BOREL STRUCTURES: A BRIEF SURVEY

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Abstract. We survey some research aiming at a theory of effective structures of size the continuum. The main notion is the one of a Borel presentation, where the domain, equality and further relations and functions are Borel. We include the case of uncountable languages where the signature is Borel. We discuss the main open questions in the area.

1. Introduction

When looking at structures of size continuum from an effective viewpoint, the following definition is a natural generalization of ideas from computable model theory.

Definition 1.1. Let X be either 2^ω, ω^ω or $\mathbb{R}$, and let $\mathcal{C}$ be a (complexity) class of relations on X. A *$\mathcal{C}$-presentation of a structure $\mathcal{A}$* is a tuple of relations $\mathcal{S} = (D, E, R_1, \dots, R_n)$ such that

- All $D, E, R_1, \dots, R_n$ are in $\mathcal{C}$;
- $D \subseteq X$ and E is an equivalence relation on D (D is called the *domain*);
- $R_1, \dots, R_n$ are relations compatible with E.

$\mathcal{S}$ is a $\mathcal{C}$-representation of $\mathcal{A}$ if $\mathcal{A} \cong \mathcal{S}/E$. When E is the identity on D, we say that $\mathcal{S}$ is an *injective* $\mathcal{C}$-presentation of $\mathcal{A}$.

There are various possible choices for $\mathcal{C}$. In this paper we concentrate on the case that $\mathcal{C}$ is the class of Borel relations. Given a topological space X as above, the σ-algebra of *Borel sets* is the smallest σ-algebra containing the open sets. That is, the Borel sets are the ones obtainable from the open sets by closing under complementation, countable unions, and intersections. A structure $\mathcal{A}$ is a *Borel structure* if it has a Borel presentation. The number of classes of a Borel equivalence relation is either countable or $2^{\aleph_0}$ (Silver's Theorem; see [Hjo11]). Thus, the same statement holds for the sizes of Borel structures. We give some examples of Borel structures.

1. The fields $(\mathbb{R}, +, \times)$ and $(\mathbb{C}, +, \times)$ are Borel structures.
2. All Büchi automatic structures (see [HKMN08]) are Borel structures.
3. The Boolean algebra $\mathcal{B}$ which is $(\mathcal{P}(\mathbb{N}), \subseteq)$ modulo finite differences of sets is a Borel structure.

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(4) For a countable structure in a countable functional signature, the lattice of substructures, the congruence lattice, and the automorphism group are Borel structures.

In fact, structures of size at most the continuum one finds in books related to analysis or algebra are usually Borel. In contrast, the well-ordering $(2^{\aleph_0}, \leq)$ is not Borel. For assume it is. Let $\mathcal{S}$ be a Borel presentation. The class $\mathcal{G}$ of linear orderings of $\mathbb{N}$ which embed in $\mathcal{S}$ is $\mathbf{\Sigma}^1_1$. On the other hand, it is exactly the class of countable well-orderings, and hence Π^1_1 complete (for each Π^1_1 class $\mathcal{C} \subseteq \mathcal{P}(\omega)$, there is a total Turing functional Ψ such that $X \in \mathcal{C} \leftrightarrow \Psi(X) \in \mathcal{G}$). Contradiction. The same argument shows that $(\omega_1, \leq)$ is not Borel.

History. Borel structures were first considered by Friedman in unpublished work dating from the late 1970s; [Ste85b] refers to Friedman's unpublished notes [Fri78, Fri79]. After that, they appear in a few papers till the late 1990s. In the last few years, the authors, together with Hjorth and Khoussainov [HKMN08, HN11], brought the topic up again, prompted by a question on Büchi automatic structures. After describing some of the earlier work, we survey this more recent research. Many questions, and even whole research directions, remain open.

Friedman [Fri78, Fri79] studied a logical system where the language is enriched by one of the following quantifiers: "for all but countably many x...", "for all x in a co-meager set ...", or "for all x in a full-measure set ...". Borel structures are very appropriate to model logics that use these quantifiers. Friedman then studied axiomatizations, completeness, decidability, etc., in these extended languages. A survey including all these results was written by Steinhorn [Ste85b]. In [Ste85a] he continued to work in this direction.

Further relevant work was on Borel linear orderings. Some very interesting results were obtained. Harrington and Shelah [HS82] showed that for every Borel linear ordering $\mathcal{A}$ there exists $\xi < \omega_1$ such that $\mathcal{A} \preceq 2^\xi$, where 2^ξ is ordered by the lexicographical ordering and $\preceq$ is the embeddability relation. As a corollary, no Borel linear ordering contains a copy of ω_1 or ω_1^*.

Later on, Louveau [Lou89], extending work of Marker, showed the following unexpected result: For every Borel linear ordering $\mathcal{A}$ and $\xi < \omega_1$, either $\mathcal{A} \preceq 2^{\omega\cdot\xi}$ or $2^{\omega\cdot\xi+1} \preceq \mathcal{A}$. Another surprising result is that under the assumption of hyperprojective determinacy, the Borel suborderings of $\mathbb{R}^\omega$ are well-quasi-ordered under $\preceq$. This last result, due to Louveau and Saint-Raymond [LSR90], shows how one can obtain interesting properties of a class of structures if one eliminates the pathological cases and restricts oneself to Borel structures.

There was also a considerable amount of work on Borel partial orderings. For example, Harrington, Marker and Shelah [HMS88] showed that every thin Borel partial ordering can be written as the countable unions of Borel chains (to be thin means that there is no uncountable Borel antichain). Kanovei [Kan98] studied under which conditions a Borel partial ordering has a Borel linearization.

The more recent work of Hjorth, Khoussainov, Montalbán and Nies [HKMN08] used Borel structures to answer a question on injective presentations that was originally posed only for Büchi presentable structures. The paper of Hjorth and Nies [HN11] concentrates on theories of Borel structures in uncountable languages.

2. Effective content of the completeness theorem

The completeness theorem states that each consistent first-order theory T has a model $\mathcal{M}$ no larger than the size of the language. In this section we will study the effective content of this theorem.

Let us first recall the computable case (where the language is countable). Every computable complete theory has a computable model. On the other hand, there is a computable theory without a computable model. Thus, the completeness theorem in the computable setting fails because there are computable theories without computable completions.

We will now look at the completeness theorem in the Borel setting, and still for a countable language. Of course, each countable structure is Borel, so in this case the completeness theorem works for Borel structures. On the other hand, interesting Borel structures have size the continuum. When Friedman introduced Borel structures he obtained the following result.

Theorem 2.1. [Fri78, Ste85b] *Every theory in a countable language with infinite models has an injective Borel model of size $2^{\aleph_0}$. The model can be chosen so that its elementary diagram is Borel.*

Sketch of a proof. Extend T to a complete theory T_1 in some countable language $L_1 \supseteq L$ such that T_1 has Skolem functions. (See [Chang, Keisler; Section 3.3].) Consider constants

$$C = \{c_x : x \in \omega \times \mathbb{R}^{\geq 0}\},$$

where $\mathbb{R}^{\geq 0}$ denotes the non-negative reals. Order these constants as $\mathbb{R}^{\geq 0}$ many copies of ω. Let $\mathcal{U}$ be an L_1-model of T_1 which has C as a set of order indiscernibles and such that every element of $\mathcal{U}$ is a term in the language L_1 using constants from C. Such a model $\mathcal{U}$ is obtained as in [Chang, Keisler; Thm 3.3.11].

Let $\mathcal{U}_0$ be the elementary submodel of $\mathcal{U}$ generated by $C_0 = \{c_x : x \in \omega \times \{0\}\}$. Note that $\mathcal{U}_0$ is countable. Using the theory S of $\mathcal{U}_0$ as a real parameter, we will construct an injective presentation of $\mathcal{U}$ that is $\Delta^1_1(S)$, and hence Borel.

Let $\mathbb{T}$ be the set of all the terms in the language L_1 with constants from C substituted for the free variables. We define an equivalence relation on $\mathbb{T}$ by

$$t_0(\bar{d}_0) \equiv t_1(\bar{d}_1) \iff \mathcal{U} \models t_0(\bar{d}_0) = t_1(\bar{d}_1),$$

where $\bar{d}_0$ and $\bar{d}_1$ are tuples from C. This equivalence relation is $\Delta^1_1(S)$: to tell whether $t_0(\bar{d}_0) \equiv t_1(\bar{d}_1)$, we can consider tuples of constants $\bar{e}_0$ and $\bar{e}_1$ from C_0 which are in the same order as $\bar{d}_0$ and $\bar{d}_1$. Then we have that $t_0(\bar{d}_0) \equiv t_1(\bar{d}_1) \iff \mathcal{U}_0 \models t_0(\bar{e}_0) = t_1(\bar{e}_1)$. In a similar way we can calculate the effect of functions and relations of L_1 over the equivalence classes of terms in $\mathbb{T}$. It is then clear that the Borel presentation with domain $\mathbb{T}/\equiv$ is isomorphic to $\mathcal{U}$.

Now we want to build an injective $\Delta^1_1(S)$ presentation of $\mathcal{U}$. We show that the equivalence relation $\equiv$ on $\mathbb{T}$ has a Borel choice function. Consider some enumeration of the terms in L_1 (without using the constants from C). So, for every $u \in \mathbb{T}$ there is a least term t_0 in this enumeration such that $\mathcal{U} \models u = t_0(\bar{d})$ for some constants $\bar{d} \in C$. However, there might be many possible choices for $\bar{d}$. Let n be the length of the tuple $\bar{d} = (d_1, ..., d_n)$. We order C^n lexicographically (so it has order type $(\omega \times \mathbb{R}^{\geq 0})^n$).

Claim 2.2. For every $u \in \mathbb{T}$ and each term t_0 as above there is a least tuple $\bar{e}$ such that $u \equiv t_0(\bar{e})$. Further, the term $t_0(\bar{e}) \in \mathbb{T}$ can be chosen in a Borel way.

Without loss of generality, suppose $u = t_0(d_1, ..., d_n)$ with $d_0 \leq d_1 \leq ... \leq d_n \in C$. First we claim that there exists a least e such that $\mathcal{U} \models u = t_0(e, \bar{c})$ for some $\bar{c} \in C^{n-1}$ where $e \leq c_1 \leq \ldots \leq c_n$. If there is such an e in C_0, there is clearly a least one. Otherwise there is a unique such e: if $\mathcal{U} \models t(e_1, \bar{c}_1) = t(e_2, \bar{c}_2)$ for $e_1 < e_2$, then, since the constants in C are order indiscernibles, for any $e \in C_0$ with $e \leq e_1$ we have that $\mathcal{U} \models t(e, \bar{c}) = t(e_2, \bar{c}_2)$ where $\bar{c}$ is obtained from $\bar{c}_1$ by changing the occurrences of e_1 to occurrences of e. In either case, there is a least such e; call it e_0. Now fix e_0. Note that the linear order induced on $\{x \in C \colon x \geq e_0\}$ is isomorphic to C. Hence, by a similar argument as for e_0, there is a least e_1 such that $\mathcal{U} \models u = t_0(e_0, e_1, \bar{c})$ for some $\bar{c} \in C^{n-2}$ where $e_0 \leq e_1 \leq c_2 \leq \ldots \leq c_n$, and so on. In this way we obtain $\bar{e}$ as desired.

To verify the second part of the claim, it suffices to show that the set of $\bar{e} \in C^n$ which are the least ones determining a value of t_0 is Borel. Suppose there exists $\bar{c}$ which is below $\bar{e}$ in C^n such that $\mathcal{U} \models t_0(\bar{e}) = t_0(\bar{c})$. This can happen if and only if there exist $\bar{e}_1, \bar{e}_2 \in C_0^n$ which are ordered in the same configuration as $\bar{e}, \bar{c}$ such that $\mathcal{U}_0 \models t_0(\bar{e}_1) = t_0(\bar{e}_2)$. Since we are using the theory S of $\mathcal{U}_0$ as a parameter, this property is Borel. □

By a *Borel automorphism* of a structure with an injective Borel representation we mean an automorphism of the structure with a Borel pre-image on $D \times D$ where D is the domain of the presentation. For instance, conjugation is a Borel automorphism of the field of complex numbers with the natural presentation. The structure with the Borel presentation obtained in Theorem 2.1 has many Borel automorphisms. Thus we obtain:

Corollary 2.3. *Every theory in a countable language with infinite models has an injective Borel model of size* $2^{\aleph_0}$ *with* $2^{\aleph_0}$ *many Borel automorphisms.*

Proof. There are $2^{\aleph_0}$ many Borel automorphisms g of the linear order $\omega \times \mathbb{R}^{\geq 0}$ (obtained by extending automorphisms of the linear order of the positive rationals). Each automorphism g of this linear order extends to an automorphism $\widehat{g}$ of the structure $\mathcal{U}$: $\widehat{g}$ is well-defined via the equation

$$\widehat{g}(t(\bar{d})) = t(g(\bar{d}))$$

for each term $t \in L_1$ and each tuple $\bar{d}$ from C. If g is Borel then so is $\widehat{g}$ for the injective presentation of $\mathcal{U}$ obtained above. For, in the setting of Claim 2.2 , if t_0 is a term and $\bar{e}$ is the least tuple such that $u \equiv t_0(\bar{e})$, then $g(\bar{e})$ is the least tuple such that $\widehat{g}(u) \equiv t_0(g(\bar{e}))$. □

A natural question is what happens to the completeness theorem in the Borel setting with the language the size of the continuum. A little more care is needed here with the basic definitions. We follow [HN11]. For generality, we allow arbitrary Polish spaces as domains. Thus, *Borel set* will mean a Borel subset of some Polish space. A *Borel signature* is a Borel set $\mathcal{L}$ of function and relation symbols (coded for instance as reals) such that the arity function is Borel.

Using prefix (or Polish) notation one can naturally identify formulas in the resulting first-order language with finite strings in

$$\mathcal{L} \cup \{\neg, \vee, \wedge, \forall, \exists, v_0, v_1, ...\},$$

where $v_0, v_1, ...$ are our variable symbols. The collection of well-formed first-order formulas, $\mathcal{L}_{\omega,\omega}$, is a Borel subset of

$$(\mathcal{L} \cup \{\neg, \vee, \wedge, \forall, \exists, v_0, v_1, ...\})^{<\omega}.$$

Let $\mathcal{L}$ be a Borel signature. Then a *Borel first-order theory in* $\mathcal{L}$ is a Borel subset T of $\mathcal{L}_{\omega,\omega}$. It is not hard to see that the closure under logical inference of a Borel theory is analytical, but may fail to be Borel.

Definition 2.4. Suppose a Borel signature $\mathcal{L}$ has been fixed. Let $\mathcal{M}$ be an $\mathcal{L}$ structure with domain M. We say that $\mathcal{M}$, together with a Polish space X and a Borel equivalence relation $E \subset X \times X$, is a *Borel presentation* if

$$M = X/E = \{[x]_E : x \in X\},$$

and $\{(a_0, ..., a_{n-1}, R) \in X^n \times \mathcal{L} :$

$$R \text{ is an } n\text{-ary relation symbol of } \mathcal{L} \ \& \ \mathcal{M} \models R([a_0]_E, ..., [a_{n-1}]_E)\}$$

is Borel as a subset of $X^n \times \mathcal{L}$;
further, $\{(a_0, ..., a_{n-1}, b, f) \in X^{n+1} \times \mathcal{L} :$

$$f \text{ is an } n\text{-ary function symbol of } \mathcal{L} \ \& \ \mathcal{M} \models f([a_0]_E, ..., [a_{n-1}]_E) = [b]_E\}$$

is Borel as a subset $X^{n+1} \times \mathcal{L}$. We say that a structure $\mathcal{N}$ is *Borel* if there is a Borel presentation $\mathcal{M}$ which is isomorphic to $\mathcal{N}$.

We will usually denote presentations as $(X, E; ...)$ where $\mathcal{M} = X/E$ and the $(...)$ refers to the interpretations of the various non-logical symbols of $\mathcal{L}$.

For a structure over a finite language, being Borel in the sense of the present definition is clearly equivalent to being Borel in the sense of Section 1. If the language is uncountable, the present definition is more restrictive than merely requiring that each individual relation or function be Borel: the relations and functions need to be "uniformly Borel".

For instance, the fields $\mathbb{R}, \mathbb{C}$ in the extended language with names for all elements and for all continuous functions from the field to itself are Borel structures.

Just as in the computable setting, we can't always find a completion of a Borel theory that is Borel. This result of [HN11] first came up in work related to [HKMN08].

Theorem 2.5. *There exists a consistent Borel theory with no Borel completion.*

Proof. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. We will consider a Borel subtheory of the atomic diagram of the structure $(P(\mathbb{N}), \mathcal{U})$, such that any model of it codes a free ultrafilter on $\mathbb{N}$. The existence of a Borel completion of this theory would contradicts the easy fact that there are no free Borel ultrafilters on $\mathbb{N}$: on the one hand such a filter would have measure 1/2. On the other hand, being closed under finite variants it would have measure 0 or 1 by the 0-1 law. Also see [Kec95, Exercise 8.50].

To give more detail, the signature of our theory contains a unary predicate U, and for each $A \subseteq \mathbb{N}$ a constant symbol c_A. The theory contains the axioms $c_A \neq c_B$, for every $A \neq B \subseteq \mathbb{N}$. Furthermore, it contains axioms saying that U determines a free ultrafilter as far as the elements named by the c_A are concerned. Thus the theory contains the following axioms: $U(c_{\mathbb{N}})$; $U(c_A) \to U(c_B)$, for every pair of sets such that $A \subseteq B \subseteq \mathbb{N}$; $U(c_A) \leftrightarrow \neg U(c_{\mathbb{N} \setminus A})$, for every $A \subseteq \mathbb{N}$; $U(c_A)$ & $U(c_B) \to U(c_{A \cap B})$, for every $A, B \subseteq \mathbb{N}$; $\neg U(c_A)$, for each finite set $A \subseteq \mathbb{N}$.

Clearly, this theory is Borel. The theory is consistent, because it has the model $(P(\mathbb{N}), \mathcal{U})$ extended by constants naming each subset of $\mathbb{N}$. If T is a completion of our theory which is Borel, then

$$\{A \subseteq \mathbb{N} : T \models U(c_A)\}$$

is a Borel free ultrafilter. Contradiction. □

The theory above also does not have any Borel model $\mathcal{X}$: otherwise, $\{A \subseteq \mathbb{N} : c_A^{\mathcal{X}} \in U^{\mathcal{X}}\}$ would be a Borel free ultrafilter on $\mathbb{N}$.

Even if we have a complete theory, the Borel version of the completeness theorem fails. This contrasts with the computable case.

Theorem 2.6 (Hjorth, Nies [HN11])**.** *There exists a complete and consistent Borel theory which has no Borel model.*

This theorem relies on a well known fact from descriptive set theory. For a proof see Example 1.6 in [Hjo11].

Fact 2.7. *There is no Borel function $F\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that*

$$X =^* Y \Leftrightarrow F(X) = F(Y)$$

for each $X, Y \subseteq \mathbb{N}$.

Broadly speaking, to prove Theorem 2.6 one builds a theory such that any Borel model would contain a function contradicting the foregoing fact. Part of the difficulty is that one also has to rule out non-injective Borel presentations.

A *Borel algebraic closure* of a Borel field F of characteristic m is a Borel model of the complete theory which is axiomatized by the atomic diagram of F together with ACF_m. In other words, one embeds F in a Borel way into an algebraically closed Borel field.

The usual construction of an algebraic closure uses a construction of Artin akin to the proof of the completeness theorem (see [Lan65]). In particular, one needs the axiom of choice when the given field is uncountable.

Question 2.8. Does each Borel field F have a Borel algebraic closure?

A negative answer would yield a further example, somewhat more natural than Theorem 2.6, of a complete Borel theory without a Borel model.

3. Borel presentability

In the computable case there is no need to differentiate between injective and non-injective presentations: if E is a computable equivalence relation, then the quotient under E can be represented computably by taking the first (in $\mathbb{N}$) element of each equivalence class. However, this is not possible in the Borel case.

Theorem 3.1 (Hjorth, Khoussainov, Montalbán, Nies [HKMN08])**.** *There is a Borel structure in a finite language without an injective Borel presentation.*

They actually build a Büchi automatic structure without an injective Borel presentation. Recall from the introduction that $\mathcal{B}$ denotes the Boolean algebra $\mathcal{P}(\mathbb{N})$ modulo finite differences. The structure built in the proof of Theorem 3.1 is the disjoint union of the Boolean algebras $\mathcal{P}(\mathbb{N})$ and $\mathcal{B}$, together with the canonical projection map $p\colon \mathcal{P}(\mathbb{N}) \to \mathcal{B}$. They apply Fact 2.7 to show that this structure has no injective Borel presentation.

While not very complicated, this example had to be built for the proof; it is not a structure that appears naturally in mathematics. Such a structure would be obtained by answering the following question in the negative.

Question 3.2. Does the Boolean algebra $\mathcal{P}(\mathbb{N})$ modulo finite differences have an injective Borel presentation?

4. Borel dimension

To say that two Borel presentations are equivalent, it is not enough to use the classical notion of isomorphism. We need Borel isomorphism.

Definition 4.1. Two Borel presentations $(X, E; ...), (Y, F; ...)$ are said to be *Borel isomorphic* if there is an isomorphism $\Phi : X/E \to Y/F$ such that the preimage on $X \times Y$

$$\widehat{\Phi} = \{\langle x, y\rangle :\ \Phi([x]_E) = [y]_F\}$$

is Borel.

Borel isomorphism is easily verified to be an equivalence relation on Borel presentations; for transitivity, one uses the Lusin separation theorem to show that the composition of two isomorphisms with Borel preimage also has a Borel preimage.

We could also introduce a slightly stronger notion of Borel isomorphism where we require in 4.1 that both Φ and its inverse are induced by Borel functions on the domains. In many examples of Borel presentations the equivalence relation has only countable classes. In this case, the two definitions are equivalent by the Lusin-Novikov uniformization theorem (see [Kec95]).

A Borel structure $\mathcal{M}$ is *Borel categorical* if any two Borel presentation of it are Borel isomorphic. More generally, one can define the *Borel dimension* of a Borel structure $\mathcal{M}$ to be the number of equivalence classes modulo Borel isomorphism on the set of Borel presentations of $\mathcal{M}$. This is analogous to the notion of computable dimension in the area of recursive model theory. It was suggested by Bakhadyr Khoussainov.

Note that $\mathcal{M}$ is Borel categorical if and only if it has Borel dimension 1. Examples of Borel categorical structures are:

(1) The linearly ordered set $(\mathbb{R}, \leq)$.
(2) The Boolean algebra $(\mathcal{P}(\mathbb{N}), \subseteq)$.
(3) The field $(\mathbb{R}, +, \times)$.

In fact, for these examples, *each* isomorphism between two Borel presentations of the structure has a Borel graph.

An example of a non-Borel categorical structure is the group $(\mathbb{R}, +)$, as shown in [HKMN08]. In [HN11] the stronger result was obtained that its Borel dimension is $2^{\aleph_0}$. To see this, for a real $p > 1$ recall the Banach space

$$\ell^p = \{\vec{x} \in \mathbb{R}^{\mathbb{N}} : \sum_n |x_n|^p < \infty\},$$

where the norm is $|\vec{x}|_p = (\sum_n |x_n|^p)^{1/p}$. Let G_p be (the canonical injective Borel presentation of) the abelian group underlying ℓ_p. Clearly, as abstract groups these are isomorphic for all p, being vector spaces of dimension $2^{\aleph_0}$ over $\mathbb{Q}$. However, they are not Borel isomorphic. For, any isomorphism between the group structure of two Polish groups that is Borel must be a homeomorphism. (See for instance [BK96, Section 1.2] or [Kec95, Thm. 9.10]; note that each Borel map is Baire measurable.)

Hence it would be linear. But for $1 < p < q$ there is no continuous linear bijection between ℓ^p and ℓ^q. See [LZ96, top of pg. 54].

A further example of a non-Borel categorical structure is given by the following result of Nies and Shore.

Theorem 4.2. *The field* $(\mathbb{C}; +, \times)$ *is not Borel categorical even for injective Borel presentations.*

Proof. Let $T = ACF_0$ be the theory of algebraically closed fields of characteristic 0. Then T is ω_1-categorical, so every model of size $2^{\aleph_0}$ is classically isomorphic to the field $\mathbb{C}$.

By Corollary 2.3, T has an injective Borel model of size $2^{\aleph_0}$ with $2^{\aleph_0}$ many Borel automorphisms. On the other hand, any Borel automorphism of $\mathbb{C}$ with the natural presentation is continuous by the result mentioned in the foregoing proof. The only continuous automorphisms of $\mathbb{C}$ are conjugation and identity. Hence the two injective Borel presentations are not Borel isomorphic. □

Nies and Shore actually gave a direct construction of an injective Borel model of ACF_0, of size $2^{\aleph_0}$, and with $2^{\aleph_0}$ many Borel automorphisms. Let B be an uncountable closed set of algebraically independent reals. Then the real closure of B in $\mathbb{R}$ is Borel. The first step is adding the roots of odd-degree polynomials with coefficients in B. One can identify the elements with the polynomials and their roots in order; the roots are computable in the coefficients. Now one iterates the process. The real closed subfield of $\mathbb{R}$ constructed in this way has an injective Borel presentation. Finally one adjoins a solution to $X^2 = -1$ to obtain the required Borel model of ACF_0. There are $2^{\aleph_0}$ many Borel automorphisms for this presentation. They are induced by the Borel permutations of B.

If one chooses B as in the proof of Theorem 5.2 below (namely, reals whose binary presentations are the paths on a perfect tree T such that the effective disjoint union of finitely many paths is arithmetically generic), then by an argument similar to the one given below, the field one obtains is actually Borel as a subfield of $\mathbb{C}$.

The following question remains open.

Question 4.3. Is there a Borel structure of Borel dimension strictly between 1 and $2^{\aleph_0}$?

5. Borel models of Peano arithmetic

Recall that a set $\mathcal{S} \subseteq 2^\omega$ is a *Scott set* if $\mathcal{S}$ is closed downwards under Turing reducibility, closed under joins, and each infinite binary tree $T \in \mathcal{S}$ has an infinite path in $\mathcal{S}$. Scott sets occur for instance in reverse mathematics as the ω-models of WKL_0.

Let M be a model of PA. The *standard system* of M consists of the standard parts of M-definable sets. Thus, the standard system is the class

$$\{D \cap \omega \colon\ D \subseteq M \text{ is parameter definable in } M\}$$

(here we think of M as extending ω).

For $n \in \mathbb{N}$ let p_n denote the n-th prime. Let M be a nonstandard model of PA. It is well-known [Kay91] that each set in the standard system has the form $\{n \in \mathbb{N} \colon\ p_n \mid a\}$ for some $a \in M$.

Scott (see [Kay91, Section 13.1]) showed that the countable Scott sets are precisely the standard systems of countable models of Peano arithmetic. Knight and

Nadel [KN82] proved the analoguous result for the size ω_1. For the size 2^ω, the analogous statement is open. Motivated by this, H. Woodin asked the following effective version of this question.

Question 5.1. If a Scott set is Borel, is it already the standard system of a Borel model of Peano arithmetic?

For an upper bound on the complexity, note that the standard system of any Borel model of PA is analytic.

A *jump ideal* is an ideal K in the Turing degrees that is closed under the jump. Note that the sets with degree in K form a Scott set. Thus the following yields an uncountable non-trivial Scott set that is Borel.

Theorem 5.2 (Slaman, 2010)**.** *There is a proper uncountable jump ideal K in the Turing degrees that is Borel.*

Proof. K is the jump ideal generated by the degrees of paths on a perfect tree T such that the effective disjoint union of finitely many paths is arithmetically generic. This jump ideal is proper because it does not contain the degree of $\emptyset^{(\omega)}$.

We sketch the argument why K is Borel. For a simple case, consider whether X is recursive in some path G in T via the Turing functional Φ. Let $G \in [T]$. For each n, the value of $\Phi(n, G)$ is determined by a finite initial segment of G, including the value "undefined" since G is generic.

So the set of $G \in [T]$ such that $\Phi(G) = X$ is a $\Pi^0_1(X \oplus T)$ subset of $[T]$. By the compactness of Cantor space, whether this set is nonempty is arithmetic in T and X. The definition of this set only depends on Φ and T. Thus, the set of X such that X is recursive in a path through T by functional Φ is arithmetic in T.

Note that $G^{(n)} \equiv_T G \oplus \emptyset^{(n)}$ for each arithmetically generic G. Similar to the argument above, one can show that it is arithmetic in T whether X is recursive in a sequence of paths of a fixed length and $0^{(n)}$ by functional Φ. The jump ideal generated by $[T]$ is the countable union of these sets, so it is Borel, too. $\square$

A recent preprint of Ali Enayat focuses on Borel models of PA. In particular, he has probed the "Borel content" of a result of Schmerl [KS06, Theorem 6.4.3] to show that Theorem 5.2 above can be strengthened: there is a proper uncountable Borel jump ideal K in the Turing degrees that can be realized as the standard system of some model of PA.

It would be interesting to determine which Borel upper semilattices with least element and the countable predecessor property are isomorphic to Borel (or analytic) ideals of the Turing degrees.

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