\xi r_{y,m}$. Therefore, $r_x \leq_\xi r_y$, and if $r_x = r_y$, then $m_x \leq m_y$. For the last part consider $x, y \in P_{m,\gamma}$, and suppose, toward a contradiction, that there is an embedding $f\colon \mathcal{L}_m \hookrightarrow (x, y)_P$. We shall define an embedding, g, of $T^{m+1}_{x,\mathcal{L}}$ into $T^m_{y,\mathcal{L}}$ such that $g(\emptyset) \supsetneq \emptyset$. This will imply that the rank of $T^{m+1}_{x,\mathcal{L}}$ is strictly smaller than the rank of $T^m_{y,\mathcal{L}}$, and therefore $r_x \leq_\xi r_{x,m+1} <_\xi r_{y,m} = r_y$. This would contradict the assumption that $r_x = r_y = \gamma$. Given $\sigma = \langle \pi_0, ..., \pi_{n-1}\rangle \in T^{m+1}_{x,\mathcal{L}}$, let

$$g(\sigma) = \langle f \restriction n + 1, \pi_0, ..., \pi_{n-1}\rangle \in T^m_{y,\mathcal{L}}.$$

It is not hard to check that g is as wanted. □

6.4. **One step iteration.** Now we join the previous two constructions into one. The partition we define in this subsection is the one that we will iterate later to construct a linearization of $\mathcal{P}$.

Let $\mathcal{L} = \mathrm{lin}(T)$ be h-indecomposable to the left. The case when $\mathcal{L}$ is $\rightarrow$ is analogous. First suppose that $\mathcal{L} \neq \omega^*$ and that $\mathcal{L} = \sum_{m\in\omega^*} \mathcal{L}_m$, where $\mathcal{L}_m = \mathrm{lin}(T_{(m)_0})$.

Definition 6.12. *For* $m, n \in I\!N$ *and* $\gamma \in \xi$, *let*

$$P_{m,\gamma,n} = \{x \in P_n : \mathop{\mathrm{rk}}_{\mathcal{P}_n,\mathcal{L}}(x) = \langle m, \gamma\rangle\},$$

where $\mathcal{P}_n$ *is as defined in 6.5. Note that the definition of* $P_{m,\gamma,n}$ *depends also on* $\mathcal{L}$.

Lemma 6.13. *If* $\mathbf{1} + \mathcal{L} + \mathbf{1} \not\preccurlyeq \mathcal{P}$ *and* $\mathcal{P}$ *is hyperarithmetic, then*

(1) *For* $\gamma \in \xi \smallsetminus \omega_1^{CK}$, $P_{m,\gamma,n} = \emptyset$.
(2) $\{P_{m,\gamma,n}\}_{\gamma\in\xi, m,n\in I\!N}$ *is a partition of* P.
(3) *if* $x \leq_P x'$, $x \in P_{m,\gamma,n}$ *and* $x' \in P_{m'\gamma'n'}$ *then* $\langle m, \gamma, n\rangle \leq_{\omega\times\xi\times\omega^*} \langle m', \gamma', n'\rangle$. *i.e.* $n \geq n'$ *or* $n = n'$ *and either* $\gamma <_\xi \gamma'$ *or* $\gamma = \gamma'$ *and* $m \leq m'$.
(4) $\mathbf{1} + \mathcal{L}_m + \mathbf{1} \not\preccurlyeq \mathcal{P}_{m,\gamma,n}$.

Proof. For each part, first apply Lemma 6.6 and then Lemma 6.11. □

The case $\mathcal{L} = \omega$ or $= \omega^*$ is a little different. Suppose $\mathcal{L} = \omega^*$. First define $\langle P_n\rangle_{n\in I\!N}$ exactly as in Definition 6.5. So, we have that if $\mathbf{1}+\omega^* \not\preccurlyeq \mathcal{P}$, then $\omega^* \not\preccurlyeq \mathcal{P}_n$ for any n. Let $\mathrm{rk}_{\mathcal{P}_n,\omega^*}(x) = \langle x, \mathrm{rk}(\mathcal{P}_{n(\leq x)})\rangle \in \omega \times \xi$. (We are using here that $P \subseteq I\!N$.) Here, $\mathrm{rk}(\mathcal{P}_{n(<x)})$ is the usual rank of the well founded partial ordering $\mathcal{P}_{n(<x)}$. Since $\mathcal{P}_n$ is hyperarithmetic, $\mathrm{rk}(\mathcal{P}_{n(<x)}) \in \omega_1^{CK}$. Let $P_{m,\gamma,n} = \{x \in P_n : \mathrm{rk}_{\mathcal{P}_n,\omega^*}(x) = \langle m, \gamma\rangle\}$. In other words, $P_{m,\gamma,n} = \{m\}$ if $m \in \mathcal{P}_n$ and $\mathrm{rk}(\mathcal{P}_{n(\leq x)}) = \gamma$, and $P_{m,\gamma,n} = \emptyset$ otherwise.

Lemma 6.14. *If* $\mathbf{1} + \omega^* + \mathbf{1} \sim \mathbf{1} + \omega^* \not\preccurlyeq \mathcal{P}$ *and* $\mathcal{P}$ *is hyperarithmetic, then*

(1) *For* $\gamma \in \xi \smallsetminus \omega_1^{CK}$, $P_{m,\gamma,n} = \emptyset$.
(2) $\{P_{m,\gamma,n}\}_{\gamma\in\xi, m,n\in I\!N}$ *is a partition of* P.
(3) *if* $x \leq_P x'$, $x \in P_{m,\gamma,n}$ *and* $x' \in P_{m'\gamma'n'}$ *then* $\langle m, \gamma, n\rangle \leq_{\omega\times\xi\times\omega^*} \langle m', \gamma', n'\rangle$.

(4) *Each $P_{m,\gamma,n}$ has at most one element.*

Proof. Parts (1), (2) and (4) follow from the fact that for all x and n, $\mathrm{rk}_{\mathcal{P}_n,\omega^*}(x) \in \omega_1^{CK}$ and that $P_{m,\gamma,n} \subseteq \{m\}$. Part (3) it is also immediate from the definition of the sets $P_{m,\gamma,n}$. □

The idea now, to linearize $\mathcal{P}$, is to keep on partitioning each piece we get in this fashion. First we partition $\mathcal{P}$ into $\{\mathcal{P}_{m,\gamma,n}\}_{\gamma\in\xi,m,n\in\mathbb{N}}$. Then, we partition each $\mathcal{P}_{m,\gamma,n}$, which is not a singleton, into $\{\mathcal{P}_{\langle\langle m,\gamma,n\rangle,\langle m',\gamma',n'\rangle\rangle}\}_{\gamma'\in\xi,m',n'\in\mathbb{N}}$ so that $\mathbf{1}+\mathcal{L}_{\langle(m)_0,(m')_0\rangle}+\mathbf{1} \not\preccurlyeq \mathcal{P}_{\langle\langle m,\gamma,n\rangle,\langle m',\gamma',n'\rangle\rangle}$, where $\mathcal{L}_\sigma = \mathrm{lin}(T_\sigma)$. We keep on doing this until we get a partition of $\mathcal{P}$ into singletons. The problem is that, to iterate this process, we need a uniform way of getting $\{\mathcal{P}_{m,\gamma,n}\}_{\gamma\in\xi,m,n\in\mathbb{N}}$ from $\mathcal{P}$. Note that the definition we gave of $\mathcal{P}_{m,\gamma,n}$ only makes sense when $\mathbf{1}+\mathcal{L}+\mathbf{1} \not\hookrightarrow \mathcal{P}$. Using the fact that the rank function is Δ^1_1 we get that $\{\mathcal{P}_{m,\gamma,n}\}_{\gamma\in\xi,m,n\in\mathbb{N}}$ is Δ^1_1 in $\mathcal{P}$. So, there is a Σ^1_1 formula $\varphi^\Sigma(\mathcal{P},\mathcal{L},\langle m,\gamma,n\rangle,x)$ and a Π^1_1 formula $\varphi^\Pi(\mathcal{P},\mathcal{L},\langle m,\gamma,n\rangle,x)$ such that if $\mathbf{1}+\mathcal{L}+\mathbf{1} \not\preccurlyeq \mathcal{P}$, then for all m,γ, n, and x,

$$\varphi^\Sigma(\mathcal{P},\mathcal{L},\langle m,\gamma,n\rangle,x) \Leftrightarrow \varphi^\Pi(\mathcal{P},\mathcal{L},\langle m,\gamma,n\rangle,x) \Leftrightarrow x \in P_{m,\gamma,n}.$$

We will use these formulas later to define the iteration process.

6.5. **The complement of a linear ordering.** Now we construct the structure over which we are going to iterate the process of partitioning $\mathcal{P}$.

For each h-indecomposable linear ordering $\mathcal{L} = \mathrm{lin}(T)$ we will define another linear ordering $\mathrm{com}(T)$, and a Π^1_1 subclass of it, $\mathrm{com}^{CK}(T)$, that we call *the complement of* $\mathbf{1}+\mathcal{L}+\mathbf{1}$. The name "complement" is inspired by the following property. Suppose that T is recursive, then for every recursive linear ordering $\mathcal{A}$ we have that

$$\mathcal{A} \preccurlyeq \overset{CK}{\mathrm{com}}(T) \Leftrightarrow \mathbf{1}+\mathcal{L}+\mathbf{1} \not\preccurlyeq \mathcal{A}.$$

The implication from left to right will follow from Lemma 6.17, and the other direction from the main result of this section, Proposition 6.18.

The idea of the definition of $\mathrm{com}(T)$ is like the one of the definition of $\mathrm{lin}(T)$ (Definition 2.6), but instead of taking ω (or ω^*) sums we take $\omega^*\times\xi^*\times\omega$ (or $\omega\times\xi\times\omega^*$) sums. Thus, for example, if $s_T(\emptyset) = +$, then

$$\mathrm{com}(T) = \sum_{\langle m,\gamma,n\rangle\in\omega^*\times\xi^*\times\omega} \mathrm{com}(T_{(m)_0}),$$

and if $s_T(\emptyset) = -$, then

$$\mathrm{com}(T) = \sum_{\langle m,\gamma,n\rangle\in\omega\times\xi\times\omega^*} \mathrm{com}(T_{(m)_0}).$$

Definition 6.15. *Given a recursive signed tree T, let*

$$\mathrm{com}(T) = \{\sigma \in \mathop{\mathrm{Seq}}_{\omega^*\times\xi^*\times\omega} : \sigma \neq \emptyset \;\&\; l(\sigma^-) \text{ is an end node of } T\},$$

where $l(\langle\langle m_0,a_0,n_1\rangle,\dots,\langle m_k,a_k,n_k\rangle\rangle) = \langle(m_0)_0,\dots,(m_k)_0\rangle$ and $\sigma^- = \sigma\restriction|\sigma|-1$. Now we define the ordering on $\mathrm{com}(T)$. Consider $\sigma_1 \neq \sigma_2 \in \mathrm{com}(T)$. Let $\tau \in \mathrm{Seq}_{\omega^*\times\xi^*\times\omega}$ and $x_1 \neq x_2 \in \omega^*\times\xi^*\times\omega$ be such that $\tau^\frown x_1 \subseteq \sigma_1$ and $\tau^\frown x_2 \subseteq \sigma_2$. We define

$$\sigma_1 \leq_{\mathrm{com}(T)} \sigma_2 \Leftrightarrow \begin{cases} x_1 \leq_{\omega^*\times\xi^*\times\omega} x_2 & \& \; s_T(l(\tau)) = + \quad \text{or} \\ x_1 \geq_{\omega^*\times\xi^*\times\omega} x_2 & \& \; s_T(l(\tau)) = -. \end{cases}$$

Let $\mathrm{com}^{CK}(T)$ *be the class of all* $\sigma \in \mathrm{com}(T)$ *such that for all* $i < |\sigma|$, $(\sigma(i))_1 \in \omega_1^{CK}$. *Let* $\widetilde{\mathrm{com}}(T)$ *be the downward closure of* $\mathrm{com}(T)$, *i.e.*

$$\widetilde{\mathrm{com}}(T) = \{\sigma \in \underset{\omega^*\times\xi^*\times\omega}{\mathrm{Seq}} : \exists \tau \supseteq \sigma(\tau \in \mathrm{com}(T))\}.$$

Observe that $\widetilde{\mathrm{com}}(T)$ *is a tree and* $\mathrm{com}(T)$ *is the set of end nodes of* $\widetilde{\mathrm{com}}(T)$.

Example 6.16. *Let us look at one of the simplest cases.* $T = \{\emptyset\}$ *and* $s_T(\emptyset) = -$. *So* $\mathrm{lin}(T) = \omega^*$, $\mathrm{com}^{CK}(T)$, *the complement of* $\mathbf{1}+\omega^*+\mathbf{1} \sim \mathbf{1}+\omega^*$, *is* $\omega\times\omega_1^{CK}\times\omega^* \sim \omega_1^{CK} \times \omega^*$. *On the one hand, observe that* $\mathbf{1}+\omega^*$ *does not embed in* $\omega \times \omega_1^{CK} \times \omega^*$. *Because otherwise we would have an embedding of* ω^* *into a popper final segment of* $\omega \times \omega_1^{CK} \times \omega^*$, *but every proper final segment of it is well ordered, since it is included in a segment of the form* $\omega \times \omega_1^{CK} \times \mathbf{n}$. *Therefore, if* $\mathbf{1}+\omega^* \preccurlyeq \mathcal{A}$, *then* $\mathcal{A} \not\preccurlyeq \mathrm{com}^{CK}(T)$. *On the other hand, consider a recursive linear ordering* $\mathcal{A}$ *such that* $\mathbf{1}+\omega^* \not\preccurlyeq \mathcal{A}$. *We can decompose* $\mathcal{A}$ *into a sum* $\sum_{i\in\omega^*} \mathcal{A}_i$ *such that each* $\mathcal{A}_i$ *is recursive and well ordered. Decompose* $\mathcal{A}$ *in the same way we partitioned* $\mathcal{P}$ *in Definition 6.5, but now we get* $\mathcal{A} = \sum_{i\in\omega^*} \mathcal{A}_i$ *because* $\mathcal{A}$ *is linearly ordered.) Then, each* $\mathcal{A}_i$ *embeds in* ω_1^{CK}, *so we have an embedding of* $\mathcal{A}$ *into* $\omega \times \omega_1^{CK} \times \omega^*$.

Lemma 6.17. *Let* T *be a recursive signed tree and* $\mathcal{L} = \mathrm{lin}(T)$. *Then* $\mathbf{1}+\mathcal{L}+\mathbf{1} \not\preccurlyeq \mathrm{com}^{CK}(T)$.

Proof. Suppose g is an embedding of $\mathbf{1}+\mathcal{L}+\mathbf{1}$ into $\mathrm{com}(T)$ such that for all $x \in \mathbf{1}+\mathcal{L}+\mathbf{1}$, $g(x) \in \mathrm{com}^{CK}(T)$. For each n, we define $\sigma_n \in T$ and a recursive embedding $g_n \colon \mathbf{1}+\mathcal{L}_{\sigma_n}+\mathbf{1} \hookrightarrow \mathrm{com}(T_{\sigma_n})$, uniformly in $0''$ (where $\mathcal{L}_\sigma = \mathrm{lin}(T_\sigma)$). We will define the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ such that for all n, $\sigma_n \subsetneq \sigma_{n+1}$, contradicting the well-foundedness of T. Let $\sigma_0 = \emptyset$ and $g_0 = g$. Suppose we have defined σ_n and $g_n \colon \mathbf{1}+\mathcal{L}_{\sigma_n}+\mathbf{1} \hookrightarrow \mathrm{com}(T_{\sigma_n})$. If for some n we have that $T_{\sigma_n} = \{\emptyset\}$ and $\mathcal{L}_{\sigma_n}$ is ω^*, then we get a contradiction because we have an embedding of $\mathbf{1}+\omega^*$ into $\omega \times \omega_1^{CK} \times \omega^*$. Analogously if for some n we have that $\mathcal{L}_{\sigma_n}$ is ω. To fix ideas assume that $s_T(\sigma_n) = -$. So

$$L_{\sigma_n} = \sum_{m\in\omega^*} \mathcal{L}_{\sigma_n^\frown(m)_0} \quad \text{and} \quad \mathrm{com}(T_{\sigma_n}) = \sum_{\langle m,\gamma,n\rangle\in\omega\times\xi\times\omega^*} \mathrm{com}(T_{\sigma_n^\frown(m)_0}).$$

Think of $\omega \times \xi \times \omega^*$ and $\omega^* \times \xi^* \times \omega$ as having the same domain but opposite orderings. For each m, let x_m be a member of $\mathcal{L}_{\sigma_n^\frown(m)_0}$, the mth term in the first sum above. So $\langle x_m\rangle_{m\in\mathbb{N}}$ is co-initial in $\mathcal{L}$. Let $a_m \in \omega^*\times\xi^*\times\omega$ be the first entry of the sequence $g_n(x_m) \in \mathrm{Seq}_{\omega^*\times\xi^*\times\omega}$. So $g_n(x_m)$ belongs to the a_mth term in the second sum above. Let $b \in \omega^*\times\xi^*\times\omega$ be the first entry of $g_n(x) \in \mathrm{Seq}_{\omega^*\times\xi^*\times\omega}$, where x is the first element of $\mathbf{1}+\mathcal{L}_{\sigma_n}+\mathbf{1}$. Note that

$$a_0 \leq_{\omega^*\times\xi^*\times\omega} a_1 \leq_{\omega^*\times\xi^*\times\omega} a_2 \leq_{\omega^*\times\xi^*\times\omega} \cdots \leq_{\omega^*\times\xi^*\times\omega} b.$$

Let $a = \lim_m(a_m)$ (with the discrete topology). The limit has to exist, because otherwise we would have an embedding of $\omega+\mathbf{1}$ into $\omega^*\times\xi^*\times\omega$, or equivalently of $\mathbf{1}+\omega^*$ into $\omega\times\xi\times\omega^*$, contradicting what is said in the example above. Let $\sigma_{n+1} = \sigma_n^\frown((a)_0)_0$. Find $\bar{m}$ such that $\forall m \geq \bar{m}(a_m = a)$. Then, we have that

$$\sum_{m\in\omega^*,m>\bar{m}} \mathcal{L}_{\sigma_n^\frown(m)_0} \preccurlyeq \mathrm{com}(T_{\sigma_{n+1}})$$

Now, pick a copy of $\mathbf{1}+\mathcal{L}_{\sigma_{n+1}}+\mathbf{1}$ inside $\sum_{m\in\omega^*,m>\bar{m}} \mathcal{L}_{\sigma_n^\frown(m)_0}$ and construct g_{n-1} as the restriction of g_n to it.

□

6.6. **The linearization.** Now we describe the partition process that we mentioned earlier. Let $\mathcal{L} = \text{lin}(T)$ be a recursive h-indecomposable linear ordering. Consider $\mathcal{P}$, a recursive partial ordering such that $\mathbf{1} + \mathcal{L} + \mathbf{1} \not\preccurlyeq \mathcal{P}$. We will define a hyperarithmetic family $\{P_\sigma\}_{\sigma \in \widetilde{\text{com}}(T)}$ of subsets of P indexed by $\widetilde{\text{com}}(T)$, such that

(C1) If $\sigma \in \text{com}(T) \setminus \text{com}^{CK}(T)$, then $P_\sigma = \emptyset$.
(C2) $\{P_\sigma\}_{\sigma \in \text{com}^{CK}(T)}$ is a partition of P.
(C3) If $\sigma, \tau \in \text{com}(T)$, $x \in P_\sigma$, $y \in P_\tau$ and $x \leq_P y$, then $\sigma \leq_{\text{com}(T)} \tau$.
(C4) For $\sigma \in \widetilde{\text{com}}(T) \setminus \text{com}(T)$, $\mathbf{1} + \mathcal{L}_{l(\sigma)} + \mathbf{1} \not\preccurlyeq \mathcal{P}_\sigma$ and $\{P_{\sigma^\frown x}\}_{x \in \omega^* \times \xi^* \times \omega}$ is the partition of P_σ given by Definition 6.12 with respect to $L_{l(\sigma)}$.
(C5) For $\sigma \in \text{com}(T)$, P_σ is either empty or a singleton.

Then we can construct a map from $\mathcal{P}$ to $\text{com}^{CK}(T)$ which preserves order. Just map $x \in P$ to the $\sigma \in \text{com}(T)$ such that $P_\sigma = \{x\}$. Therefore we have a linearization of $\mathcal{P}$ which, by Lemma 6.17, does not embed $\mathbf{1} + \mathcal{L} + \mathbf{1}$. This will prove the following proposition.

Proposition 6.18. *Given a recursive h-indecomposable linear ordering $\mathcal{L} = \text{lin}(T)$ and a recursive partial ordering $\mathcal{P}$ such that $\mathbf{1}+\mathcal{L}+\mathbf{1} \not\preccurlyeq \mathcal{P}$, there is a hyperarithmetic linearization $\mathcal{Q}$ of $\mathcal{P}$ such that $\mathcal{Q} \preccurlyeq \text{com}^{CK}(T)$, and therefore $\mathbf{1} + \mathcal{L} + \mathbf{1} \not\preccurlyeq \mathcal{Q}$.*

Theorem 6.1 now follows from the relativized version of the previous proposition and Proposition 6.7.

The obvious definition of $\{P_\sigma\}_{\sigma \in \widetilde{\text{com}}(T)}$ by recursion using the construction of 6.4, would use a too complicated recursion that is not available in ATR_*. The problem is that the definition of the partition of each P_σ only makes sense when we know that $\mathbf{1} + \mathcal{L}_{l(\sigma)} + \mathbf{1} \not\preccurlyeq \mathcal{P}_\sigma$. But to prove that, we have to have already defined P_{σ^-} and proved that $\mathbf{1} + \mathcal{L}_{l(\sigma^-)} + \mathbf{1} \not\preccurlyeq \mathcal{P}_\sigma$.

The main tool to construct this partition of $\mathcal{P}$ is the following lemma.

Lemma 6.19. *(ATR_*) Let $\psi^\Sigma(X, x)$ be a Σ^1_1 formula and $\psi^\Pi(X, x)$ and $\chi(X)$ be Π^1_1 formulas. Suppose that we know that for every set X,*

$$\text{(X)} \qquad \chi(X) \Rightarrow \quad \begin{array}{l} \forall y(\psi^\Sigma(X, y) \Leftrightarrow \psi^\Pi(X, y)) \ \& \\ \forall Y(Y = \{y : \psi^\Sigma(X, y)\} \Rightarrow \chi(Y)). \end{array}$$

Let X_0 be a given set such that $\chi(X)$. Then, there exists a sequence $\langle R_n : n \in \mathbb{N}\rangle$ such that

(1) *$R_0 = X_0$,*
(2) *for every n, $R_{n+1} = \{y : \psi^\Sigma(R_n, y)\} = \{y : \psi^\Pi(R_n, y)\}$, and*
(3) *for every n, $\chi(R_n)$.*

First we show how this implies Proposition 6.18.

Proof of Proposition 6.18. We have to construct a family $\{P_\sigma\}_{\sigma \in \widetilde{\text{com}}(T)}$ of subsets of P indexed by $\widetilde{\text{com}}(T)$ satisfying conditions (C1)-(C5). We have already seen how this implies the proposition. We apply the Lemma above to construct a sequence $\langle R_n : n \in \mathbb{N}\rangle$ such that $R_n = \{P_\sigma\}_{\sigma \in \widetilde{\text{com}}(T), |\sigma| = n}$. All we need to do is to define ψ^Σ, ψ^Π, χ and X_0. Let $X_0 = \{P\}$. Let Γ be either Σ or Π. We let $\psi^\Gamma(X, x)$ be the formula that says the following: X is of the form $\{Q_\sigma : \sigma \in \widetilde{\text{com}}(T), |\sigma| = n\}$ for some n, x is of the form $\langle \tau, y\rangle$ for some $\tau \in \widetilde{\text{com}}(T)$ with $|\tau| = n + 1$ and $y \in P$, and

$$\varphi^\Gamma(Q_{\tau^-}, \mathcal{L}_{l(\tau^-)}, \tau(n), y).$$

(The formulas $\varphi^\Gamma(Q,\mathcal{L},x,y)$ were defined at the end of Subsection 6.4.) Let $\chi(X)$ be the formula that says that X is hyperarithmetic and of the form $\{Q_\sigma : \sigma \in \widetilde{\mathrm{com}}(T), |\sigma| = n\}$ for some n, and for each $\tau \in \widetilde{\mathrm{com}}(T)$ with $|\tau| = n$, $\mathbf{1} + \mathcal{L}_{l(\tau)} + \mathbf{1} \not\preccurlyeq Q_\tau$.

Note that ψ^Σ is Σ^1_1 and ψ^Π and χ are Π^1_1. Condition (X) follows from Lemmas 6.13 and 6.14 and the comments on φ^Σ and φ^Π at the end of subsection 6.4. □

Proof of Lemma 6.19. Let Γ be either Σ or Π and $\bar\Gamma$ be the other one. We say that a sequence $\bar R = \langle R_i : i < n\rangle$, with $n \le \omega + 1$, is *acceptable*$^\Gamma$ if $R_0 = X_0$, and for all $i < n - 1$

$$\forall y(y \in R_{i+1} \Rightarrow \psi^\Gamma(R_i, y)) \text{ and } \forall y(\psi^{\bar\Gamma}(R_i, y) \Rightarrow y \in R_{i+1}).$$

We say that $\bar R$ *satisfies* χ if $\forall i < n(\chi(R_i))$. We make three observations.

The first observation is that if $\bar R$ is acceptable$^\Pi$ and satisfies χ, then it is also acceptable$^\Sigma$: For each i, since $\chi(R_i)$, $\forall y(\psi^\Sigma(R_i, y) \Leftrightarrow \psi^\Pi(R_i, y))$, and therefore $R_{i+1} = \{y : \psi^\Sigma(R_i, y)\} = \{y : \psi^\Pi(R_i, y)\}$.

The second observation is that if $\bar R$ is acceptable$^\Pi$ and satisfies χ, $\bar Q$ is either acceptable$^\Sigma$ or acceptable$^\Pi$ and $|\bar R| = |\bar Q|$, then $\bar R = \bar Q$: Use arithmetic induction. If $R_i = Q_i$, since $\chi(R_i)$ we have that

$$Q_{i+1} = \{y : \psi^\Pi(Q_i, y)\} = \{y : \psi^\Pi(R_i, y)\} = R_{i+1}.$$

These two observations imply that if there is an $\bar R$ which is acceptable$^\Pi$ and satisfies χ, then it is the unique acceptable$^\Pi$ sequence and also the unique acceptable$^\Sigma$ sequence.

The last observation is that for every n there exists an $\bar R$ of length n which is hyperarithmetic in R_0, acceptable$^\Pi$ and satisfies χ. We prove this using Σ^1_1-IND. By Lemma 5.11, the formula we are proving by induction is equivalent to a Π^1_1 one. For the induction basis consider $\langle R_0\rangle$. For the induction step assume we have $\bar R$ of length $n \ge 1$ which is hyperarithmetic in R_0, acceptable$^\Pi$ and satisfies χ. Since $\chi(R_{n-1})$, because of condition (X) we can define

$$R_n = \{y : \psi^\Sigma(R_{n-1}, y)\} = \{y : \psi^\Pi(R_{n-1}, y)\},$$

by Δ^1_1-CA (which holds in ATR_*; [Sim99, Lemma VII.4.1]). Since R_{n-1} is hyperarithmetic, R_n is too. Now, $\bar R^\frown R_n$ has length $n+1$, is hyperarithmetic in R_0, is acceptable$^\Pi$ and satisfies χ.

Now we want to define $\bar R$ of length ω, acceptable$^\Pi$, acceptable$^\Sigma$ and satisfying χ. We define it by Δ^1_1-CA as follows: We let $\langle n, x\rangle \in \bar R$ if and only if there exists a sequence $\langle Q_0, ..., Q_n\rangle$, hyperarithmetic in R_0 and acceptable$^\Pi$, such that $x \in Q_n$, which is equivalent to a Π^1_1 formula by Lemma 5.11. Equivalently, $\langle n, x\rangle \in \bar R$ if and only if there exists a sequence $\langle Q_0, ..., Q_n\rangle$, acceptable$^\Sigma$ such that $x \in Q_n$, which is a Σ^1_1 formula. It follows from the observations above that these two definitions are equivalent and that $\bar R$ is as required. □

6.7. **Extendibility of η.** The proof theoretic strength of the fact that η^* is extendible was studied by Downey, Hirschfeldt, Lempp and Solomon in [DHLS03]. They proved the extendibility of η in Π^1_2-CA_0 and give a modification of their proof, due to Howard Becker, that uses only Π^1_1-CA_0. Becker's modification is based in the observation that if $\eta \not\preccurlyeq \mathcal{P}$ and $\mathcal{P}$ is recursive, then $\mathcal{P}$ has a hyperarithmetic linearization which does not embed η. This observations allowed him to use Lemma 5.11 to reduce the complexity of certain formulas used in the proof. We prove now that the extendibility of η is provable in ATR_*. Notice that ATR_* is strictly weaker than Π^1_1-CA_0. (It is weaker because Π^1_1-CA_0 implies ATR_0 and Σ^1_1-IND. It is strictly weaker because every β-model is a model of ATR_* but there is a β-model which is not a model of Π^1_1-CA_0. See [Sim99, Chapters VI and VII].) Joseph Miller [Mil] proved that the extendibility of η implies WKL_0 and that over Σ^1_1-AC_0, it implies ATR_0. Whether the extendibility of η is equivalent to ATR_0 over RCA_0 is still an open question.

Theorem 6.20. *(ATR_*) η is extendible.*

Proof. Take a partial ordering $\mathcal{P}$ such that $\eta \not\preccurlyeq \mathcal{P}$. Consider the class of all the recursive trees T such that, if $s_T\colon T \to \{+,-\}$ is the constant function equal to $+$, then $\mathrm{lin}(T) = \mathrm{lin}(\langle T, s_T\rangle) \preccurlyeq \mathcal{P}$. (Note that the definition of $\mathrm{lin}(T)$ did not require T to be well founded.) Only consider the trees T that also satisfy that for every $\sigma \in T$, σ has an extension which is an end node of T. This is a Σ^1_1 class of trees, and therefore different from the class of well founded recursive trees (see [Sim99, Theorem V.1.9]). We claim that there is no tree T in this class with $\mathrm{lin}(T) \preccurlyeq \mathcal{P}$ which is not well founded. Suppose, toward a contradiction that $\mathrm{lin}(T) \preccurlyeq \mathcal{P}$ and $\langle a_i\rangle_{i\in\mathbb{N}}$ is a path through T. We will show that then, there is an embedding of η into $\mathcal{P}$. Consider the left-to-right ordering, $\leq_{LR}$, on Seq_2 which has order type η. Given $\sigma \in \mathrm{Seq}_2$, define $\bar\sigma \in \mathrm{Seq}_3$ of length $|\sigma|+1$ by letting, for $i < |\sigma|$, $\bar\sigma(i) = 0$ if $\sigma(i) = 0$ and $\bar\sigma(i) = 2$ if $\sigma(i) = 1$ and let $\bar\sigma(|\sigma|) = 1$. Now define $f(\sigma)$ to be a string in $\mathrm{lin}(T) \subseteq \mathrm{Seq}$ extending

$$\langle\langle a_0, \bar\sigma(0)\rangle, \langle a_1, \bar\sigma(1)\rangle, ..., \langle a_{|\sigma|}, \bar\sigma(|\sigma|)\rangle \in \widehat{T},$$

which exist by our assumption on T. Note that if $\sigma <_{LR} \tau$, then $f(\sigma) <_{\mathrm{lin}(T)} f(\tau)$. So, we have that $\eta \preccurlyeq \mathrm{lin}(T) \preccurlyeq \mathcal{P}$, contradicting our assumptions.

Hence, there has to be some well founded T such that $\mathrm{lin}(T) \not\preccurlyeq \mathcal{P}$. By Theorem 6.1, $\mathrm{lin}(T)$ is extendible, and therefore, there is a linearization of $\mathcal{P}$ which does not embed $\mathrm{lin}(T)$. But then, this linearization cannot embed η either. □

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Department of Mathematics, Cornell University, Ithaca, NY 14853, USA
E-mail address: `antonio@math.cornell.edu`