A MINIMAL PAIR OF K-DEGREES

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Abstract. We construct a minimal pair of K-degrees. We do this by showing the existence of an unbounded nondecreasing function f which forces K-triviality in the sense that $\gamma \in 2^\omega$ is K-trivial if and only if for all n, $K(\gamma \upharpoonright n) \leq K(n) + f(n) + \mathcal{O}(1)$.

1. Introduction and Notation

K-reducibility is defined with the intention of measuring the relative randomness of infinite binary strings, which we refer to as reals. This reducibility was defined using a function, K, that assigns to each finite binary string the length of its shortest description, in a sense we will specify. The idea being that if a string is random, there should not be any short way of describing it. The precise definition of K is given below, though the proofs presented in this paper use only the two properties of K listed at the end of this section.

The *prefix-free Kolomogorov complexity* of a string $\sigma \in 2^{<\omega}$ is defined to be the length of the shortest program $p \in 2^{<\omega}$ such that $U(p) = \sigma$, where U is a universal prefix-free Turing machine. That is, U is universal for machines V with the property that if $V(\tau)\downarrow$, then $V(\tau')\uparrow$ for all $\tau' \supset \tau$. We denote the Kolmogorov complexity of σ by $K(\sigma)$. This definition is independent of the choice of universal machine U, up to additive constant. The advantage of restricting to prefix-free machines is that otherwise the Kolmogorov complexity would contain extra information about the length of the string. For more background on Kolmogorov complexity, see Li and Vitányi [LV97], and Downey and Hirschfeldt [DH].

Prefix-free Kolmogorov complexity is used to define a notion of randomness for real numbers. A real $\gamma \in 2^\omega$ is *K-random* (or Levin-Gaćs-Chaitin random) if for all n, $K(\gamma \upharpoonright n) \geq n - \mathcal{O}(1)$. This notion has been extensively studied and coincides with other notions of randomness based on measure theory or unpredictability [DH], [DHNT]. We can also use K to define what it means for a real to be far from being random. We say a real is *K-trivial* if for all n, $K(\gamma \upharpoonright n) \leq K(n) + \mathcal{O}(1)$; that is, every initial segment is as simple as possible. But what of relative randomness of reals? K-reducibility was introduced to study notions of relative randomness. For two reals α and β in 2^ω we let

$$\alpha \leq_K \beta \iff (\forall n)\ K(\alpha \upharpoonright n) \leq K(\beta \upharpoonright n) + \mathcal{O}(1),$$

Date: November 19, 2004.
1991 *Mathematics Subject Classification.* Primary 03D30.
Key words and phrases. minimal pair, relative randomness.
We thank Denis R. Hirschfeldt for bringing this question to our attention. The second author was partially supported by NSF Grant DMS-0100035.

i.e., if there exists a constant C such that $(\forall n)\ K(\alpha \upharpoonright n) \leq K(\beta \upharpoonright n) + C$. The K-degrees are defined as equivalence classes under this quasiordering.

As is usual when considering a reducibility, we want to understand the structure of the K-degrees. We know that the K-degrees have a bottom element that corresponds to the K-degree of the K-trivial reals. Yu, Ding, and Downey showed that there are uncountably many K-degrees, indeed $2^{\aleph_0}$ many among the K-random reals ([YDD04], see [DHNT]). When restricting attention to c.e. reals (reals with nice approximations), Downey, Hirschfeldt, and LaForte have shown density and existence of join [DHL04]. A result of Solovay is that K-reducibility does not imply Turing reducibility (see [DH]).

A natural question to ask when studying a reducibility is if there exists a minimal pair. Rod Downey and Denis Hirschfeldt asked this question for the K-degrees. That is, they asked whether there exist non-K-trivial reals α and β in 2^ω such that whenever $\gamma \in 2^\omega$ is such that $\gamma \leq_K \alpha$ and $\gamma \leq_K \beta$ then γ is K-trivial. Here we answer this question affirmatively with a simple and elegant construction of a minimal pair. We do it by first constructing a unbounded nondecreasing function f which forces K-triviality in the sense that a real γ is K-trivial if and only if $(\forall n)\ K(\gamma \upharpoonright n) \leq K(n) + f(n) + \mathcal{O}(1)$. This function will likely be useful in showing other results about K-reducibility.

If a real is K-trivial, then there is some constant which witnesses its K-triviality. We say a real γ is *K-trivial(C)* if for all n, $K(\gamma \upharpoonright n) \leq K(n) + C$, where $K(n) = K(0^n)$. Then, we have that γ is K-trivial if and only if it is K-trivial(C) for some C. We say that γ *appears to be K-trivial(C) at n* if for all $m \leq n$, $K(\gamma \upharpoonright m) \leq K(m) + C$. We say that γ *stops appearing K-trivial(C) at n* if it appears K-trivial(C) at $n-1$ but not at n. Throughout the paper, γ will always denote a real, i.e. $\gamma \in 2^\omega$.

The properties of K that we will use are.

Property 1 (Zambella—see [DHNS03]). *For every C, there are only finitely many reals that are K-trivial(C).*

Property 2. *For any $\sigma \in 2^{<\omega}$, $\sigma^\frown 0^\omega$ is K-trivial, and hence K-trivial(C) for some C.*

2. Construction of a minimal pair

Theorem 1. *There exists a minimal pair of K-degrees.*

To prove our theorem, we will use the following lemma, which is interesting in itself, and may have other applications.

Lemma 1. *There exists a unbounded nondecreasing function f such that for all reals $\gamma \in 2^\omega$, the following are equivalent.*

1. *γ is K-trivial.*
2. *For almost every n, $K(\gamma \upharpoonright n) \leq K(n) + f(n)$.*
3. $(\forall n)\ K(\gamma \upharpoonright n) \leq K(n) + f(n) + \mathcal{O}(1)$.

Before proving Lemma 1, we show how Theorem 1 follows from it.

Proof of Theorem 1. Let f be as in Lemma 1. We will construct two non-K-trivial reals α and β such that $\min\{K(\alpha \upharpoonright n), K(\beta \upharpoonright n)\} \leq K(n) + f(n)$. This will give us a minimal pair because if $\gamma \leq_K \alpha$ and $\gamma \leq_K \beta$, then $K(\gamma \upharpoonright n) \leq K(n) + f(n) + \mathcal{O}(1)$, and hence γ is K-trivial.

We construct α and β as the limits of two sequences of finite strings, $\{\alpha_s\}_{s\in\omega}$ and $\{\beta_s\}_{s\in\omega}$, which satisfy that, for every s, $\alpha_s \subset \alpha_{s+1}$, $\beta_s \subset \beta_{s+1}$ and $|\alpha_s| = |\beta_s|$. We denote $|\alpha_s|$ by n_s. To get $\min\{K(\alpha \restriction n), K(\beta \restriction n)\} \leq K(n) + f(n)$, we ensure that if $n_s \leq n < n_{s+1}$, then $K(\alpha \restriction n) \leq K(n) + f(n)$ if s is odd, and $K(\beta \restriction n) \leq K(n) + f(n)$ if s is even. To make α and β non-K-trivial, we ensure that for every s there is some n, $n_s \leq n < n_{s+1}$, such that either $K(\alpha \restriction n) > K(n) + s$, or $K(\beta \restriction n) > K(n) + s$ depending or whether s is even or odd.

Construction. Stage 0: Let $\alpha_0 = \beta_0 = \emptyset$. Stage $s+1$: Suppose first that s is even. Let $\alpha'_{s+1} \supset \alpha_s$ be such that $K(\alpha'_{s+1}) \geq K(|\alpha'_{s+1}|) + s$. Such an α'_{s+1} must exist because not every extension of α_s is K-trivial$(s-1)$. Let C_{s+1} be such that $\alpha'_{s+1}\widehat{\ } 0^\omega$ is K-trivial(C_{s+1}). Choose $n_{s+1} > |\alpha'_{s+1}|$ such that $f(n_{s+1}) \geq C_{s+1}$. Finally, let $\alpha_{s+1} = \alpha'_{s+1}\widehat{\ } 0^\omega \restriction n_{s+1}$ and $\beta_{s+1} = \beta_s\widehat{\ } 0^\omega \restriction n_{s+1}$. If s is odd do the same as above but with roles of α and β reversed.

It is clear from the construction that for s even there is some n, $n_s \leq n < n_{s+1}$, such that $K(\alpha \restriction n) > K(n) + s$, namely $|\alpha'_{s+1}|$. Also, for every n, $n_{s+1} \leq n < n_{s+2}$,

$$\begin{aligned} K(\alpha \restriction n) = K(\alpha_{s+2} \restriction n) = K(\alpha_{s+1}\widehat{\ } 0^\omega \restriction n) &= K(\alpha'_{s+1}\widehat{\ } 0^\omega \restriction n) \\ &\leq K(n) + C_{s+1} \leq K(n) + f(n_{s+1}) \leq K(n) + f(n). \end{aligned}$$

Analogously for s odd. □

Proof of Lemma 1. Clearly (1) ⇒ (2) and (2) ⇒ (3) for any unbounded nondecreasing function. We now show that (3) ⇒ (1). That is, we construct an unbounded nondecreasing function f such that, for any real γ, if $(\forall n)\ K(\gamma \restriction n) \leq K(n) + f(n) + \mathcal{O}(1)$, then γ is K-trivial.

We first define an unbounded nondecreasing function f_0 such that $(\forall n)\ K(\gamma \restriction n) \leq K(n) + f_0(n)$ implies that γ is K-trivial(0). We do it by defining a sequence $n_0 < n_1 < n_2 < \cdots$, and letting $f_0(n) = k$ for every n such that $n_{k-1} < n \leq n_k$ (where $n_{-1} = -1$).

As there are only finitely many reals that are K-trivial(2), we can choose n_0 such that any γ that is K-trivial(2), but not K-trivial(0), has stopped appearing K-trivial(0) by n_0. Suppose now that we have already defined n_k. Let n_{k+1} be such that any γ that is K-trivial$(k+3)$, but not K-trivial(0), has stopped appearing K-trivial(0) by n_{k+1}. We can do this because there are only finitely many reals that are K-trivial$(k+3)$. Except when $k = 0$, we also require n_{k+1} to be such that any γ which stopped appearing K-trivial(0) at some m, $n_{k-1} < m \leq n_k$, does not appear to be K-trivial$(k+1)$ by n_{k+1}. Note that such n_{k+1} has to exist. Indeed, by definition of n_{k-1}, $\gamma \restriction m$ can have no K-trivial$(k+1)$ real extending it. So by König's Lemma, the tree of apparently K-trivial$(k+1)$ extensions of $\gamma \restriction m$ must be finite.

We claim that f_0 is as wanted. Suppose that γ is a real such that $(\forall n)\ K(\gamma \restriction n) \leq K(n) + f_0(n)$; we want to show that actually $(\forall n)\ K(\gamma \restriction n) \leq K(n)$. Clearly γ appears to be K-trivial(0) up to length n_0. Assume for a contradiction that γ is not K-trivial(0). Let $k > 0$ be least such that γ stops appearing K-trivial(0) at some m, $n_{k-1} < m \leq n_k$. Then by definition of n_{k+1}, γ stops appearing K-trivial$(k+1)$ by n_{k+1}. That means that there is some $m \leq n_{k+1}$ such that $K(\gamma \restriction m) \geq K(m) + k + 2 > K(m) + f_0(m)$, a contradiction.

There is nothing special about 0 in this proof. In the same way we can construct, for each i, a function f_i such that $f_i(0) = i$ and $(\forall n)\ K(\gamma \restriction n) \leq K(n) + f_i(n)$ implies that γ is K-trivial(i). Just choose n_0 such that any γ that is K-trivial$(i+2)$, but not K-trivial(i), has stopped appearing K-trivial(i) by n_0. Then given n_k, let n_{k+1} be such that any γ that is K-trivial($i+k+3$), but not K-trivial(i), has stopped appearing K-trivial(i) by n_{k+1}. For $k \neq 0$, also require n_{k+1} to be such that any γ which stopped appearing K-trivial(i) at some m, $n_{k-1} < m \leq n_k$, does not appear to be K-trivial($i+k+1$) by n_{k+1}. Let $f_i(n) = i + k$ for every n such that $n_{k-1} < n \leq n_k$.

For each $n \in \omega$, let $f(n) = \min\{f_{2i}(n) - i : i \in \omega\}$, which exists because $(\forall i, n) f_{2i}(n) - i \geq i$. Note that f is a nondecreasing function. It is also unbounded because for each j, if we let n be such that $(\forall i < j) f_{2i}(n) > 2j$, then $j \leq f(n)$. Now, suppose that γ is a real such that $(\forall n)\ K(\gamma \upharpoonright n) \leq K(n) + f(n) + i$ for some i. Then $(\forall n)\ K(\gamma \upharpoonright n) \leq K(n) + f_{2i}(n)$, and hence γ is K-trivial($2i$). So every γ such that $K(\gamma \upharpoonright n) \leq K(n) + f(n) + \mathcal{O}(1)$ is K-trivial. □

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