Boolean Algebras, Tarski Invariants, and Index Sets

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Abstract

Tarski defined a way of assigning to each boolean algebra, B, an invariant $\mathrm{inv}(B) \in \mathtt{In}$, where $\mathtt{In}$ is a set of triples from $\mathbb{N}$, such that two boolean algebras have the same invariant if and only if they are elementarily equivalent. Moreover, given the invariant of a boolean algebra, there is a computable procedure that decides its elementary theory. If we restrict our attention to dense Boolean algebras, these invariants determine the algebra up to isomorphism. In this paper we analyze the complexity of the question "Does B have invariant x?". For each $x \in \mathtt{In}$ we define a complexity class Γ_x, that could be either Σ_n, Π_n, $\Sigma_n \wedge \Pi_n$, or $\Pi_{\omega+1}$ depending on x, and prove that the set of indices for computable boolean algebras with invariant x is complete for the class Γ_x. Analogs of many of these results for computably enumerable Boolean algebras were proven in [Sel90] and [Sel91]. According to [Sel03] similar methods can be used to obtain the results for computable ones. Our methods are quite different and give new results as well. As the algebras we construct to witness hardness are all dense, we establish new similar results for the complexity of various isomorphism problems for dense Boolean algebras.

1 Introduction

A common theme in mathematical investigations is the classification of structures (within a specified class) and the characterization of the (sub)classes delineated. Indeed, Hodges

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[Hod93] offers the classification process (along with constructions of specified types of structures) as the essence of model theory. Of course, the general endeavor pervades many branches of mathematics. Our topic in this paper has its origin in such a study of the class of Boolean algebras. It begins with Tarski's classification [Tar49] of Boolean algebras into countably many classes each consisting of the models of a complete extension of the basic theory. (Of course, this classifies Boolean algebras up to elementary equivalence.) His motivation was to prove that the theory of Boolean algebras was decidable and he did this by producing a uniformly computable list of axioms for (each of) the complete extensions corresponding to his classification.

Given such a classification (or the prospect of one), one may well want to characterize membership in each subclass in some way and analyze the complexity of the classes (i.e. of membership in each). The algebraist asks for invariants corresponding to structural properties that determine membership in each class. The model theorist might ask for the (simplest) axioms that insure such membership. The descriptive set theorist or recursion theorist wants to determine the location of the classes in some standard hierarchy. The former, expresses the results as completeness properties for the classes of countable structures at levels of the Borel hierarchy. The latter, takes the lightface approach of proving completeness of the subclasses of the computable structures in the arithmetic, hyperarithmetic or analytic hierarchy. (Typically, relativization of such lightface characterizations produces the boldface Borel ones.)

For the classification of Boolean algebras up to elementary equivalence, Tarski [Tar49] (see also [Ers64], [Gon97, Ch. 2] and [Mon89, Ch. 7]) provides the structural information by describing algebraic invariants as well as axiomatizations for each class. The determination of the simplest form of such axiom systems (in the sense syntactic complexity) is given by Wasziewicz [Was74]. In this paper, we provide the recursion (and so descriptive set) theoretic characterizations of these classes as complete at specified levels of the arithmetic hierarchy and a bit more. The classes provide not only index sets complete at the Σ_n or Π_n level for each $n < \omega$ but also for level $\Pi_{\omega+1}$ (the sets co-c.e. in $0^{(\omega)}$) and even more unusually for the classes $\Sigma_n \wedge \Pi_n$ (the sets which are intersections of one in Σ_n and one in Π_n) for $n \equiv 1, 2 \pmod 4$. As a by-product of our analysis we reprove the results of [Was74] as well.

A standard question related to classifying the complexity of membership in such subclasses is how to characterize the complexity of the isomorphism problem (when two structures are isomorphic) for structures in the class or specified subclasses. Again, there are natural descriptive set theoretic as well as recursion theoretic versions of this problem. For the class of all Boolean algebras the isomorphism problem is as complicated as possible, i.e. Σ_1^1 complete, and so one typically says that there is no way to classify all Boolean algebras up to isomorphism or provide isomorphism invariants. There is, however, an algebraically defined class of Boolean algebras, the dense Boolean algebras (see Definition 4.2), for which elementary equivalence is the same as isomorphism. (So model theoretically these are the saturated Boolean algebras.) We construct dense Boolean algebras as witnesses for all the hardness results for membership in each of the elementary classes. Thus we can deduce analogous results for isomorphism problems on these classes of Boolean algebras. (Some care needs to be taken as being dense is itself a complicated property.) We present the results in terms of typical *strong* index set notation, e.g. $(\Sigma_n, \Pi_n) \leq_{\mathtt{m}} (\mathcal{DB}_r, \mathcal{DB}_s)$ (where $\mathcal{DB}_r$ and $\mathcal{DB}_s$ are classes of dense Boolean algebras) as in Soare [Soa87, IV.3.1] and explained in Definition 2.9. This easily translates into the terminology proposed by Knight of the isomorphism relation being, e.g. Π_n, within some class of dense Boolean algebras. (See Definition 2.11 and also [Cal] for further discussion of this notion.) Thus our results also supply examples of classes complete (in a strong way) at the same syntactic levels for a collection of isomorphism problems. (Isomorphism problems at certain higher levels of the hyperartihmetical hierarchy are provided by classes of reduced Abelian p-groups as shown in Calvert [Cal].)

While all of these issues are natural in their own right, we should note that we came to the particular questions addressed here from the problem of classifying the complexity of related issues in terms of Reverse Mathematics. The question raised in [Sho04] is the proof theoretic complexity of the existence of invariants for (countable) Boolean Algebras classifying them up to elementary equivalence. Answers to such questions are often provided by index set type results. Indeed, as explained in [Sho04] it seemed plausible, because of the nature of the results and the proof theoretic issues, that one might need such results in this case. As it turned out, weaker hardness theorems for membership in some of the classes sufficed to reach the desired proof theoretic system of ACA_0^+ (corresponding to the existence of $X^{(\omega)}$ for every set X). Nonetheless, the recursion theoretic questions remained interesting. In particular, the class at level $\Pi_{\omega+1}$ plays no role in the proof theoretic analysis and we thank Jim Schmerl for raising the corresponding question.

As we were about to submit this paper for publication, we came across [Sel03], a survey of positive (i.e. computably enumerable) structures. Selivanov describes there (Theorems 4.5.5-4.5.7) a number of results on index sets for computably enumerable Boolean algebras which, along with many others, appear in [Sel90] and [Sel91]. He also states (Remark 1 following Theorem 4.5.7) that analogs of the results mentioned may be proven for the computable Boolean algebras (with the index sets one step lower in every case) by straightforwardly generalizing his proofs for the computably enumerable ones. The analogs of the results mentioned in [Sel03] and appearing in [Sel91] cover our completeness results for the finitely axiomatizable classes of Boolean algebras. Others in [Sel90, Lemma 12], if also generalized to the computable case, would cover the other cases except for the nonarithmetic class at level $\Pi_{\omega+1}$. (The explicit results of [Sel90, Lemma 12] give the strong index set form of the results corresponding to the first four lines of our table in Theorem 2.10. The general ones for finitely axiomatizable classes as in [Sel91, p. 168] provide completeness results but do not explicitly give the strong form of the index set results as in the fifth and sixth lines of our table.) The question corresponding to the nonarithmetic class of computably enumerable Boolean algebras is explicitly left open in [Sel90]. All of our proofs, including the nonarithmetic case, immediately supply the corresponding results for computably enumerable algebras. (The index sets are one level higher in the arithmetic hierarchy than those for computable Boolean algebras in the arithmetic cases and at the same level ($\Pi_{\omega+1}$) in the nonarithmetic one. To see this, note that one can go from computable to computably enumerable at the cost of one level in the hierarchy by simply relativizing to algebras computable in $0'$ as every Δ^0_2 (i.e. computable in $0'$) Boolean algebra is isomorphic to a uniformly constructed Σ^0_1, i.e. computably enumerable, one (essentially by [Fei67] according to [Dow97, Cor. 3.10] or explicitly by [OS89, Th. 2]). Of course, $\Pi_{\omega+1}$ relativized to $0'$ is still $\Pi_{\omega+1}$ and so the result is the same for the computably enumerable algebras as for the computable ones in this case.) Thus we also reprove some of the results of [Sel90] and [Sel91].

Our methods are quite different from Selivanov's. We use no representations as tree algebras but extensively exploit the back and forth relations and notions of k-friendliness of [AK00] to unify and simplify our analysis in the arithmetic cases. The nonarithmetic case also needs some specific constructions using interval algebras. All our results are proven for dense Boolean algebras and so also provide new results on index sets for the isomorphism problem for these algebras as mentioned above.

We provide the basic definitions for Boolean algebras needed to define our classes and state the main index set type theorems in Section 2. We prove the easy, quantifier counting aspect of our complexity results in Section 3. We define dense Boolean algebras in Section 4 and present some useful lemmas about them. Section 5 introduces the back and forth relations of Ash and Knight [AK00] and their notion of k-friendly structures. The remaining sections prove the hardness results for the various classes of Boolean algebras: Σ_n or Π_n for every $n < \omega$; $\Sigma_n \wedge \Pi_n$ for $n \equiv 1, 2 (\mathrm{mod}\, 4)$; and, finally, $\Pi_{\omega+1}$.

We refer the reader to Monk [Mon89] (especially Ch. 7) and Goncharov [Gon97] (especially Ch. 2) for general background about Boolean algebras. For recursion theory, we suggest Soare [Soa87].

2 Definitions and Theorems

We begin with some basic definitions.

Definition 2.1. Let B be a Boolean algebra. We use the usual notation of constants 0 and 1 and operations $\wedge$, $\vee$, and $\neg$. We define the following abbreviations. We let $x \leq y$ abbreviate $x \wedge y = x$; $x - y$ abbreviate $x \wedge \neg y$; and $x \triangle y$ abbreviate $(x - y) \vee (y - x)$. We say that $x \in B$ is an *atom* if $x \neq 0 \,\&\, \forall z < x (z = 0)$; x is *atomic* if for every non-zero element $z < x$, there is an atom $y \leq z$; x is *atomless* if it has no atoms below it.

Let $\mathcal{I}(B)$ denote the ideal of all elements x of B such that $x = y \vee z$, where y is atomic and z is atomless. Let $B^{[0]} = B$, and $B^{[n+1]} = B^{[n]}/\mathcal{I}(B^{[n]})$. We now define the *invariant* of B to be $\operatorname{inv}(B) = \langle p, q, r \rangle$, where $p \leq \omega$, $q \leq \omega$, $r \leq 1$, and

$$p = \begin{cases} \min\{n : B^{[n+1]} = 0\} & \text{if it exists,} \\ \omega & \text{otherwise,} \end{cases}$$

$$q = \begin{cases} \sup\{n : B^{[p]} \text{ has at least } n \text{ atoms}\} & \text{if } p < \omega, \\ 0 & \text{if } p = \omega, \end{cases}$$

$$r = \begin{cases} 1 & \text{if } p < \omega \text{ and } B^{[p]} \text{ contains an atomless element,} \\ 0 & \text{otherwise.} \end{cases}$$

If $\operatorname{inv}(B) = \langle p, q, r \rangle$, we write $\operatorname{inv}_1(B) = p$, $\operatorname{inv}_2(B) = q$, and $\operatorname{inv}_3(B) = r$. We let $\mathtt{In}$ be the set of possible invariants. That is, $\mathtt{In}$ is the set of triplets $\langle p, q, r \rangle \in (\omega+1) \times (\omega+1) \times 2$ such that if $p = \omega$ then $q = r = 0$ and if $p < \omega$ then q and r are not both 0.

The original theorem showing that these are invariants for elementary equivalence is Tarski's:

Theorem 2.2 (Tarski). *[Tar49] If A and B are Boolean algebras, then* $\operatorname{inv}(A) = \operatorname{inv}(B)$ *if and only if A and B are elementarily equivalent.*

To simplify our notation we assign names to the classes (of computable algebras) corresponding to each invariant and an additional *level* value that will roughly correspond to the level of the associated index sets.

Definition 2.3. Given $\langle p, q, r \rangle \in \mathtt{In}$, we let $\mathcal{B}_{\langle p,q,r \rangle}$ be the set of indices of computable Boolean algebras with invariant $\langle p, q, r \rangle$. To each $x \in \mathtt{In}$ we assign a *level*, $l(x) \in \omega + 1$, as follows

$$l(x) = \begin{cases} 4p+1 & \text{if } x \in \{\langle p, q, 0 \rangle : q < \omega\}, \\ 4p+2 & \text{if } x \in \{\langle p, q, 1 \rangle : q < \omega\}, \\ 4p+3 & \text{if } x = \langle p, \omega, 0 \rangle, \\ 4p+4 & \text{if } x = \langle p, \omega, 1 \rangle, \\ \omega & \text{if } x = \langle \omega, 0, 0 \rangle. \end{cases}$$

For a Boolean algebra B, we let $l(B) = l(\operatorname{inv}(B))$. Given $n \in \omega$, we let $\mathcal{B}_n$ be $\mathcal{B}_{\langle p,1,0 \rangle}$ if $n = 4p+1$, $\mathcal{B}_{\langle p,0,1 \rangle}$ if $n = 4p+2$, $\mathcal{B}_{\langle p,\omega,0 \rangle}$ if $n = 4p+3$, and $\mathcal{B}_{\langle p,\omega,1 \rangle}$ if $n = 4p+4$. We let $\mathcal{B}_{\langle p,\bar{q},r \rangle} = \cup\{\mathcal{B}_{\langle p,q',r \rangle} | q' \neq q, \omega\}$ and $\mathcal{B}_{\langle \bar{p},q,r \rangle} = \cup\{\mathcal{B}_{\langle p',q,r \rangle} | p' \neq p\}$.

We can now formulate our main results in terms of characterizing the complexity of these index sets. First, we deal with the standard levels of the arithmetic hierarchy.

Theorem 2.4. *For every n, $\mathcal{B}_n$ is Σ_n-complete if 4 divides n and Π_n-complete if 4 does not divide n.*

Next, we turn to completeness results that fall between some of the Σ_n and Σ_{n+1} levels.

Definition 2.5. A set S is in $\Sigma_n \wedge \Pi_n$ if there are $\phi \in \Sigma_n$ and $\psi \in \Pi_n$ such that $x \in S \leftrightarrow \phi(x) \,\&\, \psi(x)$.

Theorem 2.6. *For every $p < \omega$, and $1 < q < \omega$, $\mathcal{B}_{\langle p,q,0\rangle}$ is $\Pi_n \wedge \Sigma_n$-complete, where $n = 4p+1 = l(\langle p,q,0\rangle)$. For every $p < \omega$, and $0 < q < \omega$, $\mathcal{B}_{\langle p,q,1\rangle}$ is $\Pi_n \wedge \Sigma_n$-complete, where $n = 4p+2 = l(\langle p,q,1\rangle)$.*

Finally, we reach the level beyond the arithmetic ones.

Definition 2.7. A set S is $\Sigma^0_{\omega+1}$ if it is c.e. in $0^{(\omega)}$, and it is $\Pi^0_{\omega+1}$ if its complement is $\Sigma_{\omega+1}$.

(Note that we here follow the notation used in Soare [Soa87, XII.4]. In Ash-Knight [AK00], these classes are called Σ^0_ω and Π^0_ω, respectively.)

It is well known, and not hard to prove, that a set S is $\Pi_{\omega+1}$ if there is a computable f such that $n \in S \iff \forall j(f(n,j) \notin 0^{(j)})$. A set S is $\Sigma_{\omega+1}$ iff $\overline{S}$ is $\Pi_{\omega+1}$, that is, if there is a computable f such that $n \in S \iff \exists j(f(n,j) \in 0^{(j)})$.

Theorem 2.8. *$\mathcal{B}_{\langle\omega,0,0\rangle}$ is $\Pi_{\omega+1}$-complete.*

In fact, in every case our proofs will show more.

Definition 2.9. For any class Γ (of subsets of ω) and its complementary class $\breve{\Gamma} = \{\bar{S} | S \in \Gamma\}$ and for any $A, B \subseteq \omega$, $(\Gamma, \breve{\Gamma}) \leq_{\mathtt{m}} (A, B)$ means that for every $S \in \Gamma$ there is a computable function f such that $\forall x(x \in S \to f(x) \in A)$ and $\forall x(x \notin S \to f(x) \in B)$.

Our constructions will control the outcomes required in the proofs of hardness so as to improve the hardness conclusion. We can summarize our results as follows:

Theorem 2.10. *For each $x \in \mathtt{In}$, $\mathcal{B}_x$ is in Γ_x where Γ_x is specified in the second column of the table below. Moreover, $\mathcal{B}_x$ is complete for Γ_x and, indeed, complete in the sense of a reduction for $(\Gamma_x, \breve{\Gamma}_x)$ as given by the third column:*

x	Γ_x	$(\Gamma_x, \breve{\Gamma}_x) \leq_{\mathtt{m}}$
$\langle p,1,0\rangle$	Π_{4p+1}	$(\mathcal{B}_{\langle p,1,0\rangle}, \mathcal{B}_{\langle p,0,1\rangle})$
$\langle p,0,1\rangle$	Π_{4p+2}	$(\mathcal{B}_{\langle p,0,1\rangle}, \mathcal{B}_{\langle p,\omega,0\rangle})$
$\langle p,\omega,0\rangle$	Π_{4p+3}	$(\mathcal{B}_{\langle p,\omega,0\rangle}, \mathcal{B}_{\langle p,\omega,1\rangle})$
$\langle p,\omega,1\rangle$	Σ_{4p+4}	$(\mathcal{B}_{\langle p,\omega,1\rangle}, \mathcal{B}_{\langle p+1,1,0\rangle})$
$\langle p,q,0\rangle,\ 1<q<\omega$	$\Sigma_{4p+1} \wedge \Pi_{4p+1}$	$(\mathcal{B}_{\langle p,q,0\rangle}, \mathcal{B}_{\langle p,\bar{q},0\rangle})$
$\langle p,q,1\rangle,\ 0<q<\omega$	$\Sigma_{4p+2} \wedge \Pi_{4p+2}$	$(\mathcal{B}_{\langle p,\omega,1\rangle}, \mathcal{B}_{\langle p,\bar{\omega},1\rangle})$
$\langle\omega,0,0\rangle$	$\Pi_{\omega+1}$	$(\mathcal{B}_{\langle\omega,0,0\rangle}, \mathcal{B}_{\langle\bar{\omega},\omega,0\rangle})$

In addition, in every case we will also be able to restrict the sets of (indices for) Boolean algebras in the third column to the (indices for) dense ones (Definition 4.2) in the same classes. (When we say that $\mathcal{B}_x$ is in Γ_x for $\langle 0,1,0\rangle$ and $\langle 0,1,1\rangle$ we mean that there formulas of the form specified by Γ_x such that any Boolean algebra satisfying them is in $\mathcal{B}_x$. The issue here is that to say that a number is an index of a Boolean algebra (or even a structure at all) is already Π_2.)

As Goncharov [Gon97, 2.3.2] proves that any two countable dense Boolean algebras with the same invariant are isomorphic, we can restate some of these results in terms of the terminology introduced by Knight (see [GK02] and [Cal, 3.1, 3.2] and the accompanying discussion) for classifying the complexity of the problem of determining if two structures are isomorphic.

Definition 2.11. Let Γ be a class of subsets of ω (e.g. a complexity class such as Π_n), $A \subseteq B \subseteq \omega$ (e.g. the sets of indices of some subclass and class, respectively, of structures). We say that A is Γ *complete within* B if, for any $S \in \Gamma$, there is a computable function $f : \omega \to B$ such that $\forall n(n \in S \Leftrightarrow f(n) \in A)$.

Corollary 2.12. *The isomorphism problem for dense Boolean algebras, i.e. the set $A = \{\langle i,j\rangle |$ i and j are indices of isomorphic dense computable Boolean algebras$\}$, is $\Pi_{\omega+1}$ complete within $\mathcal{DB}$, the set of indices of dense computable Boolean algebras. Indeed, for $x \in \mathtt{In}$, the finer problem of being isomorphic to the dense Boolean algebra D_x of level $l(x)$, $\{i|i$ is an index of a dense Boolean algebra isomorphic to $D_x\}$, is Γ_x complete within $\mathcal{DB}$ for Γ_x as specified in the table in Theorem 2.10.*

The results of Wasziewicz [Was74] are also derived along the way and slightly improved. (See Section 3 and the final remarks of Section 6 for the proofs.)

Theorem 2.13. *[Was74] If $x \in \mathtt{In}$ and $l(x) = n < \omega$, then the class of Boolean algebras B with $\mathrm{inv}(B) = x$ is axiomatized as follows:*

x	*Axioms*
$\langle p,1,0\rangle$	one $\forall_{4p+1}$
$\langle p,0,1\rangle$	one $\forall_{4p+2}$
$\langle p,\omega,0\rangle$	one $\forall_{4p+3}$ and a computable set of $\exists_{4p+2}$
$\langle p,\omega,1\rangle$	one $\exists_{4p+4}$ and a computable set of $\exists_{4p+2}$
$\langle p,q,0\rangle$, $1 < q < \omega$	one $\exists_{4p+1}$ and one $\forall_{4p+1}$
$\langle p,q,1\rangle$, $0 < q < \omega$	one $\exists_{4p+2}$ and one $\forall_{4p+2}$
$\langle \omega,0,0\rangle$	one $\forall_n$ for each n

By [Tar49], each class corresponds to a complete theory and so, for any $m < \omega$, if $l(B), l(B') \leq m$ and $B \equiv_m B'$ (i.e. they satisfy the same $\exists_m$ sentences) then $B \equiv B'$. On the other hand, if $l(B), l(B') > m$ then $B \equiv_m B'$.

Corollary 2.14. *[Was74] The class of Boolean algebras B with* $\mathrm{inv}(B) = x$ *are not axiomatizable by sentences in $\exists_{n-1}$ and $\forall_{n-1}$ where $n = l(x) < \omega$. The classes with invariants $\langle p, \omega, 0\rangle$ and $\langle p, \omega, 1\rangle$ are not finitely axiomatizable. The class of Boolean algebras with invariant $\langle \omega, 0, 0\rangle$ is not axiomatizable by sentences at any bounded level of the $\exists_n$ hierarchy.*

3 Counting Quantifiers

In this section, we prove that, for each $x \in \mathtt{In}$, $\mathcal{B}_x$ is in Γ_x. In fact, we will also analyze the complexity of the axioms needed to guarantee that a Boolean algebra is in $\mathcal{B}_x$. We will prove that $\mathcal{B}_x$ is Γ_x-hard in the following sections.

Definition 3.1. We define unary predicates $\mathcal{I}_n$, $Atom_n$, $Atomless_n$ and $Atomic_n$ and the associated formulas in the language of Boolean algebras by induction:

$$\begin{aligned}
\mathcal{I}_0(x) &\iff x = 0;\\
Atom_n(x) &\iff \neg\mathcal{I}_n(x) \ \&\ \forall y \leq x(\mathcal{I}_n(y) \vee \mathcal{I}_n(x-y));\\
Atomless_n(x) &\iff \neg\exists y \leq x(Atom_n(y));\\
Atomic_n(x) &\iff \neg\exists y \leq x(\neg\mathcal{I}_n(y) \ \&\ Atomless_n(y));\\
\mathcal{I}_{n+1}(x) &\iff \exists y, z(Atomless_n(y) \ \&\ Atomic_n(z) \ \&\ x = y \vee z).
\end{aligned}$$

Let B be a Boolean algebra. Note that $\mathcal{I}_n(B) = \{x \in B : B \models \mathcal{I}_n(x)\}$ is the ideal of B such that $B^{[n]} = B/\mathcal{I}_n(B)$. Let $[x]_n$ denote the equivalence class of x in $B/\mathcal{I}_n(B)$, that is $[x]_n = \{y \in B : x \triangle y \in \mathcal{I}_n(B)\}$. Then $Atom_n(x)$ holds iff $[x]_n$ is an atom of $B^{[n]}$, $Atomless_n(x)$ holds iff $[x]_n$ is atomless in $B^{[n]}$, and $Atomic_n(x)$ holds iff $[x]_n$ is atomic in $B^{[n]}$. Observe that the formulas $\mathcal{I}_n$, $Atom_n$, $Atomless_n$ and $Atomic_n$ are $\exists_{4n}$, $\forall_{4n+1}$, $\forall_{4n+2}$ and $\forall_{4n+3}$ respectively in the language of Boolean algebras. (Of course, that a computable Boolean algebra B satisfies a $\exists_n$ or $\forall_n$ formula is a Σ_n or Π_n relation, respectively.)

Definition 3.2. For $p, q < \omega$, we let $\mathcal{B}_{\langle p, \leq q, r\rangle} = \bigcup_{i \leq q} \mathcal{B}_{\langle p, i, r\rangle}$. Also let $l(\langle p, \leq q, r\rangle) = l(\langle p, q, r\rangle)$.

Lemma 3.3. *For $p, q < \omega$, x equal to either $\langle p, \leq q, 0\rangle$, $\langle p, \leq q, 1\rangle$ or $\langle p, \omega, 0\rangle$ and $n = l(x)$, $\mathcal{B}_x$ is in Π_n. Moreover, the corresponding classes of Boolean algebras are axiomatized by a $\forall_n$ sentence, a $\forall_n$ sentence, a $\forall_n$ sentence and a computable set of $\exists_{n-1}$ sentences (but not by any finite set of axioms), respectively. If $x = \langle p, \omega, 1\rangle$, then $\mathcal{B}_x$ is in Σ_n. Moreover, the corresponding class of Boolean algebras is axiomatized by a $\exists_n$ sentence and a computable set of $\exists_{n-2}$ sentences but is not finitely axiomatizable. (Of course, $\mathcal{B}_{\langle p,1,0\rangle} = \mathcal{B}_{\langle p,\leq 1,0\rangle}$ and $\mathcal{B}_{\langle p,0,1\rangle} = \mathcal{B}_{\langle p,\leq 0,1\rangle}$.)*

Proof. Consider $x = \langle p, \leq q, 0\rangle$, and let B be a computable Boolean algebra. B is in $\mathcal{B}_x$ if and only if B has first invariant at least p, but no more than q atoms in $B^{[p]}$, and no atomless members in $B^{[p]}$. Now, q atoms can generate at most 2^q non-equivalent members, so to say that there are at most q atoms it suffices to say

$$\neg\exists x_0, ..., x_{2^q}(\forall i, j \leq 2^q \neg\mathcal{I}_p(x_i \triangle x_j)),$$

which is a Π_{4p+1} predicate of B and indeed clearly equivalent to the truth of a $\forall_{4p+1}$ sentence. (Replace the bounded quantification by the corresponding conjunction.) This sentence also implies there are no atomless elements in $B^{[p]}$. For B to be in $\mathcal{B}_x$ we still need to say that B has first invariant at least p, i.e. $\neg\mathcal{I}_p(1)$ which is a $\forall_{4p}$ sentence.

Now consider $x = \langle p, \leq q, 1\rangle$. B is in $\mathcal{B}_x$ if and only if B has first invariant at least p, no more than q atoms in $B^{[p]}$, but more than 2^q elements in $B^{[p]}$. This is expressed by $\neg\mathcal{I}_p(1)$,

$$\neg\exists x_0, ..., x_q(\forall i \leq q(Atom_p(x_i)) \ \&\ \forall i < j \leq q(\neg\mathcal{I}_p(x_j \triangle x_i))),$$

and

$$\exists x_0, ..., x_{2^q}(\forall i, j \leq 2^q \neg\mathcal{I}_p(x_i \triangle x_j)).$$

Note that this is clearly equivalent to the truth in B a $\forall_{4p+2}$ sentence.

Consider now $x = \langle p, \omega, 0\rangle$. B is in $\mathcal{B}_x$ if and only if $[1]_p$ is atomic in $B^{[p]}$ and there are infinitely many atoms:

$$Atomic_p(1) \ \&$$
$$\forall m \exists x_1, ..., x_m(\forall i \leq m(Atom_p(x_i)) \ \&\ \forall i < j \leq m(\neg\mathcal{I}_p(x_j \triangle x_i))).$$

Observe that this is a Π_{4p+3} predicate on B which is equivalent to the truth of a $\forall_{4p+3}$ sentence and a computable set of Σ_{4p+2} sentences (one for each m). If this class were finitely axiomatizable then, by the completeness of the associated theory, some finite subset of this list of axioms would suffice to axiomatize the class. This, however, is obviously impossible since any finite subset has a an algebra with invariant $\langle p, q, 0\rangle$ for some q.

Finally let $x = \langle p, \omega, 1\rangle$. Then $B \in \mathcal{B}_x$ if and only if, in $B^{[p]}$, 1 is the sum of an atomless element and an atomic element, and there are infinitely many atoms. This is expressed by

$$\exists yz(1 = y \vee z \ \&\ Atomic_p(y) \ \&\ Atomless_p(z) \ \&\ \neg\mathcal{I}_p(z)) \ \&$$
$$\forall m \exists x_1, ..., x_m(\forall i \leq m(Atom_p(x_i)) \ \&\ \forall i < j \leq m(\neg\mathcal{I}_p(x_j \triangle x_i))))$$

which is a Σ_{4p+4} predicate on B which is equivalent to the truth of a $\exists_{4p+4}$ sentence and a computable set of Σ_{4p+2} sentences. The argument that this class is not finitely axiomatizable is the same as for $\langle p, \omega, 0\rangle$. □

It follows that, for all $n \in \omega$, $\mathcal{B}_n$ is in Σ^0_n if 4 divides n and it is in Π^0_n otherwise.

Lemma 3.4. *For $x = \langle p, q, r\rangle$ with $p < \omega$ and either $r = 0$ & $1 < q < \omega$, or $r = 1$ & $0 < q < \omega$, $\mathcal{B}_x$ is in $\Sigma_n \wedge \Pi_n$, where $n = l(x)$. Moreover, the corresponding classes of Boolean algebras are axiomatized by a sentence in $\exists_n$ and one in $\forall_n$.*

Proof. For $r = 0$, observe that $\mathcal{B}_{\langle p,q,0\rangle}$ consists of the Boolean algebras B in $\mathcal{B}_{\langle p,\leq q,0\rangle}$ which are not in $\mathcal{B}_{\langle p,\leq q-1,0\rangle}$. By Lemma 3.3, B in $\mathcal{B}_{\langle p,\leq q,0\rangle}$ is guaranteed by a $\forall_{4p+1}$ sentence and B not in $\mathcal{B}_{\langle p,\leq q-1,0\rangle}$ is expressible by a $\exists_{4p+1}$ sentence. Similarly, for $r = 1$, observe that $\mathcal{B}_{\langle p,q,1\rangle}$ consists of the Boolean algebras B in $\mathcal{B}_{\langle p,\leq q,1\rangle}$ (guaranteed by a $\forall_{4p+2}$ sentence) which are not in $\mathcal{B}_{\langle p,\leq q-1,1\rangle}$ (expressible by a $\exists_{4p+2}$ sentence). $\square$

Lemma 3.5. *$\mathcal{B}_{\langle \omega,0,0\rangle}$ is in $\Pi_{\omega+1}$. The corresponding class of Boolean algebras is axiomatized by a computable set of $\forall_n$ sentences with one for each n.*

Proof. A computable Boolean algebra B is in $\mathcal{B}_{\langle \omega,0,0\rangle}$ if for all p, $B^{[p]}$ is non-empty. In other words if

$$\forall p < \omega(\neg \mathcal{I}_p(1)).$$

Since $0^{(\omega)}$ knows whether $\mathcal{I}_p(1)$ for each p uniformly in p, $\mathcal{B}_{\langle \omega,0,0\rangle}$ is co-c.e. in $0^{(\omega)}$, or equivalently $\Pi^0_{\omega+1}$. $\square$

Note that these Lemmas establish the axiomatizabilty of the classes of Boolean algebras by sentences of the complexity required in Theorem 2.13. The second part of this theorem follows from Theorem 6.1(2).

Now that we have that, for each x, $\mathcal{B}_x$ is in Γ_x, we turn to proving that $\mathcal{B}_x$ is Γ_x-hard. We first need to introduce the concepts of dense Boolean algebras and back-and-forth relations.

4 Dense Boolean Algebras

We start by defining the Tarski invariants on elements of a Boolean Algebra.

Definition 4.1. Let B be a Boolean algebra and $a \in B$. We let $B \restriction a$ be the Boolean algebra whose domain is $\{b \in B : b \leq a\}$, $1_{B\restriction a} = a$, $0_{B\restriction a} = 0$, $\vee_{B\restriction a}$ and $\wedge_{B\restriction a}$ are the restrictions of the corresponding operations in B, and the complement of b in $B \restriction a$ is $a - b$. We let $\mathrm{inv}^B(a) = \mathrm{inv}(B \restriction a)$. When no confusion should arise, we may might write $\mathrm{inv}(a)$ instead of $\mathrm{inv}^B(a)$.

Definition 4.2. A Boolean algebra B is *dense* if for every $b \in B$,

1. $\forall k < \mathrm{inv}_1(b)(\exists a \leq b(\mathrm{inv}(a) = \langle k, \omega, 0\rangle))$ and

2. if $\mathrm{inv}_1(b) = \omega$ or $\mathrm{inv}_2(b) = \omega$, then there is an $a \leq b$ such that $\mathrm{inv}_1(a) = \mathrm{inv}_1(b) = \mathrm{inv}_1(b - a)$ and $\mathrm{inv}_2(a) = \mathrm{inv}_2(b) = \mathrm{inv}_2(b - a)$.

Goncharov [Gon97, 2.3.2] proves that any two countable dense Boolean algebras with the same invariant are isomorphic. Moreover, he proves that every countable Boolean algebra B has an elementary extension B^* which is dense. This then shows that any two countable Boolean algebras with the same invariant are elementarily equivalent and so establishes Tarski's theorem.

We let D_x denote the dense Boolean algebra with invariant x. All of them are computably (even decidably) presentable by Morozov [Mor82].

Definition 4.3. We define an addition operation on the set `In` of invariants as follows:

$$\sum_{i \leq m} \langle p_i, q_i, r_i \rangle = \langle p_0, q_0, r_0 \rangle + ... + \langle p_m, q_m, r_m \rangle = \langle p, q, r \rangle,$$

where

$$\begin{aligned} p &= \max\{p_i : i \leq m\}, \\ q &= \sum \{q_i : i \leq m \ \& \ p_i = p\}, \\ r &= \max\{r_i : i \leq m \ \& \ p_i = p\}. \end{aligned}$$

We then say that $\langle p_0, q_0, r_0 \rangle, ..., \langle p_m, q_m, r_m \rangle$ is a *partition* of $\langle p, q, r \rangle$. (Here, we are using the convention that $\omega + q = q + \omega = \omega$.)

Definition 4.4. We say that $a_0, ..., a_m \in B$ form a *partition* of $a \in B$ if $\bigvee_{i \leq m} a_i = a$ and for all $i \leq m$,

$$a_i \wedge \bigvee_{j \leq m, j \neq i} a_j = 0.$$

Observe that if $a_0, ..., a_m$ form a partition of 1, then $B \cong B \upharpoonright a_0 \times ... \times B \upharpoonright a_m$. We then say that $B \upharpoonright a_0, ..., B \upharpoonright a_m$ form a *partition* of B.

Now consider an arbitrary tuple $\bar{b} = (b_0, \ldots, b_n)$ of members of B. This generates a partition of B as follows. Let $A_0 = \{b_0, 1 - b_0\}$. Let $A_i = \{a - b_i, b_i \cap a : a \in A_i\}$ for $0 \leq i \leq n$. Then $\{B \upharpoonright a : a \in A_n, a \neq 0\}$ is *the partition of B generated by $\bar{b}$*.

Lemma 4.5. *If $a_0, ..., a_{m-1}$ form a partition of a, then $\mathrm{inv}(a_0), ..., \mathrm{inv}(a_{m-1})$ form a partition of $\mathrm{inv}(a)$.*

Proof. See [Gon97, Lemma 2.2.4] for a proof of the lemma when $m = 2$. The general case follows easily by induction. □

When we are dealing with dense Boolean algebras, the converse of the previous lemma also holds.

Lemma 4.6. *A Boolean algebra B is dense if and only if, for every $b \in B$ and every partition $x_0, ..., x_m$ of $\mathrm{inv}(b)$, there exists a partition $a_0, ..., a_m$ of b such that, for each $i \leq m$, $\mathrm{inv}(a_i) = x_i$.*

Proof. The denseness conditions are just special cases of the partition property.

To see that, if B is dense, then B has the partition property, make use of the denseness conditions along with Lemma 4.7 below. □

Lemma 4.7. *[Gon97, Lemma 2.2.6] Let B be a Boolean algebra, $b \in B$, and $x = \langle p, q, r\rangle \in \mathtt{In}$.*

1. *If $p < \mathrm{inv}_1(b)$, $q < \omega$, and $r \leq 1$, then there is an $a \leq b$ such that $\mathrm{inv}(b-a) = \mathrm{inv}(b)$.*
2. *If $p = \mathrm{inv}_1(b)$, $q \leq \mathrm{inv}_2(b)$, and $r \leq \mathrm{inv}_3(b)$, then there is an $a \leq b$ such that $\mathrm{inv}(a) = x$. Moreover, if $q < \mathrm{inv}_2(b)$ or $r = 1$, then we can also require that $\mathrm{inv}_1(b-a) = \mathrm{inv}_1(b)$, $\mathrm{inv}_2(b-a) = q$, and $\mathrm{inv}_3(b-a) = \mathrm{inv}_3(b)$, where we take $\omega - \omega$ to be 0.*

Corollary 4.8. *The product of dense Boolean algebras is dense.*

Proof. Consider $x, y \in \mathtt{In}$. We want to prove that $D_x \times D_y \cong D_{x+y}$. The element 1 of D_{x+y} has invariant $x + y$. So, by the lemma above, there exists a partition a, b of 1 such that $\mathrm{inv}(a) = x$ and $\mathrm{inv}(b) = y$. Since a, b is a partition of 1, $D_{x+y} \cong D_{x+y} \upharpoonright a \times D_{x+y} \upharpoonright b$. Since D_{x+y} is dense, so are $D_{x+y} \upharpoonright a$ and $D_{x+y} \upharpoonright b$. Therefore

$$D_x \times D_y \cong D_{x+y} \upharpoonright a \times D_{x+y} \upharpoonright b \cong D_{x+y}.$$

□

5 Back-and-Forth relations

In this section we define back-and-forth relations between structures and state the properties about them that we need. We refer the reader to [AK00] for more information on these relations.

Definition 5.1. Let K be a class of structures for a fixed language. For each $n < \omega$, we define the *standard back-and-forth relation* $\leq_n$ on pairs $(A, \bar{a})$, where $A \in K$ and $\bar{a}$ is a tuple in A. First suppose that $\bar{a}$ in A and $\bar{b}$ in B are tuples of the same length. Then,

1. $(A, \bar{a}) \leq_1 (B, \bar{b})$ if and only if all Σ_1 formulas true of $\bar{b}$ in B are true of $\bar{a}$ in A.
2. For $n > 1$, $(A, \bar{a}) \leq_n (B, \bar{b})$ if and only if for each $\bar{d}$ in B, and each $1 \leq k < n$, there exists a $\bar{c}$ in A with $|\bar{c}| = |\bar{d}|$ such that $(B, \bar{b}, \bar{d}) \leq_k (A, \bar{a}, \bar{c})$.

Now, we extend the definition of $\leq_n$ to tuples of different lengths. For $\bar{a}$ in A and $\bar{b}$ in B, let $(A, \bar{a}) \leq_n (B, \bar{b})$ if and only if $|\bar{a}| \leq |\bar{b}|$ and for the initial segment $\bar{b}'$ of $\bar{b}$ of length $|\bar{b}|$, we have $(A, \bar{a}) \leq_n (B, \bar{b}')$. We may write $A \leq_n B$ instead of $(A, \emptyset) \leq_n (B, \emptyset)$.

One observation that might give the reader some intuition about the back-and-forth relation is that $(A, \bar{a}) \leq_n (B, \bar{b})$ if and only if all the Π_n infinitary formulas true of $\bar{a}$ in A are true of $\bar{b}$ in B. (See [AK00, Proposition 15.1]; see [AK00, Chapter 6] for information on infinitary formulas.) Also observe that if $k < n$ and $(A, \bar{a}) \leq_n (B, \bar{b})$ then $(A, \bar{a}) \equiv_k (B, \bar{b})$, where $(A, \bar{a}) \equiv_k (B, \bar{b})$ if and only if $(A, \bar{a}) \leq_k (B, \bar{b})$ and $(A, \bar{a}) \geq_k (B, \bar{b})$.

The only structures we will be dealing with are Boolean algebras. The following lemma gives us a way of computing the back-and-forth relations on Boolean algebras without having to refer to the definition given above.

Lemma 5.2. *[AK00, 15.13] Suppose that A and B are Boolean algebras. Then $A \leq_1 B$ if and only if A is infinite or can be split into at least as many disjoint parts as B (i.e., if A is generated by p atoms, then B is generated by k atoms, for some $k \leq p$). For $n > 1$, $A \leq_n B$ if and only if, for any l with $1 \leq l < n$ and any finite partition of B into $B_1, ..., B_k$, there is a corresponding partition of A, $A_1, ..., A_k$, such that $B_i \leq_l A_i$.*

We will be interested in analyzing the back-and-forth relation among the dense Boolean algebras. Since each isomorphism type of a dense Boolean algebra is determined by its invariant, we translate the back-and-forth relation to one on the set of invariants:

Definition 5.3. Given $x, x' \in \mathtt{In}$ and $n < \omega$ we let $x \leq_n x'$ if $D_x \leq_n D_{x'}$.

The back-and-forth relations on the set of invariants can be computed using the following lemma.

Lemma 5.4. *Consider $x, x' \in \mathtt{In}$. Then $x \leq_1 x'$ if and only if either $l(x) > 1$, or $x = \langle 0, q, 0 \rangle$, $x' = \langle 0, q', 0 \rangle$ and $q \geq q'$. For $n > 1$, $x \leq_n x'$ if and only if, for any partition $y'_1, ..., y'_k$ of x', there is a corresponding partition $y_1, ..., y_k$ of x such that $y'_i \leq_{n-1} y_i$.*

Proof. Immediate from Lemma 5.2, noting that D_x is infinite if and only if $l(x) > 1$, and that if $l(x) = 0$ then $x = \langle 0, q, 0 \rangle$ for some $1 \leq q < \omega$, so for D_x to be such that it can be split into at least as many disjoint parts as D'_x we must have $x' = \langle 0, q', 0 \rangle$ for some $q' \leq q$. □

The above considerations reduce computing the back-and-forth relations on $\mathtt{In}$ to a combinatorial task, which we will do in Theorem 6.1. To complete the proofs of our hardness results we also make use of the concept of k-friendliness., which we now introduce. Again, we refer the reader to [AK00, Chapter 15] for more information.

Definition 5.5. A pair of structures $\{A_0, A_1\}$ is *k-friendly* if the structures A_i are computable, and for $n < k$, the standard back-and-forth relations $\leq_n$ on $(A_i, \overline{a})$, for $\overline{a} \in A_i$, are c.e., uniformly in n.

Theorem 5.6. *[AK00, 18.6] Let A_0 and A_1 be structures such that $A_1 \leq_k A_0$ and $\{A_0, A_1\}$ is k-friendly. Then for any Π^0_k set S, there is a uniformly computable sequence of structures $\{C_n\}_{n\in\omega}$ such that*

$$C_n \cong \begin{cases} A_0 & \text{if } n \in S \\ A_1 & \text{otherwise} \end{cases}$$

This theorem can be restated as follows.

Corollary 5.7. *Let A_0 and A_1 be n-friendly structures and $\mathcal{B}_{A_0}$ and $\mathcal{B}_{A_1}$ be subsets of ω such that every index of a computable copy of A_0 is in $\mathcal{B}_{A_0}$ and every index of a computable copy of A_1 is in $\mathcal{B}_{A_1}$. Then*

$$A_1 \leq_n A_0 \implies (\Sigma_n, \Pi_n) \leq_{\mathtt{m}} (\mathcal{B}_{A_1}, \mathcal{B}_{A_0}).$$

6 The Σ_n and the Π_n cases (Theorem 2.4)

We start by giving a complete analysis of the back-and-forth relations on the set of invariants, or equivalently, on the dense Boolean algebras. The proof of the following theorem is purely combinatorial and all it uses about the back-and-forth relations on `In` is Lemma 5.4.

Theorem 6.1. *Let $x = \langle p, q, r\rangle$ and $x' = \langle p', q', r'\rangle$ be invariants with $l(x) = l$ and $l(x') = l'$, and let $n \geq 1$. The following conditions determine whether $x \leq_n x'$.*

Case 1: If $l < n \vee l' < n$, then $x \leq_n x'$ iff $x = x'$.

Case 2: If $l > n$ & $l' > n$, then $x \leq_n x'$ always.

Case 3: If $l = n$ & $l' = n$, then $x \leq_n x'$ iff $q \geq q'$.

Case 4: If $l > n$ & $l' = n$, then $x \leq_n x'$ iff $n \neq 4p' + 4$.

Case 5: If $l = n$ & $l' > n$, then $x \leq_n x'$ iff $n = 4p + 4$.

Proof. The proof is by induction on n. The case $n = 1$ follows trivially from Lemma 5.4 (recall l, $l' \geq 1$ by definition of level). Consider $n > 1$ and assume the theorem holds for all $m < n$.

Case 1: Suppose that either $l < n$ or $l' < n$. Clearly if $x = x'$ then $x \leq_n x'$. Now suppose $x \leq_n x'$. Then $x \equiv_{n-1} x'$. By induction hypothesis this can only happen either if $x = x'$ or if $l > n - 1$ and $l' > n - 1$. Therefore, since either $l < n$ or $l' < n$, we must have $x = x'$.

Case 2: Suppose $l > n$ and $l' > n$. We have to show that given any finite partition $y'_1, ..., y'_k$ of x', there is a corresponding partition $y_1, ..., y_k$ of x, such that $y'_i \leq_{n-1} y_i$. Assume $y'_1, ..., y'_k$ are ordered such that for some $j \leq k$, $y'_1, ..., y'_j$ have level $\geq n$ and

$y'_{j+1}, ..., y'_k$ have level $< n$. It is not hard to observe that, whatever n is, since $l(x) > n$, it is always the case that there exists $y_1, ..., y_j$ of level $\geq n$ such that $\sum_{i \leq j} y_i = x$. Note that by induction hypothesis, since $l(y_i) > n-1$ and $l(y'_i) > n-1$, $y'_i \leq_{n-1} y_i$ for all $i \leq j$. For $i > j$ let $y_i = y'_i$. Another easy general observation is that for every $y, z \in \mathtt{In}$ with $l(y) \leq l(z) - 2$, $z + y = z$. Then $\sum_{i \leq k} y_i = x + \sum_{i=j+1}^{k} y'_i = x$. So $y_1, ..., y_k$ is the desired partition of x.

Case 3: Assume $l = l' = n$. Note that $p = p'$ and $r = r'$. Also if $n = 4p+3$ or $n = 4p+4$, then $q = q' = \omega$ and therefore $x = x'$. So suppose n is either $4p+1$ or $4p+2$.

First suppose $q \geq q'$; we want to show that $x \leq_n x'$. Consider a partition $y'_1, ..., y'_k$ of x' with $y'_i = \langle p_i, q_i, r_i \rangle$. Note that, necessarily, for some $i \leq k$, $p_i = p$ and $r_i = r$; without loss of generality suppose that $p_1 = p$ and $r_1 = r$. Let $y_1 = \langle p, q_1 + (q - q'), r \rangle = \langle p, q - q', r \rangle + y'_1$, and for $i > 1$ let $y_i = y'_i$. Observe that

$$\sum_{i \leq k} y_i = \langle p, q - q', r \rangle + \sum_{i \leq k} y'_i = \langle p, q - q', r \rangle + x' = x.$$

Also, since $l(y_1) = l(y'_1) = n > n - 1$, by Case 2 of the inductive hypothesis $y'_1 \leq_{n-1} y_1$. So $y_1, ..., y_k$ is the desired partition.

Now suppose $q < q'$; we want to show that $x \not\leq_n x'$.

If $n = 4p+1$ or equivalently $r = 0$, consider the partition $y'_i = \langle p, 1, 0 \rangle$ for $i \leq q'$ of $x' = \langle p, q', 0 \rangle$. It is not hard to see that any partition, $y_1, ..., y_{q'}$, of x cannot have more than q elements at level n. So, for some $i \leq q'$, $l(y_i) < n = l(y'_i)$. Then, by either case 1 or case 4 of the induction hypothesis, $y'_i \not\leq_{n-1} y_i$.

If $n = 4p+2$, consider the partition $y'_i = \langle p, 1, 0 \rangle$ for $i \leq q'$ and $y_{q'+1} = \langle p, 0, 1 \rangle$ of $x' = \langle p, q', 1 \rangle$. Suppose toward a contradiction that there is a partition $y_1, ..., y_{q'+1}$ of x such that for all $i \leq q'+1$, $y'_i \leq_{n-1} y_i$. By induction hypothesis, $\langle p, 1, 0 \rangle \leq_{n-1} y_i$ implies that $y_i = \langle p, 1, 0 \rangle$. So, for all $i \leq q'$, $y_i = \langle p, 1, 0 \rangle$. But then, since $q < q'$, it cannot be the case that $\sum_{i < q'+1} y_i = \langle p, q, 1 \rangle = x$.

Case 4: Suppose now that $l > n$ and $l' = n$.

First suppose that $n \neq 4p' + 4$. Then, any partition $y'_1, ..., y'_k$ of x' must have some member at level n. Assume $l(y'_1) = n$. Note that there is a y_1 of level $l > n$ such that $y_1 + \sum_{1 < i \leq k} y'_i = x$. Also observe that since $l(y_1) > n - 1$ and $l(y'_1) > n-1$, $y'_1 \leq_{n-1} y_i$. Then, if for $1 < i \leq k$ we let $y_i = y'_i$, we obtain the desired partition of x.

Now suppose that $n = 4p'+4$ and hence $x' = \langle p', \omega, 1 \rangle$; we want to show that $x \not\leq_n x'$. Consider the following partition of x': let $y'_1 = \langle p', \omega, 0 \rangle$ and $y'_2 = \langle p', 0, 1 \rangle$. Suppose toward a contradiction that y_1, y_2 is a partition of x such that $y'_i \leq_{n-1} y_i$ for $i \leq 2$. Then by induction hypothesis we must have $y_1 = \langle p', \omega, 0 \rangle$ and $y_2 = \langle p', 0, 1 \rangle$. But then $l(y_1 + y_2) = n < l(x)$, contradicting $y_1 + y_2 = x$.

Case 5: The last case is $l = n$ and $l' > n$.

Suppose first that $n = 4p+4$, so $x = \langle p, \omega, 1 \rangle$. Let $y'_1, ..., y'_k$ be a partition of x'. Let $y_i = y'_i$ if $l(y'_i) < n$ and $y_i = \langle p, \omega, 1 \rangle$ otherwise. Note that $\sum_{i \leq k} y_k = x$, and that for i with $l(y'_i) > n$, since $l(y_i) > n-1$, $y'_i \leq_{n-1} y_i$. So, $y_1, ..., y_k$ is the desired partition.

Note that if $l(x^*) \geq n+1$ and $l(x') \geq n+1$, then $x' \leq_n x^*$ by Case 2. So to show $x \not\leq_n x'$ it suffices to show $x \not\leq_n x^*$ for any x^* of level $n+1$.

Suppose $n = 4p+1$ or $n = 4p+2$, so $x = \langle p, q, r \rangle$ with $q < \omega$ and x' has level $n+1$; we want to show that $x \not\leq_n x'$. By case 3, $\langle p, q, r \rangle \not\leq_n \langle p, q+1, r \rangle$. By case 4, $x' \leq_n \langle p, q+1, r \rangle$. So we must have $\langle p, q, r \rangle \not\leq_n x'$.

Last, suppose $n = 4p+3$. Then $x = \langle p, \omega, 0 \rangle$ and let $x' = \langle p, \omega, 1 \rangle$; we want to show that $x \not\leq_n x'$. Consider the partition $y'_1 = \langle p, \omega, 0 \rangle$ and $y'_2 = \langle p, 0, 1 \rangle$ of x'. Suppose toward a contradiction that there is a partition y_1, y_2 of x such that $y'_i \leq_{n-1} y_i$. Now, $\langle p, 0, 1 \rangle \leq_{n-1} y_2$ implies, by induction hypothesis, that $y_2 = \langle p, 0, 1 \rangle$. But then y_2 cannot be part of a partition of $\langle p, \omega, 0 \rangle$. Contradiction. □

Corollary 6.2. *Let A and B be computable presented Boolean algebras such that the functions* inv^A *and* inv^B *are computable. Then $\{A, B\}$ is n-friendly for every $n < \omega$.*

Proof. Let A_0 and A_1 be in $\{A, B\}$, $\bar{a}_0$ be a tuple in A_0, $\bar{a}_1$ be a tuple in A_1, and $n < \omega$. We will show how to decide whether $(A_0, \bar{a}_0) \leq_n (A_1, \bar{a}_1)$ computably. If $|a_0| > |a_1|$, then $(A_0, \bar{a}_0) \not\leq_n (A_1, \bar{a}_1)$. So suppose $|a_0| \leq |a_1|$. By truncating $\bar{a}_1$ if necessary, we can assume without loss of generality that they have the same length. Each tuple $\bar{a}_i$ generates a partition of A_i. We can then effectively compute the invariants of the partition, $y_{i,0}, \ldots, y_{i,k}$. By [AK00, Lemma 15.12], $(A_0, \bar{a}_0) \leq_n (A_1, \bar{a}_1)$ iff $y_{0,j} \leq_n y_{1,j}$ for $0 \leq j \leq k$. Then we can use Theorem 6.1 to decide this. □

In [Mor82], Morozov uniformly constructs dense Boolean algebras of each invariant which are decidable. While decidability does not quite give the computability of the inv functions on these algebras, it is not hard to see that they are in fact computable. (The argument is a tedious one by induction with several cases. Enough of it to give the ideas is carried out in Shore [Sho04, Proposition 6.5] when that proof is specialized to these algebras.) Therefore, by Corollary 6.2, these Boolean algebras are n-friendly for each n. Then, from Corollary 5.7 we obtain the following:

Corollary 6.3. *For every $p < \omega$,*

$$\begin{aligned}
(\Sigma_{4p+1}, \Pi_{4p+1}) &\leq_{\mathtt{m}} (\mathcal{DB}_{\langle p,0,1\rangle}, \mathcal{DB}_{\langle p,1,0\rangle}) \\
(\Sigma_{4p+2}, \Pi_{4p+2}) &\leq_{\mathtt{m}} (\mathcal{DB}_{\langle p,\omega,0\rangle}, \mathcal{DB}_{\langle p,0,1\rangle}) \\
(\Sigma_{4p+3}, \Pi_{4p+3}) &\leq_{\mathtt{m}} (\mathcal{DB}_{\langle p,\omega,1\rangle}, \mathcal{DB}_{\langle p,\omega,0\rangle}) \\
(\Sigma_{4p+4}, \Pi_{4p+4}) &\leq_{\mathtt{m}} (\mathcal{DB}_{\langle p,\omega,1\rangle}, \mathcal{DB}_{\langle p+1,1,0\rangle})
\end{aligned}$$

Theorem 2.4 and the corresponding lines of Theorem 2.10 now follow from this corollary and Lemma 3.3.

We can also now derive the second part of Theorem 2.13 and Corollary 2.14. As remarked above, $D_x \equiv_n D_{x'}$ implies that the same $\forall_n$ formulas are true in D_x and $D_{x'}$ ([AK00, Proposition 15.1]). Case (2) of Theorem 6.1 then implies that, for every $m < \omega$, if $l(B), l(B') > m$ then $B \equiv_m B'$ as required for the second part of Theorem 2.13. As for Corollary 2.14, if D_x were axiomatized by sentences in $\exists_m$ and $\forall_m$ for $m < l(x)$ then, by the second part of Theorem 2.13, $D_{x'} \equiv_m D_x$ for any x' with $l(x') > n$ and so we would have $D_{x'} \equiv D_x$ for a contradiction.

7 The $\Sigma_n \wedge \Pi_n$ cases (Theorem 2.6)

Now we prove that, for $x = \langle p, q, r\rangle$ with $0 < q + r < \omega$, $\mathcal{DB}_x$ is $\Sigma^0_{l(x)} \wedge \Pi^0_{l(x)}$-hard. We first prove it for $x \neq \langle p, 2, 0\rangle$. Later, using a more complicated proof, we prove it for $x = \langle p, 2, 0\rangle$.

Lemma 7.1. *For $2 < q < \omega$, $\mathcal{DB}_{\langle p,q,0\rangle}$ is $(\Sigma_{4p+1} \wedge \Pi_{4p+1})$-hard. For $0 < q < \omega$, $\mathcal{DB}_{\langle p,q,1\rangle}$ is $(\Sigma_{4p+2} \wedge \Pi_{4p+2})$-hard. Moreover, the reductions proving hardness produce, in the case that n is not in the $\Sigma_{4p+1} \wedge \Pi_{4p+1}$ or $\Sigma_{4p+2} \wedge \Pi_{4p+2}$ set, an index in $\mathcal{DB}_{\langle p,\bar{q},0\rangle}$ or $\mathcal{DB}_{\langle p,\bar{q},1\rangle}$, respectively, as required in Theorem 2.10.*

Proof. Let $2 < q < \omega$. Consider two Σ_{4p+1} formulas $\phi(n)$ and $\psi(n)$. We want to construct a computable function f such that $\forall n(\phi(n) \,\&\, \neg\psi(n)) \iff f(n) \in \mathcal{DB}_{\langle p,q,0\rangle})$. Since $q > 2$, by Theorem 6.1, $\langle p, q, 0\rangle \leq_{4p+1} \langle p, 1, 0\rangle$ and $\langle p, q-1, 0\rangle \leq_{4p+1} \langle p, 1, 0\rangle$. So by Corollary 5.7 there are computable g and h such that $\phi(n) \Rightarrow g(n) \in \mathcal{DB}_{\langle p,q-1,0\rangle}$, $\neg\phi(n) \Rightarrow g(n) \in \mathcal{DB}_{\langle p,1,0\rangle}$, $\psi(n) \Rightarrow h(n) \in \mathcal{DB}_{\langle p,q,0\rangle}$, and $\neg\psi(n) \Rightarrow h(n) \in \mathcal{DB}_{\langle p,1,0\rangle}$. Associating Boolean algebras with their indices, let $f(n) = g(n) \times h(n)$ and note that, by Corollary 4.8, $f(n)$ is an index for a dense Boolean algebra. Then if $\phi(n) \,\&\, \neg\psi(n)$, we have $\mathrm{inv}(f(n)) = \mathrm{inv}(g(n)) + \mathrm{inv}(h(n)) = \langle p, q-1, 0\rangle + \langle p, 1, 0\rangle = \langle p, q, 0\rangle$. If $\phi(n) \,\&\, \psi(n)$ then $\mathrm{inv}(f(n)) = \langle p, q-1, 0\rangle + \langle p, q, 0\rangle = \langle p, 2q-1, 0\rangle$, if $\neg\phi(n) \,\&\, \psi(n)$ then $\mathrm{inv}(f(n)) = \langle p, 1, 0\rangle + \langle p, q, 0\rangle = \langle p, q+1, 0\rangle$, and if $\neg\phi(n) \,\&\, \neg\psi(n)$ then $\mathrm{inv}(f(n)) = \langle p, 1, 0\rangle + \langle p, 1, 0\rangle = \langle p, 2, 0\rangle$. Thus f has the required properties.

Now suppose $0 < q < \omega$ and that $\phi(n)$ and $\psi(n)$ are Σ_{4p+2}. We now wish to construct a computable function f such that $\forall n(\phi(n) \,\&\, \neg\psi(n) \iff f(n) \in \mathcal{DB}_{\langle p,q,1\rangle})$. Again by Theorem 6.1 and Corollary 5.7 there are computable g and h such that $\phi(n) \Rightarrow g(n) \in \mathcal{DB}_{\langle p,q,1\rangle}$, $\neg\phi(n) \Rightarrow g(n) \in \mathcal{DB}_{\langle p,0,1\rangle}$, $\psi(n) \Rightarrow h(n) \in \mathcal{DB}_{\langle p,q+1,1\rangle}$, and $\neg\psi(n) \Rightarrow h(n) \in \mathcal{DB}_{\langle p,0,1\rangle}$. Now let $f(n) = g(n) \times h(n)$ and note that f has the required properties. Indeed, if $\phi(n) \,\&\, \neg\psi(n)$, we have $\mathrm{inv}(f(n)) = \mathrm{inv}(g(n)) + \mathrm{inv}(h(n)) = \langle p, q, 1\rangle + \langle p, 0, 1\rangle = \langle p, q, 1\rangle$. If $\phi(n) \,\&\, \psi(n)$ then $\mathrm{inv}(f(n)) = \langle p, q, 1\rangle + \langle p, q+1, 1\rangle = \langle p, 2q+1, 1\rangle$, if $\neg\phi(n) \,\&\, \psi(n)$ then $\mathrm{inv}(f(n)) = \langle p, 0, 1\rangle + \langle p, q+1, 1\rangle = \langle p, q+1, 1\rangle$, and if $\neg\phi(n) \,\&\, \neg\psi(n)$ then $\mathrm{inv}(f(n)) = \langle p, 0, 1\rangle + \langle p, 0, 1\rangle = \langle p, 0, 1\rangle$. $\square$

To finish the proof of Theorem 2.6 and the corresponding parts of Theorem 2.10, we still need to prove that, for every p, $\mathcal{DB}_{\langle p,2,0\rangle}$ is $(\Sigma_{4p+1} \wedge \Pi_{4p+1})$-hard via reductions with an appropriate outcome in the case that n is not in the given $\Sigma_{4p+1} \wedge \Pi_{4p+1}$ set. We need the following definition.

Definition 7.2. Let $\{B_i\}_{i\in\omega}$ be a sequence of Boolean algebras. We define $\prod_{i\in\omega}^{\omega} B_i$, the *weak product* of $\{B_i\}_{i\in\omega}$, to be the Boolean algebra with domain the set of infinite strings $\bar{b} = (b_0, b_1, ...)$ such that $\forall i(b_i \in B_i)$ and for some i_0, either $\forall j \geq i_0(b_j = 0)$ or $\forall j \geq i_0(b_j = 1)$. The operations and constants of $\prod_{i\in\omega}^{\omega} B_i$ are defined coordinatewise in the obvious way, with $\bar{0} = (0_{B_0}, 0_{B_1}, \ldots)$, $\bar{1} = (1_{B_0}, 1_{B_1}, \ldots)$, and so forth.

Observation 7.3. We make two observations. One is that

$$\prod_{i\in\omega}^{\omega} B_i \cong B_0 \times \prod_{i\in\omega, i>0}^{\omega} B_i \cong B_0 \times B_1 \times ... \times B_n \times \prod_{i\in\omega, i>n}^{\omega} B_i$$

The second one is that

$$\prod_{i\in\omega}^{\omega} D_{\langle p,\omega,1\rangle} \cong D_{\langle p+1,1,0\rangle}.$$

Proof. The first observation is clear. To see the second, we will show that $\mathrm{inv}(\prod_{i\in\omega}^{\omega} D_{\langle p,\omega,1\rangle}) = \langle p+1,1,0\rangle$, and that $\prod_{i\in\omega}^{\omega} D_{\langle p,\omega,1\rangle}$ is dense. Note that if $\bar{1} = \overline{x} \vee \overline{y}$, then we may assume without loss of generality that there exists i_0 such that $\forall j \geq i_0(x_j = 1)$, as either $\overline{x}$ or $\overline{y}$ must have this form. Now since $[1]_p$ is neither atomic nor atomless in $D_{\langle p,\omega 1\rangle}^{[p]}$, $[1]_p$ is neither atomic nor atomless in $(\prod_{i\in\omega}^{\omega} D_{\langle p,\omega,1\rangle})^{[p]}$. Hence $\mathrm{inv}_1(\bar{1}) > p$. Now if $\bar{b} \in \prod_{i\in\omega}^{\omega} D_{\langle p,\omega,1\rangle}$ is such that $\exists i_0 \forall j > i_0(b_j = 0)$, then $\bar{b} \in \mathcal{I}_{p+1}(\prod_{i\in\omega}^{\omega} D_{\langle p,\omega,1\rangle})$. If $\bar{b}$ is such that $\exists i_0 \forall j > i_0(b_j = 1)$, then $\bar{1} \triangle \bar{b} \in \mathcal{I}_p(\prod_{i\in\omega}^{\omega} D_{\langle p,\omega,1\rangle})$. Thus $[\bar{1}]_p$ is an atom in $(\prod_{i\in\omega}^{\omega} D_{\langle p,\omega,1\rangle})^{[p]}$, and hence $\mathrm{inv}(\prod_{i\in\omega}^{\omega} D_{\langle p,\omega,1\rangle}) = \langle p+1,1,0\rangle$. For denseness, let $\bar{b} \in \prod_{i\in\omega}^{\omega} D_{\langle p,\omega,1\rangle}$. If $\bar{b} = (b_0, \ldots, b_{i_0}, 0, 0, \ldots)$ then

$$(\prod_{i\in\omega}^{\omega} D_{\langle p,\omega,1\rangle}) \upharpoonright \bar{b} \cong D_{\langle p,\omega,1\rangle} \upharpoonright b_0 \times \cdots \times D_{\langle p,\omega,1\rangle} \upharpoonright b_{i_0},$$

which is dense by Corollary 4.8. The denseness condition for b follows. If $\bar{b} = (b_0, \ldots, b_{i_0}, 1, 1, \ldots)$ then, by the first observation,

$$\mathrm{inv}(\bar{b}) = \mathrm{inv}^{D_{\langle p,\omega,1\rangle}}(b_0) + \cdots + \mathrm{inv}^{D_{\langle p,\omega,1\rangle}}(b_{i_0}) + \mathrm{inv}^{\prod_{i\in\omega}^{\omega} D_{\langle p,\omega,1\rangle}}(\bar{1}) = \langle p+1,1,0\rangle.$$

Note that $(0, \ldots, 0, 1, 0, \ldots) < \bar{b}$ and has invariant $\langle p,\omega,1\rangle$. So the denseness condition for $\bar{b}$ follows from denseness below $(0, ..., 0, 1, 0, ...)$ and the fact that below $(0, \ldots, 0, 1, 0, \ldots)$ there is an element of invariant $\langle p,\omega,0\rangle$. $\square$

Lemma 7.4. *For every $p < \omega$, $\mathcal{DB}_{\langle p,2,0\rangle}$ is $(\Sigma_{4p+1} \wedge \Pi_{4p+1})$-hard. Moreover, the reductions proving hardness produce, in the case that n is not in the $\Sigma_{4p+1} \wedge \Pi_{4p+1}$ set, an index in $\mathcal{DB}_{\langle p,\bar{q},0\rangle}$ as required in Theorem 2.10.*

Proof. Consider two Σ_{4p+1} formulas $\phi(n)$ and $\psi(n)$. We want to construct a computable function f such that for every n

$$\phi(n) \ \& \ \neg\psi(n) \iff f(n) \in \mathcal{DB}_{\langle p,2,0\rangle}.$$

We start by finding a Π_{4p} formula $\hat{\phi}(n,x)$ such that $\phi(n) \iff \exists x\hat{\phi}(n,x)$, and such that if $\phi(n)$, then there is at most one x such that $\hat{\phi}(n,x)$. Since $\phi \in \Sigma_{4p+1}$, $\phi(n) = \exists x \forall w \bar{\phi}(n,x,w)$ for some $\bar{\phi} \in \Sigma_{4p-1}$. Let $\hat{\phi}(n,x)$ be the formula

$$x = \langle y, z\rangle \;\&\; \forall w \bar{\phi}(n,y,w) \;\&\; \forall y' < y \exists w \leq z(\neg\bar{\phi}(n,y',w)) \;\& \\ \forall z' < z \exists y' \leq y \forall w \leq z' \bar{\phi}(n,y',w) \quad (1)$$

which has the desired properties. Indeed, it is clear that if $\exists x\hat{\phi}(n,x)$ then $\phi(n)$. Now suppose $\phi(n)$ holds. So $\exists x\forall w\bar{\phi}(n,x,w)$. Choose y least such that $\forall w\bar{\phi}(n,y,w)$. Then for each $y' < y$ there is a minimal w such that $\neg\bar{\phi}(n,y',w)$. Let z be the maximum of these w. Then $\hat{\phi}(n,\langle y,z\rangle)$ holds. Suppose also $\hat{\phi}(n,\langle\tilde{y},\tilde{z}\rangle)$. Then the third condition in (1) gives $y = \tilde{y}$ and the fourth condition gives $z = \tilde{z}$.

We also define $\hat{\psi}(n,x)$ to be a Π_{4p} formula such that for all n, $\psi(n) \iff \exists x\hat{\psi}(n,x)$, but if $\psi(n)$, then there are exactly two x such that $\hat{\psi}(n,x)$. We define $\hat{\psi}$ as we did with $\hat{\phi}$ but replace "$x = \langle y,z\rangle$" with "$x = \langle y,z\rangle \vee x = \langle y,z\rangle + 1$".

Let g be a computable function such that $\forall n,x(\hat{\phi}(n,x) \implies g(n,x) \in \mathcal{DB}_{\langle p,1,0\rangle})$ and $\forall n,x(\neg\hat{\phi}(n,x) \implies g(n,x) \in \mathcal{DB}_{\langle p-1,\omega,1\rangle})$. Such a g exists by Corollary 6.3. Let h do the same with $\hat{\psi}$. Think of $g(n,x)$ and $h(n,x)$ as computable dense Boolean algebras, rather than as indices for such. For each n and x let $B_{n,x}$ be $g(n,\frac{x}{2})$ if x is even and $h(n,\frac{x-1}{2})$ if x is odd. Let $f(n) = \prod^{\omega}_{x\in\omega} B_{n,x}$. If $\phi(n) \;\&\; \neg\psi(n)$, then there is exactly one x such that $\hat{\phi}(n,x)$, so along the even components of $f(n)$ there is one copy of $D_{\langle p,1,0\rangle}$ with all others $D_{\langle p-1,\omega,1\rangle}$. As $\forall x(\neg\hat{\psi}(n,x))$, along the odd components there are copies of $D_{\langle p-1,\omega,1\rangle}$. Hence by the two observations about the product,

$$\begin{aligned}\mathrm{inv}(f(n)) &= \mathrm{inv}(D_{\langle p-1,\omega,1\rangle} \times \ldots \times D_{\langle p-1,\omega,1\rangle} \times D_{\langle p,1,0\rangle} \times \prod^{\omega}_{x\in\omega} D_{\langle p-1,\omega,1\rangle}) \\ &= \langle p-1,\omega,1\rangle + \cdots + \langle p-1,\omega,1\rangle + \langle p,1,0\rangle + \langle p,1,0\rangle \\ &= \langle p,2,0\rangle.\end{aligned}$$

Moreover, the resulting product is dense by Observation 7.3 and Corollary 4.8. Similarly, if $\neg\phi(n) \;\&\; \neg\psi(n)$, then we get only copies of $D_{\langle p-1,\omega,1\rangle}$, so $\mathrm{inv}(f(n)) = \langle p,1,0\rangle$. If $\phi(n) \;\&\; \psi(n)$, then we get three copies of $D_{\langle p,1,0\rangle}$, the rest $D_{\langle p-1,\omega,1\rangle}$, so $\mathrm{inv}(f(n)) = \langle p,4,0\rangle$, and if $\phi(n) \;\&\; \psi(n)$, then we get two copies of $D_{\langle p,1,0\rangle}$, the rest $D_{\langle p-1,\omega,1\rangle}$, so $\mathrm{inv}(f(n)) = \langle p,3,0\rangle$. Again, in every case the resulting algebra is dense by Observation 7.3 and Corollary 4.8. Thus f has the desired properties. □

8 The $\Pi_{\omega+1}$ case (Theorem 2.8)

We first prove that $\mathcal{B}_\omega = \mathcal{B}_{\langle\omega,0,0\rangle}$ is $\Pi^0_{\omega+1}$-hard. As in the previous section we will need to define some operations on Boolean Algebras.

In 8.2 we will define a binary operation, $*$, on presentations of Boolean algebras that corresponds, via the Interval Algebra operator, to the usual product on linear orderings. The only properties we will use of $*$ are the following.

Proposition 8.1. *Let B_0 and B_1 be Boolean algebras.*

1. *If* $\mathrm{inv}(B_0) = \langle p, 1, 0\rangle$ *and* $\mathrm{inv}(B_1) = \langle p_1, q_1, r_1\rangle$*, then* $\mathrm{inv}(B_0 * B_1) = \langle p + p_1, q_1, r_1\rangle$*.*
2. *If* $\mathrm{inv}(B_0) = \langle p, \omega, 0\rangle$*, then* $\mathrm{inv}(B_0 * B_1) = \langle p, \omega, 0\rangle$*.*

Moreover,

$$D_{\langle p,1,0\rangle} * D_{\langle p_1,q_1,r_1\rangle} \cong D_{\langle p+p_1,q_1,r_1\rangle} \quad \text{and} \quad D_{\langle p,\omega,0\rangle} * B_1 \cong D_{\langle p,\omega,0\rangle}.$$

We will prove Proposition 8.1 in subsection 8.2, but use it now to prove Theorem 2.8. We will also make use of the following uniform version of Theorem 5.6.

Proposition 8.2. *([AK00, 18.9]) For each k, let A_k and B_k be structures such that $A_k \leq_k B_k$ and $\{A_k, B_k\}$ is k-friendly, and let S_k be a Σ^0_k set, all uniformly in k. If $f(n,k)$ is a computable function then there is a uniformly computable sequence $\{C_{n,k}\}_{n\in\omega,k\in\omega}$ such that*

$$C_{n,k} \cong \begin{cases} A_k & \text{if } f(n,k) \in S_k \\ B_k & \text{otherwise.} \end{cases}$$

Theorem 8.3. *$\mathcal{B}_\omega$ is $\Pi^0_{\omega+1}$-hard.*

Proof. Suppose $S \in \Pi_{\omega+1}$, and f is a computable function such that

$$n \in S \iff \forall j(f(n,j) \notin 0^{(j)}).$$

We begin with a uniformly computable sequence $\langle A_{n,k} : n, k \in \omega\rangle$ of dense Boolean algebras such that

- $f(n,k) \in 0^{(k)} \implies A_{n,k} = D_{\langle k,\omega,0\rangle}$, and
- $f(n,k) \notin 0^{(k)} \implies A_{n,k} = D_{\langle k,1,0\rangle}$

Such a sequence exists by Proposition 8.2, Theorem 6.1 and the comment after Corollary 6.2.

Now define $K_{n,j}$ by recursion: $K_{n,1} = A_{n,1}$ and $K_{n,j+1} = K_{n,j} * A_{n,j+1}$. Let $K_n = \prod^{\omega}_{j\in\omega} K_{n,j}$. Let us next compute $\mathrm{inv}(K_n)$. First suppose that $n \in S$. Then, for every k, $\mathrm{inv}(A_{n,k}) = \langle k, 1, 0\rangle$, and then by Proposition 8.1

$$\mathrm{inv}(K_{n,j}) = \mathrm{inv}(A_{n,1}) + \cdots + \mathrm{inv}(A_{n,j}) = \langle 1 + 2 + \cdots + j, 1, 0\rangle = \langle \frac{j(j+1)}{2}, 1, 0\rangle.$$

Therefore $\mathrm{inv}_1(K_n) \geq \mathrm{inv}_1(K_{n,j}) = \frac{j(j+1)}{2}$ for every j. So, $\mathrm{inv}(K_n) = \langle \omega, 0, 0 \rangle$. On the other hand, if $n \notin S$ there is a first j_0 such that $f(n, j_0) \in 0^{(j_0)}$. Then, again by Proposition 8.1,

$$\mathrm{inv}(K_{n,j_0}) = \mathrm{inv}(A_{n,1}) + \cdots + \mathrm{inv}(A_{n,j_0-1}) + \mathrm{inv}(A_{n,j_0}) = \langle \frac{j_0(j_0+1)}{2}, \omega, 0 \rangle,$$

and for $j \geq j_0$, $\mathrm{inv}(K_{n,j})$ is constant and equal to $\langle \frac{j_0(j_0+1)}{2}, \omega, 0 \rangle$. Therefore, for every j, $K_{n,j}^{[\frac{j_0(j_0+1)}{2}]}$ is atomic. It is not hard to see that then $K_n^{[\frac{j_0(j_0+1)}{2}]}$ is also atomic, and hence $\mathrm{inv}(K_n) = \langle \frac{j_0(j_0+1)}{2}, \omega, 0 \rangle$. □

An interesting corollary is the following one about the complexity of deciding whether two Boolean algebras are elementarily equivalent. White [Whi00, 6.2.4] showed that for arbitrary structures this problem is as complicated as it can be. We prove the same when the structures are restricted to be Boolean algebras. Let $EE(BA)$ be the set of pairs $\langle i, j \rangle$ such that the computable Boolean algebras with indices i and j are elementarily equivalent. It clear that $EE(BA)$ is $\Pi^0_{\omega+1}$ because

$$\langle i, j \rangle \in EE(BA) \iff \forall \varphi \in \mathcal{L}^{BA}(B_i \models \varphi \iff B_j \models \varphi),$$

(where B_i and B_j are the computable Boolean algebras with indices i and j respectively and $\mathcal{L}^{BA}$ is the first order language of Boolean Algebras) and $0^{(\omega)}$ can tell whether $B_i \models \varphi$ uniformly in i and φ.

Corollary 8.4. *$EE(BA)$ is $\Pi^0_{\omega+1}$ complete.*

Proof. We already showed that $EE(BA)$ is in $\Pi^0_{\omega+1}$. We have to show that $EE(BA)$ is $\Pi^0_{\omega+1}$-hard. Consider $S \in \Pi_{\omega+1}$. Let K_n be as in the proof of the theorem above and let k_n be a computable index for K_n. Let d_ω be a computable index for $D_{\langle \omega,0,0 \rangle}$. Then

$$n \in S \iff \mathrm{inv}(K_n) = \langle \omega, 0, 0 \rangle \iff \langle d_\omega, k_n \rangle \in EE(BA). \qquad \square$$

8.1 $(\Sigma^0_{\omega+1}, \Pi^0_{\omega+1}) \leq_{\mathtt{m}} (\mathcal{DB}_{\langle \bar{\omega},\omega,0 \rangle}, \mathcal{DB}_{\langle \omega,0,0 \rangle})$

We now complete the proof of Theorem 2.10. We verify the last line of the table by improving the proof of Theorem 8.3 in which we showed that, given $S \in \Pi_{\omega+1}$, there are Boolean algebras K_n such that $n \in S \iff \mathrm{inv}_1(K_n) = \omega$. The $K_{n,j}$ as defined in the proof of Theorem 8.3 are dense because of Proposition 8.1. But when $n \in S$, K_n is not dense. We slightly modify the definition of K_n to make it dense.

Proposition 8.5. $(\Sigma^0_{\omega+1}, \Pi^0_{\omega+1}) \leq_{\mathtt{m}} (\mathcal{DB}_{\langle \bar{\omega},\omega,0 \rangle}, \mathcal{DB}_{\langle \omega,0,0 \rangle})$.

Proof. Let S and $K_{n,j}$ be as in the proof of Theorem 8.3. Now, instead of taking a product over ω, we define a componentwise product over $2^{<\omega}$. For $\sigma \in 2^{<\omega}$ let $K_{n,\sigma} = K_{n,|\sigma|}$. Let

$$\widetilde{K}_n = \prod_{\sigma \in 2^{<\omega}} K_{n,\sigma}$$

where $\prod_{\sigma \in 2^{<\omega}} B_\sigma$ is the set of $\langle b_\sigma : \sigma \in 2^{<\omega}\rangle \in \Pi_{\sigma \in 2^{<\omega}} B_\sigma$ such that for some n_0 we have that for every $\sigma \in 2^{n_0}$ either $\forall \tau \supseteq \sigma(b_\tau = 0)$ or $\forall \tau \supseteq \sigma(b_\tau = 1)$. The operations and constants for $\prod_{\sigma \in 2^{<\omega}} B_\sigma$ are defined componentwise.

As in the proof of Theorem 8.3, if $n \notin S$, then $\text{inv}(\widetilde{K}_n) = \langle k, \omega, 0\rangle$ for some $k < \omega$, and if $n \in S$ then $\text{inv}(\widetilde{K}_n) = \langle \omega, 0, 0\rangle$. If $n \notin S$, then denseness follows immediately from componentwise denseness as in Observation 7.3. Suppose $n \in S$, and $b \in \widetilde{K}_n$. Then, for each σ, $\text{inv}(K_{n,\sigma}) = \langle \frac{|\sigma|(|\sigma|+1)}{2}, 1, 0\rangle$, and hence, as in the proof of Theorem 8.3, if $\text{inv}_1(b) < \omega$ then for some n_0, for every $\sigma \in 2^{n_0}$, $\forall \tau \supseteq \sigma(b_\tau = 0)$, and if $\text{inv}_1(b) = \omega$ then for some σ, $\forall \tau \supseteq \sigma(b_\tau = 1)$. If $\text{inv}_1(b) < \omega$, then the denseness conditions for b are satisfied as in Observation 7.3. Suppose $\text{inv}(b) = \langle \omega, 0, 0\rangle$. Then, there is some $\sigma \in 2^{<\omega}$ such that $\forall \tau \supseteq \sigma(b_\tau = 1)$. Now consider a defined by $\forall \tau \not\supseteq \sigma(a_\tau = b_\tau)$, $\forall \tau \supseteq \sigma^\frown 0(a_\tau = 0)$, and $\forall \tau \supseteq \sigma^\frown 1(a_\tau = 1)$. Observe that $a \leq b$ and $\text{inv}(a) = \text{inv}(b - a) = \langle \omega, 0, 0\rangle$ as desired to prove the denseness condition for b. □

8.2 Interval Algebras and the $*$ operation

In this subsection we will show how to obtain a Boolean algebra from a linear ordering and vice versa. This will allow us to use operations on linear orderings on the corresponding Boolean algebras. We refer the reader to Monk [Mon89, I.6.15] and Goncharov [Gon97, 1.6 and 3.2] for general information on interval algebras. The goal of this section is to define a computable operator $*$ satisfying Proposition 8.1.

Definition 8.6. If L is a linear ordering with a first element, $\text{IntAlg}(L)$ is the Boolean algebra of finite unions of half open intervals $[a, b)$ of L where b can be ∞. (The understanding here is that $[a, \infty) = \{x : x \geq a\}$.)

It is clear that if L is computable then so is $\text{IntAlg}(L)$. The converse is also true:

Lemma 8.7. *[Gon97, 3.2.22] There is a computable operator* Lin *that, given a countable Boolean algebra B, returns a linear ordering* $\text{Lin}(B)$ *such that* $\mathcal{B} \cong \text{IntAlg}(\text{Lin}(B))$.

Definition 8.8. The product of linear orderings, $L_0 \cdot L_1$, is gotten by replacing each element of L_1 by a copy of L_0 (and so, it is the ordering on pairs $\langle x_1, y_1\rangle \in L_0 \times L_1$ given by $\langle x_1, y_1\rangle < \langle x_2, y_2\rangle \iff y_1 < y_2 \vee (y_1 = y_2 \;\&\; x_1 < x_2)$).

Given two Boolean algebras B_0 and B_1 we let

$$B_0 * B_1 = \text{IntAlg}(\text{Lin}(B_0) \cdot \text{Lin}(\mathcal{B}_1)).$$

Note that $B_0 * B_1$ depends on the presentations of B_0 and B_1.

Now we show how to describe the analysis of the Tarski invariants of $\mathrm{IntAlg}(L)$ in terms of L.

Definition 8.9. A subset S of L is *convex* if $x, y \in S$ and $x < z < y$ implies that $z \in S$. An equivalence relation $\sim$ on L is *convex* if every one of its equivalence classes is convex.

Proposition 8.10. *[Gon97, 1.6,3.2][Mon89, I.6.15]There is a one-one correspondence between ideals I of $\mathrm{IntAlg}(L)$ and convex equivalence relations $\sim$ on L such that $\mathrm{IntAlg}(L)/I \cong \mathrm{IntAlg}(L/\sim)$. Here $L/\sim$ is the linear ordering of equivalence classes $[x]$, $[y]$ of $\sim$ given by $[x] < [y] \iff \forall w \sim x \forall z \sim y (w < z)$. The convention here is that if a final segment of L is collapsed to a single equivalence class, then it is removed from $L/\sim$ and its role is taken by ∞. For a given ideal I, the corresponding equivalence relation $\sim$ is given by $x \sim y \iff [x, y) \in I$ for $x \leq y \in L$.*

Definition 8.11. We denote $L/\sim_T$ by $L^{[1]}$ where $\sim_T$ is the equivalence relation corresponding to $\mathcal{I}$ and so

$$\mathrm{IntAlg}(L^{[1]}) \cong \mathrm{IntAlg}(L)/\mathcal{I}(\mathrm{IntAlg}(L)) = \mathrm{IntAlg}(L)^{[1]}$$

The following lemma is key for the proof of Proposition 8.1. The sum over M, $\sum_{i \in M} L_i$, of linear orderings L_i, $i \in M$, is gotten by replacing each element i of M by a copy of L_i. Observe that when for every i, $L_i \cong L$ we have that $\sum_{i \in M} L_i \cong L \cdot M$.

Lemma 8.12. *[Sho04, 5.8] If, for every $i \in \omega$, $\mathrm{inv}_1(L_i) \geq 1$ for every L_i and $L = \sum_{i \in M} L_i$ then $L^{[1]} = \Sigma_{i \in M} L_i^{[1]}$.*

Corollary 8.13. *If $\mathrm{inv}_1(K) \geq 1$ then $(K \cdot M)^{[1]} = K^{[1]} \cdot M$.*

Lemma 8.14. *Let B_0 and B_1 be Boolean algebras.*

1. *If B_0 is the trivial Boolean algebra, i.e. $\mathrm{inv}(B_0) = \langle 0, 1, 0 \rangle$, $B_0 * B_1 \cong B_1$.*
2. *If B_0 is atomic and has infinitely many atoms, then $B_0 * B_1$ is atomic and $\mathrm{inv}(B_0 * B_1) = \langle 0, \omega, 0 \rangle$.*
3. *If $\mathrm{inv}(B_0) = \langle p, 1, 0 \rangle$ and $\mathrm{inv}(B_1) = \langle p_1, q_1, r_1 \rangle$, then $\mathrm{inv}(B_0 * B_1) = \langle p + p_1, q_1, r_1 \rangle$.*
4. *If $\mathrm{inv}(B_0) = \langle p, \omega, 0 \rangle$, then $\mathrm{inv}(B_0 * B_1) = \langle p, \omega, 0 \rangle$.*

Proof. For (1), if $\mathrm{inv}(B_0) = \langle 0, 1, 0 \rangle$, then $\mathrm{Lin}(B_0) \cong \mathbf{1}$. Hence $\mathrm{Lin}(B_0) \cdot \mathrm{Lin}(B_1) \cong \mathrm{Lin}(B_1)$, and so $B_0 * B_1 \cong B_1$.

For (2), consider a non-zero $[x, y) \subseteq \mathrm{Lin}(B_0) \cdot \mathrm{Lin}(B_1)$. There is some non-zero $[x_0, y_0) \subseteq [x, y)$ with $[x_0, y_0)$ contained in a copy of $\mathrm{Lin}(B_0)$. As B_0 is atomic, there is an atom below $[x_0, y_0)$, and hence below $[x, y)$. Thus $B_0 * B_1$ is atomic. Since B_0 has infinitely many atoms, so does $B_0 * B_1$, hence $\mathrm{inv}(B_0 * B_1) = \langle 0, \omega, 0\rangle$.

For parts (3) and (4) we first make a general observation. If $\mathrm{inv}_1(B_0) = p$, then

$$\begin{aligned}
(B_0 * B_1)^{[p]} &= \mathrm{IntAlg}(\mathrm{Lin}(B_0) \cdot \mathrm{Lin}(B_1))^{[p]} \\
&\cong \mathrm{IntAlg}((\mathrm{Lin}(B_0) \cdot \mathrm{Lin}(B_1))^{[p]}) \\
&= \mathrm{IntAlg}((\mathrm{Lin}(B_0)^{[1]} \cdot \mathrm{Lin}(B_1))^{[p-1]}) && \text{(by Corollary 8.13)} \\
&= \mathrm{IntAlg}((\mathrm{Lin}(B_0)^{[2]} \cdot \mathrm{Lin}(B_1))^{[p-2]}) && \text{(again by Corollary 8.13)} \\
&\ \ \vdots \\
&= \mathrm{IntAlg}(\mathrm{Lin}(B_0)^{[p]} \cdot \mathrm{Lin}(B_1))
\end{aligned}$$

For (3), we have that $\mathrm{Lin}(B_0)^{[p]} = 1$, so $(B_0 * B_1)^{[p]} = \mathrm{IntAlg}(1 \cdot \mathrm{Lin}(B_1)) \cong B_1$. Hence $\mathrm{inv}(B_0 * B_1) = \langle p + p_1, q_1, r_1\rangle$.

Finally, for (4), we have that $(B_0 * B_1)^{[p]} \cong \mathrm{IntAlg}(\mathrm{Lin}(B_0)^{[p]} * \mathrm{Lin}(B_1))$, and so, since $B_0^{[p]}$ is atomic and has infinitely many atoms, $\mathrm{Lin}(B_0)^{[p]} * \mathrm{Lin}(B_1)$ is also atomic and has infinitely many atoms as in part (2). The result follows. □

The first part of Proposition 8.1 follows from the lemma above. This first part was all we used in the proof of Theorem 8.3. We now prove the second part, used to prove Proposition 8.5.

Lemma 8.15. *1.* $D_{\langle p,1,0\rangle} * D_{\langle p_1,q_1,r_1\rangle} \cong D_{\langle p+p_1,q_1,r_1\rangle}$.

2. $D_{\langle p,\omega,0\rangle} * B \cong D_{\langle p,\omega,0\rangle}$.

Proof. We have seen, by Lemma 8.14, that the invariants are as claimed, so it remains to check denseness. Consider $B_0 * B_1$ where B_0 is dense, and an element of the interval algebra $b = [x, y)$ for which we want to verify the density conditions. If x and y belong to the same copy of $\mathrm{Lin}(B_0)$ in the product $\mathrm{Lin}(B_0) \cdot \mathrm{Lin}(B_1)$, then we are done by the assumed density of B_0. If they are in adjacent copies of $\mathrm{Lin}(B_0)$, then one of the two subintervals lying within single copies into which b can be decomposed is responsible for the hypothesis of the density condition holding and an application of density for that subinterval within its copy supplies the desired witness for density. Thus we may assume that there is a copy of $\mathrm{Lin}(B_0)$ between x and y.

For (1), $B_0 = D_{\langle p,1,0\rangle}$, and $B_1 = D_{\langle p_1,q_1,r_1\rangle}$, so $\mathrm{Lin}(B_0)^{[p]} = \mathbf{1}$ and so $(\mathrm{Lin}(B_0) \cdot \mathrm{Lin}(B_1))^{[p]} = \mathrm{Lin}(B_1)$. We may assume that y is ∞ or the first element of some copy of $\mathrm{Lin}(B_0)$. In either case, $\mathrm{inv}_1(b) \geq p$ and the image of b in $(\mathrm{Lin}(B_0) \cdot \mathrm{Lin}(B_1))^{[p]}$ is the interval of $\mathrm{Lin}(B_1)$ corresponding to the copies of $\mathrm{Lin}(B_0)$ starting with x and ending with y. We now take the witness for density in $\mathrm{Lin}(B_1)$ and pull it back to $\mathrm{Lin}(B_0) \cdot \mathrm{Lin}(B_1)$.

For (2), $B_0 = D_{\langle p,\omega,0\rangle}$, and $B_1 = B$. So $\mathrm{Lin}(B_0)^{[p]}$ is atomic and has infinitely many atoms. Thus $\mathrm{inv}(b) = \langle p, \omega, 0\rangle$ and the required witnesses for the first and second denseness conditions can be found within a copy of $\mathrm{Lin}(B_0)$ contained in b. □

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