# $K$-TRIVIALS ARE NEVER CONTINUOUSLY RANDOM

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## 1. Introduction

In [RS07, RS08], Reimann and Slaman raise the question "For which infinite binary sequences $X$ do there exist continuous probability measures $\mu$ such that $X$ is effectively random relative to $\mu$?". They defined the collection $\mathrm{NCR}_1$ of binary sequences for which such measures do not exist (we give formal definitions below), and showed, for example, that $\mathrm{NCR}_1$ is countable, indeed that every sequence in $\mathrm{NCR}_1$ is hyperarithmetic. In this paper we contribute toward the understanding of $\mathrm{NCR}_1$ by showing that it contains all sets which are Turing reducible to an incomplete, recursively enumerable set. In particular, $\mathrm{NCR}_1$ contains all $K$-trivial sets.

### 1.1. Randomness relative to continuous measures.

We begin by reviewing the basic definitions needed to precisely formulate this question.

**Notation 1.1.**

- For $\sigma \in 2^{<\omega}$, $[\sigma]$ is the basic open subset of $2^\omega$ consisting of those $X$'s which extend $\sigma$. Similarly, for $W$ a subset of $2^{<\omega}$, let $[W]$ be the open set given by the union of the basic open sets $[\sigma]$ such that $\sigma \in W$.
- For $U \subseteq 2^\omega$, $\lambda(U)$ denotes the measure of $U$ under the uniform distribution. Thus, $\lambda([\sigma])$ is $1/2^\ell$, where $\ell$ is the length of $\sigma$.

**Definition 1.2.** A *representation* $m$ of a probability measure $\mu$ on $2^\omega$ provides, for each $\sigma \in 2^{<\omega}$, a sequence of intervals with rational endpoints, each interval containing $\mu([\sigma])$, and with lengths converging monotonically to 0.

**Definition 1.3.** Suppose that $Z \in 2^\omega$. A *test relative to* $Z$, or $Z$*-test*, is a set $W \subseteq \omega \times 2^{<\omega}$ which is recursively enumerable in $Z$. For $X \in 2^\omega$, $X$ *passes* a test $W$ if and only if there is an $n$ such that $X \notin [W_n]$.

**Definition 1.4.** Suppose that $m$ represents the measure $\mu$ on $2^\omega$ and that $W$ is an $m$-test.

- $W$ is *correct for* $\mu$ if and only if for all $n$, $\mu([W_n]) \leq 2^{-n}$.
- $W$ is *Solovay-correct for* $\mu$ if and only if $\sum_{n\in\omega} \mu([W_n]) < \infty$.

**Definition 1.5.** $X \in 2^\omega$ is 1*-random relative to a representation* $m$ *of* $\mu$ if and only if $X$ passes every $m$-test which is correct for $\mu$. When $m$ is understood, we say that $X$ is 1-random relative to $\mu$.

By an argument of Solovay, see [Nie09], $X$ is 1-random relative to a representation $m$ of $\mu$ if an only if for every $m$-test which is Solovay-correct for $\mu$, there are infinitely many $n$ such that $X \notin [W_n]$.

**Definition 1.6.** $X \in \mathrm{NCR}_1$ if and only if there is no representation $m$ of a continuous measure $\mu$ such that $X$ is 1-random relative to the representation $m$ of $\mu$.

In [RS08], Reimann and Slaman show that if $X$ is not hyperarithmetic, then there is a continuous measure $\mu$ such that $X$ is 1-random relative to $\mu$. Conversely, Kjøs-Hanssen and Montalbán, see [Mon05], have shown that if $X$ is an element of a countable $\Pi^0_1$-class, then there is no continuous measure for which $X$ is 1-random. As the Turing degrees of the elements of countable $\Pi^0_1$-classes are cofinal in the Turing degrees of the hyperarithmetic sets, the smallest ideal in the Turing degrees that contains the degrees represented in $\mathrm{NCR}_1$ is exactly the Turing degrees of the hyperarithmetic sets.

In [RSte], Reimann and Slaman pose the problem to find a natural $\Pi^1_1$-norm for $\mathrm{NCR}_1$ and to understand its connection with the natural norm mapping a hyperarithmetic set $X$ to the ordinal at which $X$ is first constructed. As of the writing of this paper, this problem is open in general, but completed in [RSte] for $X \in \Delta^0_2$.

Suppose that $X \in \Delta^0_2$ and that for all $n$, $X(n) = \lim_{t\to\infty} X_t(n)$, where $X_t(n)$ is a computable function of $n$ and $t$. Let $g_X$ be the convergence function for this approximation, that is for all $n$, $g_X(n)$ is the least $s$ such that for all $t \geq s$ and all $m \leq n$, $X_t(m) = X(m)$. Let $f_X$ be function obtained by iterated application of $g_X$: $f_X(0) = g_X(0)$ and $f_X(n+1) = g_X(f_X(n))$.

For a representation $m$ of a continuous measure $\mu$, the granularity function $s_m$ maps $n \in \omega$ to the least $\ell$ found in the representation of $\mu$ by $m$ such that for all $\sigma$ of length $\ell$, $\mu([\sigma]) < 1/2^n$. Note that, $s_m$ is well-defined by the compactness of $2^\omega$.

**Theorem 1.7** (Reimann and Slaman [RSte])**.** *Let $X$ be a $\Delta^0_2$ set and let $f_X$ be the function defined as above. If $X$ is 1-random relative the representation $m$ of $\mu$, then the granularity function $s_m$ for $\mu$ is eventually bounded by $f_X$.*

Thus, for $\Delta^0_2$ sets $X$, there is a continuous measure relative to which $X$ is 1-random if and only if there is a continuous measure whose granularity is eventually bounded by $f_X$. The latter condition is arithmetic, again by a compactness argument.

1.2. ***K*-triviality.** $K$-triviality is a property of sequences which characterizes another aspect of their being far from random. We briefly review this notion and the results surrounding it. A full treatment is given in Nies [Nie09].

For $\sigma \in 2^{<\omega}$, let $K(\sigma)$ denote the prefix-free Kolmogorov complexity of $\sigma$. Intuitively, given a universal computable $U$ with domain an antichain in $2^{<\omega}$, $K(\sigma)$ is length of the shortest $\tau$ such that $U(\tau) = \sigma$. Similarly, for $X \in 2^\omega$, let $K^X(\sigma)$ denote the prefix-free Kolmogorov complexity of $\sigma$ relative to $X$. That is, $K^X$ is determined by a function universal among those computable relative to $X$.

**Definition 1.8.** A sequence $X \in 2^\omega$ is *$K$-trivial* if and only if there is a constant $k$ such that for every $\ell$, $K(X \restriction \ell) \leq K(0^\ell) + k$, where $0^\ell$ is the sequence of 0's of length $\ell$.

By early results of Chaitin and Solovay and later results of Nies and others, there are a variety of equivalents to $K$-triviality and a variety of properties of the $K$-trivial sets. For example, $X$ is $K$-trivial if and and only if for every sequence $R$, $R$ is 1-random for $\lambda$ if and only if $R$ is 1-random for $\lambda$ relative to $X$.

In the next section, we will apply the following.

**Theorem 1.9** (Nies [Nie09], strengthening Chaitin [Cha76])**.** *If $X$ is $K$-trivial, then there is a computably enumerable and $K$-trivial set which computes $X$.*

The following theorem follows from the work of Nies and others [Nie09]. Some versions of this property have been used by Kučera extensively, e.g. in [Kuč85].

**Theorem 1.10.** *Suppose that $X$ is $K$-trivial and $\{U_e^X : e \in \omega\}$ a uniformly $\Sigma_1^{0,X}$ family of sets. Then, there is a computable function $g$ and a $\Sigma_1^0$ set $V$ of measure less than 1 such for every $e$, if $\lambda(U_e^Z) < 2^{-g(e)}$ for every oracle $Z$, then $U_e^X \subseteq V$.*

*Proof.* Let $\left((E_i^e)\right)_{e\in\mathbb{N}}$ be a uniform sequence of all oracle Martin-Löf tests. A standard construction of a universal oracle Martin-Löf test $(T_i)$ (e.g. see [Nie09]) gives a recursive function $f$ such that $\forall Z \subseteq \omega\ (E_{f(i,e)}^{e,Z} \subseteq T_i^Z)$ for all $e, i \in \mathbb{N}$. Let $T := T_2$ and $f(e) := f(2, e)$ for all $e \in \mathbb{N}$, so that $\mu(T^Y) \leq 2^{-2}$ for all $Y \in 2^\omega$ and $E_{f(e)}^e \subseteq T$ for all $e \in \mathbb{N}$. In [KH07] it was shown that $X$ is $K$-trivial iff for some member $T$ of a universal oracle Martin-Löf test, there is a $\Sigma_1^0$ class $V$ with $T^X \subseteq V$ and $\mu(V) < 1$.

Now given a uniform enumeration $(U_e)$ of oracle $\Sigma_1^0$ classes we have the following property of $T$:

> There is a recursive function $g$ such that for each $e$,
> either $\exists Z \subseteq \omega\ (\mu(U_e^Z) \geq 2^{-g(e)-1})$, or $\forall Z \subseteq \omega\ (U_e^Z \subseteq T^Z)$.

To see why this is true, note that every $U_e$ can be effectively mapped to the oracle Martin-Löf test $(M_i)$ where $M_i^Z = U_e^Z[s_i]$ and $s_i$ is the largest stage such that $\mu(U_e^Z[s_i]) < 2^{-i-1}$ (which could be infinity). Effectively in $e$ we can get an index $n$ of $(M_i)$. It follows that if $\mu(U_e^Z) < 2^{-f(n)-1}$ for all $Z$, then $U_e^X = M_{f(n)}^X = E_{f(n)}^{n,X} \subseteq T^X \subseteq V$. So $g(e) = f(n) + 1$ is as wanted. $\square$

1.3. **Our results.** Intuitively, $X \in \mathrm{NCR}_1$ asserts that $X$ is not effectively random relative to any continuous measure and $X$ is $K$-trivial asserts that relativizing to $X$ does change the evaluation of randomness relative to the uniform distribution. In the next section, we connect the two notions.

**Theorem 1.11.** *Every $K$-trivial set belongs to* $\mathrm{NCR}_1$.

A recursively enumerable (r.e.) set $W$ is called *incomplete* if it does not compute the halting problem $\emptyset'$.

**Theorem 1.12.** *If $W$ is an incomplete r.e. set and $X \leq_{\mathrm{T}} W$, then $X \in \mathrm{NCR}_1$.*

As we mentioned above, Theorem 1.12 implies Theorem 1.11, because every $K$-trivial set is computable from a r.e. $K$-trivial set, and every $K$-trivial set is incomplete. However, the proof we give of Theorem 1.11 does not use the incompleteness of $K$-trivial sets. We believe that the direct proof technique is of independent interest.

## 2. $K$-trivial sets and $\mathrm{NCR}_1$

In this section we prove Theorem 1.11.

Let $Y$ be $K$-trivial and let $\mu$ be a continuous measure with representation $m$; we want to show $Y$ is not $\mu$-random. By Theorem 1.9, let $X$ be a computably enumerable $K$-trivial sequence that computes $Y$. Let $f$ be the iterated convergence function as defined above for the computable approximation to $Y$ given by approximating $X$'s computation of $Y$. Since $X$ is computably enumerable, $X$ can compute the convergence function for its own enumeration and hence $f$ is computable from $X$.

Let $s_m$ be the granularity function for $\mu$ as represented by $m$. By Theorem 1.7, $f$ eventually dominates $s_m$. By changing finitely many values of $f$, we may assume that $f$ dominates $s_m$ everywhere. So, we have that for every $n$

$$\mu([Y \upharpoonright f(n)]) \leq 2^{-n}.$$

Further, we may assume that $f$ can be obtained as the limit of a computable function $f(n,s)$ such that for all $s$, $f(n-1,s) \leq f(n,s) \leq f(n,s+1)$.

We will build an $m$-test $\{S_i : i \in \omega\}$ which is Solovay-correct for $\mu$ and which $Y$ does not pass, thereby concluding that $Y$ is not $\mu$-random. That is, we plan to build $\{S_i : i \in \omega\}$ to be a uniformly $\Sigma_1^{0,m}$ sequence of sets such that $\sum_{i\in\omega} \mu(S_i)$ is bounded and such that there are co-finitely $i$ for which $Y \in [S_i]$. Our construction will not be uniform.

$X$'s $K$-triviality is exploited in the form of Theorem 1.10. Let $V$ and $g$ be given by Theorem 1.10 where $\{U_e^X : e \in \omega\}$ is a listing of all $\Sigma_1^{0,X}$ sets. We will build an oracle $\Sigma_1^0$ class $U$ along the construction. We use the recursion theorem to assume that in advance we know an index $e$ such that $U = U_e$. During the construction we will make sure that for every oracle $Z$, $\lambda(U^Z) < 2^{-g(e)}$. Theorem 1.10 then implies that $U^X \subseteq V$ where $V$ is a $\Sigma_1^0$ class of measure less than 1. To simplify our notation, let $a$ denote $g(e)$. Furthermore, assume $a$ is large enough so that $\lambda(V) + 2^{-a} < 1$.

We use the approximation to $X$ as a computably enumerable set to enumerate approximations to initial segments of $Y$ into the sets $S_i$; we rely on the $K$-triviality of $X$ to keep the total $\mu$-measure of the $S_i$'s bounded.

For each $n > a$ we have a requirement $R_n$ whose task is to enumerate $Y \upharpoonright f(n)$ into $S_n$. Let $y_{n,s} = Y_s \upharpoonright f(n, s)$ the stage $s$ approximation to $Y \upharpoonright f(n)$. Let $x_{n,s}$ be the initial segment of $X_s$ necessary to compute $y_{n,s}$ and $f(n, s)$. So, if $y_{n,s+1} \neq y_{n,s}$, it is because $x_{n,s+1} \neq x_{n,s}$. In this case, $x_{n,s+1}$ is not only different than $x_{n,s}$, but also incomparable. At stage $s$, $R_n$ would like to enumerate $y_{n,s}$ into $S_n$, but before doing that it will *ask for confirmation* using the fact that $U^X \subseteq V$. Since we are constrained to keep $\lambda(U^X)$ less than or equal to $2^{-a}$, we will restrict $R_n$ to enumerate at most $2^{-n}$ measure into $U^X$. The reason why we need a bit of security before enumerating a string in $S_n$ is that we have to ensure that $\sum_i \mu(S_i)$ is bounded. For this purpose, we will only enumerate mass into $S_n$ when we see an equivalent mass going into $V$.

**Action of requirement $R_n$:**

(1) The first time after $R_n$ is initialized, $R_n$ chooses a clopen subset of $2^\omega$, $\sigma_n$, of $m$-measure $2^{-n}$, that is disjoint form $V_s$ and $U_s^{X_s}$. Note that since $V$ and $U^{X_s}$ have measure less than $\lambda(V) + 2^{-a} < 1$, we can always find such a clopen set. Furthermore we can chose $\sigma_n$ to be different from the $\sigma_i$ chosen by other requirements $R_i$, $i > a$. We note the value of $\sigma_n$ might change if $R_n$ is initialized.

(2) To *confirm* $x_{n,s}$, requirement $R_n$ enumerates $\sigma_n$ into $U^{x_{n,s}}$. Requirement $R_n$ will not be allowed to enumerate anything else into $U^{X_s}$ unless $X_s$ changes below $x_{n,s}$. This way $R_n$ is always responsible for at most $2^{-n}$ measure enumerated in $U^{X_s}$.

(3) Then, we wait until a stage $t > s$ such that
   (a) either $x_{n,s} \not\subseteq x_{n,t}$ (as strings),
   (b) or $\sigma_n \subseteq V_t$.

   Observe that if $x_{n,s}$ is actually an initial segment of $X$, then we will have $\sigma_n \subseteq U^X \subseteq V$. So, we will eventually find such a stage $t$.
   - In Case 3(a), we start over with $R_n$. Note that in this case $\sigma_n$ has come out of $U^{X_t}$, and hence $R_n$ is responsible for no measure inside $U^{X_t}$ at stage $t$.
   - In Case 3(b), if $\mu([y_{n,t}]) \leq 2^{-n}$, enumerate $y_{n,t}$ into $S_n$. (Recall that we are allowed to use the representation of $\mu$ as an oracle when enumerating $S_n$.)

Since we only enumerate $y_{n,t}$ of $\mu$-measure less than $2^{-n}$ when $\sigma_n$ is enumerated in $V$, we have that

$$\sum_i \mu(S_i) \leq \lambda(V) < 1.$$

It is not hard to check that $\lambda(U^X) \leq \sum_{n=a+1}^{\infty} 2^{-n} = 2^{-a}$, so we actually have that $U^X \subseteq V$. Also notice that once $x_{n,s}$ is a initial segment of $X$, we will eventually enumerate $\sigma_n$ into $V$ and an initial segment of $Y$ into $S_n$.

This completes the proof of Theorem 1.11.

## 3. Incomplete r.e. degrees and $\mathrm{NCR}_1$

We turn to the proof of Theorem 1.12. Let $W$ be an incomplete r.e. set, and let $X \leq_{\mathrm{T}} W$.

The fact that $W$ is recursively enumerable and $X \leq_{\mathrm{T}} W$ implies that there is a recursive approximation $X = \lim_t X_t$ such that the modulus functions $g_X$ is recursive in $W$, hence $f_X \leq_{\mathrm{T}} W$.

Suppose, for contradiction, that $X$ is 1-random relative to a representation $m$ of a continuous measure $\mu$. By Theorem 1.7, by changing $f_X$ at finitely many inputs, we obtain a function $f \leq_{\mathrm{T}} W$ which bounds the granularity function $s_m$. Let $h(n) = X \upharpoonright f(n)$. So $h \leq_{\mathrm{T}} W$, and for all $n$, $\mu([h(n)]) < 2^{-n}$.

Let $J$ be a universal partial recursive function. For $n \in \mathbb{N}$, let $U_n = \{J(n)\}$ if $n \in \operatorname{dom} J$ and $J(n)$ is a binary string such that $\mu(J(n)) < 2^{-n}$. Otherwise, $U_n$ is empty. Then the test $U$ is recursively enumerable in $m$, and is correct for $\mu$. Since $X$ must pass $U$, we see that for all $n \in \operatorname{dom} J$, $h(n) \neq J(n)$.

The function $h$ is diagonally nonrecursive. By Jockusch [Joc89], $h$ computes a fixed-point-free function. This contradicts Arslanov's completeness criterion [Ars81], which states that an incomplete r.e. set cannot compute a fixed-point-free function.

This completes the proof of Theorem 1.12.

The question of which $\Delta^0_2$ sets belong to $\mathrm{NCR}_1$ remains open.

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