Poisson brackets, Groupoids and General Relativity

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Abstract: The solutions of Dirac's constraint equations in the 3+1 formulation of Einstein's equations in general relativity form a coisotropic subvariety in the cotangent bundle of a space of metrics on a 3-dimensional manifold. This situation resembles that for the zero set of a momentum map for a hamiltonian action, but the formalism does not work when one tries to use the group Diff(M) of diffeomorphisms of a space-time M as the symmetry group. What seems to be more relevant for this problem is the groupoid DH(M) of diffeomorphisms between all pairs of hypersurfaces in M. Christian Blohmann (Regensburg), Marco Cezar Fernandes (Brasilia), and I have found several groupoids and Lie algebroids related to DH(M) which reproduce the Poisson brackets between Dirac's constraint functions. In these lectures, I will give introductions to the variational and hamiltonian formulations of the Einstein equations and to the theory of Lie algebroids and Lie groupoids, after which I will describe the use of symmetry groupoids and their Lie algebroids in relativity.

(purely "formal" — no hard analysis)

Outline

The equations and variational principle
Evolution form and constraints; Poisson brackets
Constraints as momentum map?
Diffeomorphism groupoids, Lie algebroids
Σ-universes, microverses, and

THE EQUATIONS

$\boxed{Ric(g) = 0}$ for g a metric on a 4-manifold

HILBERT ACTION PRINCIPLE

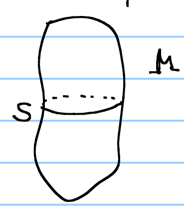
$$\delta \int_M R(g)\, vol_g = 0$$

FORMAL VARIATIONAL CALCULUS

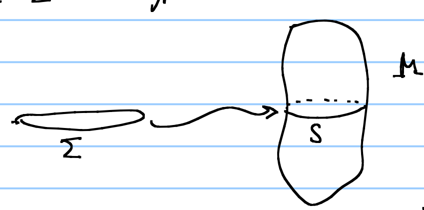
(Integration as cohomology)

EVOLUTION EQUATIONS

Evolution ("3+1") formalism due to Dirac, Arnowitt-Deser-Misner (1959), ... in terms of initial data on a space-like Cauchy surface S.



$\rightsquigarrow$ Hamiltonian system on $T^*(\mathrm{Met}\,\Sigma)$, where Σ is "typical time slice."



Initial data (γ, π): γ is riemanian metric on Σ, π is symmetric 2-form (-density) on Σ which may be interpreted as second fundamental form with respect to embedding in some M.

MAIN ISSUE : (γ, π) cannot be freely prescribed; they are subject to <u>constraints</u>

ENERGY $$\frac{1}{2}(\mathrm{Tr}_\gamma \pi)^2 - \mathrm{Tr}_\gamma \pi^2 + 2R(\gamma) = 0$$

MOMENTUM $$\mathrm{div}\big((\mathrm{Tr}_\gamma \pi)\gamma - \pi\big) = 0$$

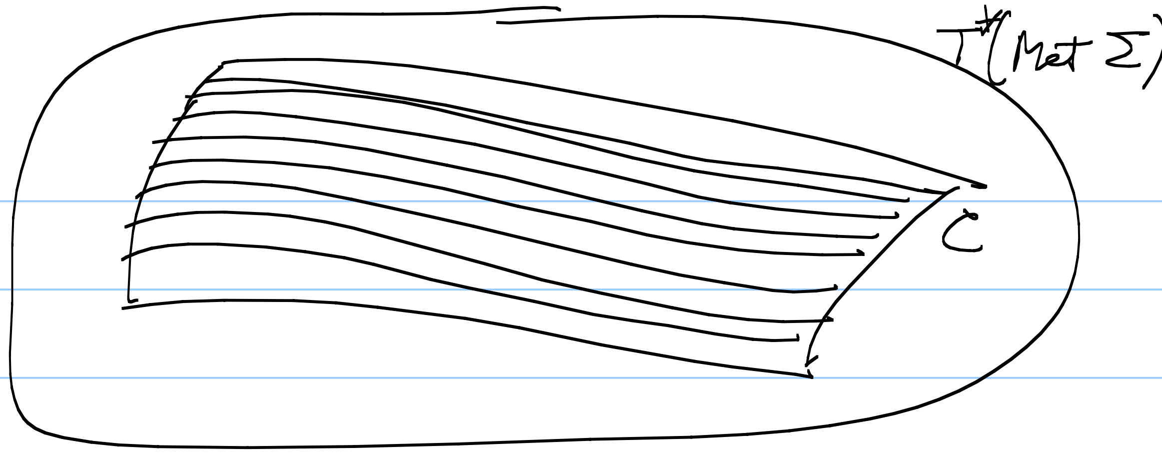
[explain; S=hamiltonian, this is pointwise]

Why the constraints?
- Coduzzi equations
- Degenerate lagrangian

Geometrically, the constraints describe a <u>coisotropic subvariety</u> $\mathcal{C} \subseteq T^*(\mathrm{Met}\ \Sigma)$. This will follow from Poisson bracket relations below.

The variety $\mathcal{C}$ has a <u>characteristic foliation</u> whose leaves define the "evolution of the gravitational field in space-time."

The evolution is multidimensional because "time is many fingered"



Roughly speaking, the points on a given leaf correspond to views of a given Ricci-flat metric at different "instants", i.e. space-like hypersurfaces.

The picture is complicated by the presence of isometries, which give singularities of $\mathcal{C}$

GOAL To realize $\mathcal{C} \subseteq T^*\mathrm{met}(\Sigma)$ as the zero set of a moment(um) map for the action of a symmetry group. This would open the way to using symplectic reduction, applications to quantization, etc. [cf. lectures of Y. Karshon]

Not only is $\mathcal{C}$ coisotropic, but it has (Arms-Gotay-Marsden) quadratic singularities like a momentum-zero variety.

Momentum zero sets for hamiltonian actions

$P \xrightarrow{J} \mathfrak{g}^*$ momentum map for G action on P. G-equivariance $\overset{(G \text{ connected})}{\Longleftrightarrow}$ Poisson map

If 0 is a regular value, then $\mathcal{C} = J^{-1}(0)$ is coisotropic, and G acts locally freely and transitively on the characteristic submanifolds. (Leaf spaces are then symplectic orbifolds.)

Characteristic foliation $\approx$ conormal bundle

$\Longrightarrow$

To recover the action from $\mathcal{C}$, could try to identify conormal spaces with a Lie algebra.
Lie algebra bracket comes from Poisson bracket on a vector space of independent constraint functions.

Let's try this on the (regular part of) the constraint manifold $\mathcal{C}$ defined by:

$$\boxed{\begin{gathered} \frac{1}{2}(\mathrm{Tr}_\gamma \pi)^2 - \mathrm{Tr}_\gamma \pi^2 + 2R(\gamma) = 0 \\ \mathrm{div}((\mathrm{Tr}_\gamma \pi)\gamma - \pi) = 0 \end{gathered}}$$

By pairing the constraints above with functions φ and vector fields ξ on Σ, we get real valued constraint functions C_φ and C_ξ on $T^*\mathrm{Met}(\Sigma)$ which are the "defining functions" of $\mathcal{C}$.

$$\boxed{\begin{aligned} C_\varphi &= \int_\Sigma \left(\tfrac{1}{2}(\mathrm{Tr}_\gamma \pi)^2 - \mathrm{Tr}_\gamma \pi^2 + 2R(\gamma)\right)\varphi \, \mathrm{vol}_\gamma \\ C_\xi &= \int_\Sigma \langle \mathrm{div}((\mathrm{Tr}_\gamma \pi)\gamma - \pi), \xi\rangle_\gamma \, \mathrm{vol}_\gamma \end{aligned}}$$

A Poisson bracket formula for these functions was found shortly after the appearance of Dirac's (1958) paper by a French physicist named Katz. His relations (rediscovered many times — present form by de Witt) are:

$$\boxed{\begin{aligned} \{C_\xi, C_\eta\} &= C_{[\xi,\eta]} \qquad \text{semidirect product, or automorphisms of trivial line bundle} \\ \{C_\xi, C_\varphi\} &= C_{\xi\cdot\varphi} \\ \{C_\varphi, C_\psi\} &= C_{\varphi\,\mathrm{grad}\,\psi - \psi\,\mathrm{grad}\,\varphi} \end{aligned}}$$

$$= C_{\gamma^\flat(\varphi\, d\psi - \psi\, d\varphi)} \qquad (\gamma^\flat : T^*\Sigma \to T\Sigma)$$

The function C_1 is "the hamiltonian for time evolution"; but ....

The brackets are metric dependent. If we freeze the metric, we find a failure of Jacobi:

$$\{C_\xi, \{C_\varphi, C_\psi\}\} + \circlearrowleft = (\mathcal{L}_\xi \gamma)^\flat (\varphi\, d\psi - \psi\, d\varphi)$$

(except when ξ is an isometry of γ).

Main problem. Understand the geometry behind these brackets.

The "candidate Lie algebra":
(functions $\oplus$ vector fields) looks like 4-vector fields. This suggests $\mathrm{Diff}(M)$ as the symmetry group. But $\mathrm{Diff}(M)$ does not act on $\mathrm{Met}(\Sigma)$ or $T^*\mathrm{Met}(\Sigma)$.
[Also, our 4-vector fields are defined on the 3 manifold Σ, not on M.]

There is a better interpretation of these 4-vector fields along 3-manifolds.

$\mathcal{E} = \mathcal{E}(\Sigma, M) =$ embeddings $\Sigma \xrightarrow{e} M$

$T_e\mathcal{E} =$ vector fields along $e = \Gamma(e^* TM)$.

But where is the Lie algebraic structure?
(where is the 3+1 splitting?)

$\mathcal{D} = \mathrm{Diff}(\Sigma)$ acts from the right.

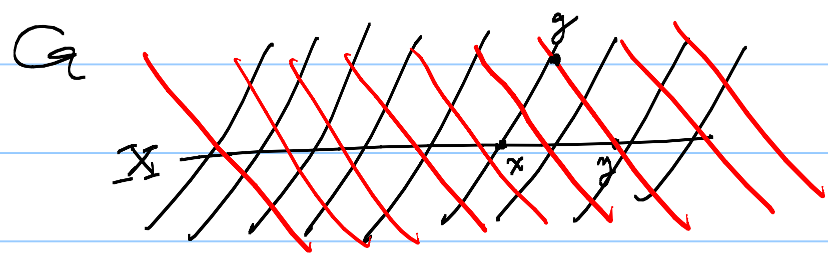
$\mathcal{E}/\mathcal{D} \simeq \mathcal{H} = \mathcal{H}_\Sigma(M) =$ hypersurfaces in M diffeomorphic to Σ.

$\mathcal{E}\times\mathcal{E}/\mathcal{D} =$ "gauge groupoid" $\simeq \mathcal{DH}_\Sigma(M)$

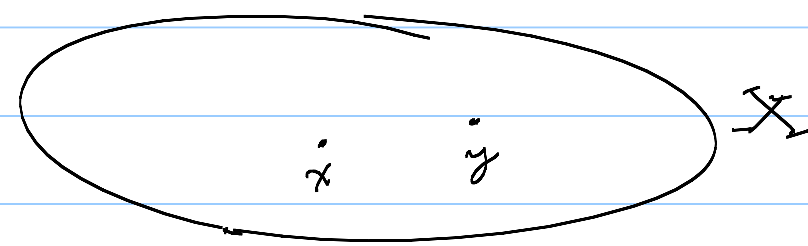
$=$ diffeomorphisms between hypersurfaces.

NEW GOAL. Establish $\mathcal{DH}_\Sigma(M)$ as the fundamental symmetry group<u>oid</u> of canonical general relativity.

GROUPOIDS



$$\begin{tikzcd} G \arrow[d, shift right, "l"'] \arrow[d, shift left, "r"] \\ X \end{tikzcd}$$



Composition $r(g) = l(h) \Rightarrow gh$ defined, $l(gh) = l(g)$, $r(gh) = r(h)$ ‖ Units, Inverses

<u>Lie groupoid</u>: l + r submersions, everything differentiable.

LIE ALGEBROIDS

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d] & TX \arrow[dl] \\ X & \end{tikzcd} \qquad [\ ,\ ] \text{ on } \Gamma(A)$$

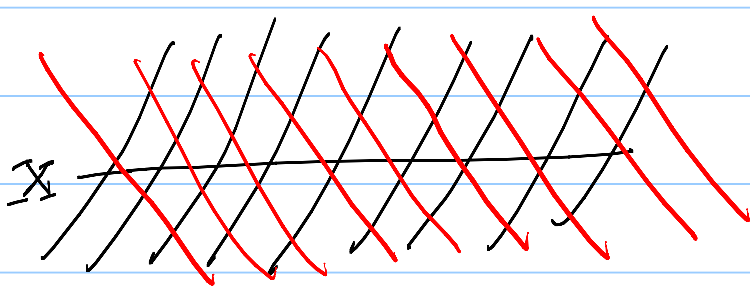
$$[a, fb] = f[a,b] + \underline{\qquad\qquad}$$

A is like "generalized tangent bundle of X."

Dual bundle A^* has natural Poisson structure, "linear on fibres", just as in special case $X = \{pt\}$.

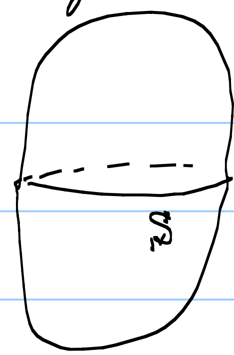
Given a Lie groupoid $\begin{tikzcd} G \arrow[d, shift right, "s"'] \arrow[d, shift left, "r"] \\ X \end{tikzcd}$

G



$A = \nu(X, G)$ is Lie algebroid, with $\Gamma(A)$ identified with left-invariant vector fields, $\rho = T\mathbf{r}$.

What is the Lie algebroid of $\mathcal{DH}_\Sigma(M)$?



A_S = 4-vector fields along S $= \Gamma(T_S M)$

(Given $e : \Sigma \to M$, identify with $e^* T_S M$.)

This is a perfectly good Lie algebroid, but its bracket (Poisson bracket) lives on sections rather than fibres (~~total~~ space rather than fibres.) Restriction to fibres requires something like a connection on A (or A^*), since the value $[a,b](x)$ depends on the 1-jets of a and b at x.

We need to introduce <u>metric</u> information.

Σ-universes, microverses, and blinks (and ?-?)

Given a 3-manifold Σ, consider cooriented embeddings $\Sigma \to M$ onto space-like hypersurfaces in Lorentz $(+++-)$ manifolds. We declare two such embeddings $\Sigma \to M$ and $\Sigma \to M'$ equivalent if there is a compatible isometry $M \to M'$. A <u>Σ-universe</u> is an equivalence class of embeddings. <u>$\mathcal{U}(\Sigma)$</u> is all of them

(Note that the isometry is <u>unique</u>.)

<u>A groupoid $G \rightrightarrows \mathcal{U}(\Sigma)$</u>

Morphisms are equivalence classes (in the sense above) of <u>pairs</u> of embeddings.

$$\begin{tikzcd}[row sep=small] & M \\ \Sigma \arrow[ur, "i'"] \arrow[dr, "i"'] & \\ & M \end{tikzcd}$$

The fibres at i of the Lie algebroid are naturally isomorphic to $\mathfrak{F}(\Sigma)\oplus\mathfrak{X}(\Sigma)$. The brackets on constant sections reproduce the Katz-deWitt relations! How is this related to constraints?

A <u>Σ-microverse</u> is a germ of a Σ-universe around Σ. Each of these has a unique gaussian representative given by a <u>germ of paths</u> of metrics on Σ.

<u>Germs</u>:

<u>Gaussian representation</u>:

Although the groupoid $G \rightrightarrows \mathcal{U}(\Sigma)$ does not descend to $m\mathcal{U}(\Sigma)$; its Lie algebroid does. We can compute the action of $\mathfrak{F}(\Sigma)\oplus\mathfrak{X}(\Sigma)$ by the process of <u>gaussian extension</u>. (This is how we recover the deWitt–Katz brackets.)

Can we push down further under the "1-jet" map $m\mathcal{U}(\Sigma) \longrightarrow T^*(\mathrm{Met}\,\Sigma)$ which takes each microverse to the induced metric on Σ and the 2nd fund. form?

One way to do this would be via a section

$$[B\mathcal{U}(\Sigma)=]\ T^*(\mathrm{Met}\,\Sigma) \longrightarrow m\mathcal{U}(\Sigma),$$ i.e. a way of extending first order data along Σ to local data.

The Einstein equations give this along C; what about elsewhere ??????

We interpolate between $\mathcal{MU}(\Sigma)$ and $T^*\mathcal{M}(\Sigma)$
the "nanoverses" $\mathcal{NU}(\Sigma)$ = "formal microverses".
The Lie algebroid G pushes down easily to these, and then we may use a section $T^*\mathcal{M}et(\Sigma) \to \mathcal{NU}(\Sigma)$ given by formally solving the evolution part of the Einstein system without imposing the constraints.

At this point, we need to know that the image of $T^*\mathcal{M}et\,\Sigma$ is invariant under our Lie algebroid on $\mathcal{NU}(\Sigma)$. Proving this uses the variational formulation of the evolution equations. Since time is "formal", we must use the formal variational calculus, since there is no actual integration over a formal variable.

WHAT NEXT?

(1) Symplectic reduction in this context?
(2) Arms-Marsden-Moncrief singularity theorem?
(3) Coordinates for analysis?
(4) Quantization?
(5) Physical meaning of space time

Dirac wrote, "I am inclined to believe...that four-dimensional symmetry is not a fundamental property of the physical world."

Pirani, writing about Dirac's paper in *Mathematical Reviews,* writes, "This reviewer finds it difficult to concur".