Note Title 7/4/2008

MODULAR CLASSES AND THE VOLUME OF A DIFFERENTIABLE STACK

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Use of the term "modular" in the sense of today's talk was first introduced in the theory of locally compact groups (by A. Weil?), based on the following "ordinary language" word.

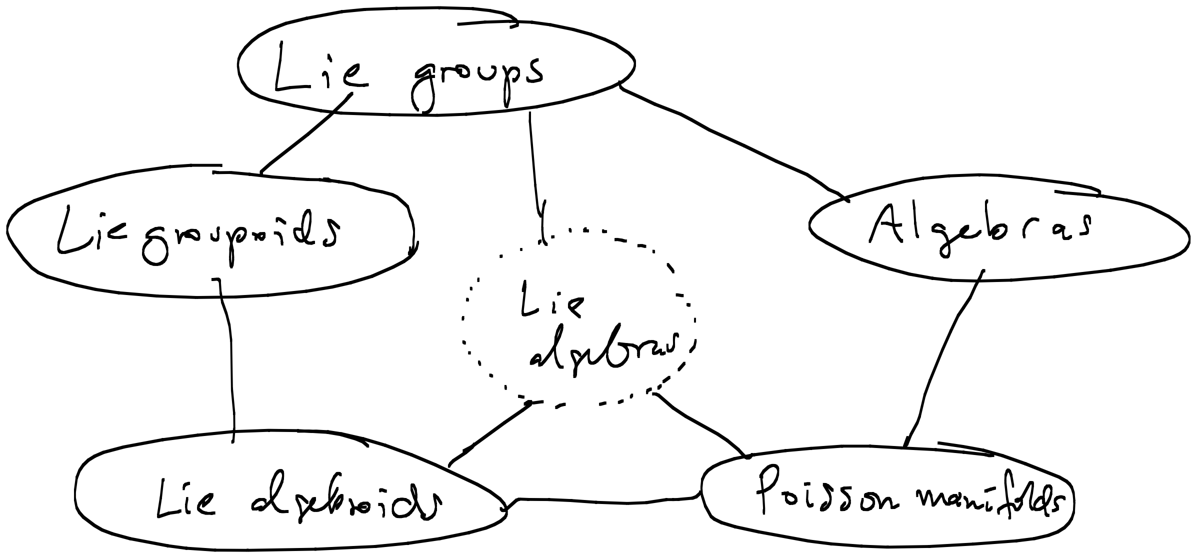
mod·ule

n.

1. A standard or unit of measurement.

2. *Architecture* The dimensions of a structural component, such as the base of a column, used as a unit of measurement or standard for determining the proportions of the rest of the construction.

From groups, it migrated to other mathematical objects.


Lie groups
Lie groupoids
Algebras
Lie algebras
Lie algebroids
Poisson manifolds

Modular (cohomology class) is obstruction to existence of some kind of "nice measure" (volume rather than length.)

- Groups :
- Associative algebras :
- Poisson manifolds :
- Lie algebras :
- Lie algebroids :
- Lie groupoids :

This talk:

- Modular classes of objects (with SE + J-H L)
- Modular classes of mappings (with YKS + CL-G)
- What can you do when the obstruction vanishes?
- The case of Poisson (including symplectic) manifolds.

Groups

Obstruction to bi-invariant measure: function $H \xrightarrow{\operatorname{mod} H} \mathbb{R}^\times$.

Given inclusion $K \xrightarrow{\Phi} H$

$\operatorname{mod} \Phi = \operatorname{mod} K / \Phi^* \operatorname{mod} H$.

$\longrightarrow$ H/K <u>K - invariant measure</u>

$\operatorname{mod} H = \det(\Lambda^{top} Ad)$ (H acting on $\Lambda^{top} \mathfrak{h}$)

Lie algebras $\quad \mathrm{tr}\,\mathrm{ad} : \mathfrak{h} \longrightarrow \mathbb{R}$

Algebras $\quad$ obstruction to a trace

$A \longrightarrow \mathbb{C} \qquad \mu(ab) = \mu(ba).$

Tomita-Takesaki-Connes

modular automorphism group

data: "state" $\quad \mu : A \to \mathbb{C}$

dependence upon data: change μ $\leadsto$ change mod by
inner automorphisms.

"Reduces" to usual in case of group algebra;
(evaluation at identity) $\qquad e^{\mathrm{mod}\, H}$

Poisson manifolds

(earlier by Gallavotti–Pulvirenti; Koszul)

Quasi-classical limit of algebra case

$f \overset{X_b}{\longmapsto} \operatorname{div} H_f$ modular vector field.

data: smooth measure b

dependence on data: $b \to \gamma b$, $\boxed{X_{\gamma b} = X_b + H_{\log|\gamma|}}$.

Reduces to standard one for $\mathfrak{g}^*$. (Relation to group algebra: "Fourier transform".)

Lie algebroids (with Evens and Lu)

$$\begin{tikzcd} A \arrow[rr, "\rho"] \arrow[dr] & & TM \arrow[dl] \\ & M & \end{tikzcd}$$

What are the data?

$$\boxed{Q_A = \Lambda^{top} A \otimes \Lambda^{top} T^*M}$$

Action (representation) of $\Gamma(A)$ by bracket.

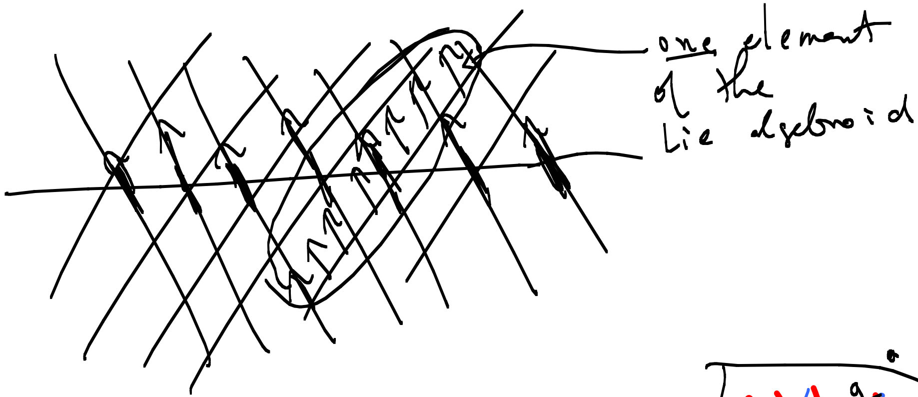
$\lambda \in \Gamma(Q_A) \rightsquigarrow$ modular cocycle $\alpha_\lambda \in \Gamma(A^*)$.

dependence on data: $\alpha_{\gamma\lambda} = \alpha_\lambda + d_A \log|\gamma|$

Special cases: Lie algebra, Poisson. $\longleftrightarrow \boxed{(T^*P)^{\otimes 2}}$

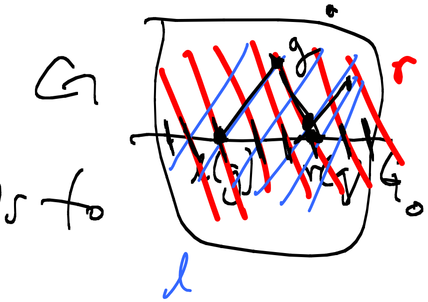
$\longrightarrow$ Dual of Lie algebroid.

$\underbrace{\text{(Trace obstruction: Neumaier-Waldmann)}}$



Groupoids

Very much like Lie algebroids.



$\boxed{G \text{ acts on } Q_A}$, section leads to cocycle $G \longrightarrow \mathbb{R}^\times$.

Note: does not reduce to Lie algebra(oid) when group(oid) is not connected. Can be nontrivial cocycle even for discrete groupoid. (Invariance condition on function on G_0.)

Relative classes (with Kosmann-Schwarzbach + Laurent-Gengoux)

$$\begin{tikzcd}
K \arrow[d, shift left] \arrow[d, shift right] & A \arrow[r, "\Phi"] \arrow[d] & B \arrow[d] & H \arrow[d, shift left] \arrow[d, shift right] \\
M & M \arrow[r, "\phi"] & N & N
\end{tikzcd}$$

A acts on $Q_\Phi = Q_A \otimes \Phi^!(Q_B^*) =$ "$Hom_\Phi(Q_B, Q_A)$"

$\mathrm{Mod}\,\Phi = \mathrm{Mod}_A - \Phi^* \mathrm{Mod}_B =$ characteristic class of Q_Φ

Meaning?

Special case:

Pullback morphism by surjective submersion = equivalence. Inverting these gives generalized morphisms, in particular, equivalences.

(In Lie algebroid case, need 1-connected fibres.)

Relative class defined for "morphisms of stacks."

$$\begin{tikzcd} & \bullet \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ \bullet & & \bullet \arrow[ll] \end{tikzcd}$$

Stacks

$$G \overset{l}{\underset{r}{\rightrightarrows}} G_0, \quad \boxed{G_0/\!\!/G} = \boxed{BG},$$

$$G_0/\!\!/G \longrightarrow G_0/G$$

$$H \rightrightarrows \bullet \qquad \bullet/\!\!/H$$

Stack is "groupoid up to equivalence."

Discrete case: G isom $\coprod_i X_i \times X_i \times H_i \rightrightarrows \coprod_i X_i$

equiv $\coprod_i H_i \rightrightarrows \coprod_i \{i\}$

Baez-Dolan define $\quad \mathrm{card}(G) = \sum_i 1/\#(H_i)$ $\qquad \#$

I prefer $\#(G_0/\!\!/G) = \#(BG) = \sum_i 1/\#(H_i)$

$$= \sum_i \#(\{i\}/\!\!/H_i)$$

In particular, $\#(pt/\!\!/H) = 1/\#(H)$.

($pt/\!\!/H$ is "$\underbrace{1/\#(H)\text{th}}$ of a point")

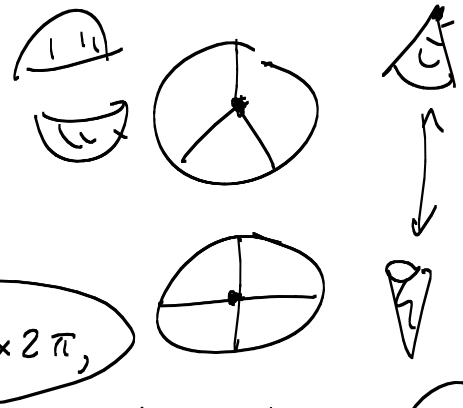
Also, $\#(X/\!\!/H) = \#(X)/\#(H)$, whatever the action.
(follows easily from a lemma below).

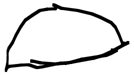
And $\#(\text{finite sets}/\!\!/\text{bijections}) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots = \underline{\underline{e}}$

Matches dfn. of Euler characteristic of orbifold.

<u>Example:</u>

$\chi\Big($  $\Big) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$



volume is  $\frac{2\pi}{3} + \frac{2\pi}{4} = \frac{7}{12} \times 2\pi,$

curvature $K = 1$, $\frac{1}{2\pi}\int K \, dVol = \frac{1}{2\pi}\left(\frac{7}{12} \cdot 2\pi\right) = \frac{7}{12} = \chi.$

PROBLEM: Extend notion of volume to more general stacks, e.g. quotients $X /\!\!/ H$, where H is a Lie group acting on a manifold H. (Don't just push measure forward!)

Corresponding groupoid here is

$$G = \underline{H} \times \underline{X} \rightrightarrows X \qquad \underline{(h,x) \overset{(l,r)}{\longmapsto} (hx, x)}.$$

$$A = \mathfrak{h} \times X \to X$$

$$\underline{Q_A = \Lambda^{top}\mathfrak{h} \otimes \Lambda^{top} T^*X}$$

Section of $\underline{Q_A}$ is suitable data, e.g. in free case, can make sense of $\underline{\underline{\mathrm{vol}(X)/\mathrm{vol}(H)}}$.

General Lie groupoid

Lemma: For $G \rightrightarrows G_0$ finite,

$$\#(G/\!\!/G_0) \stackrel{\circ}{=} \sum_{\theta \in G/G_0} \#(G_\theta)^{-1} = \sum_{y \in G_0} \#(r^{-1}(y))^{-1}.$$

This suggests, for $G \rightrightarrows G_0$ compact Lie, given $a \in \Gamma \Lambda^{top} A^*$, $b \in \Gamma \Lambda^{top} T^* G^0$

$$\mathrm{vol}_{a,b}(G_0/\!\!/G) = \int_{y \in G_0} \underbrace{\left(\int_{r^{-1}(y)} a_r \right)^{-1}}\, b.$$

Here we use identification of $\Gamma(A)$ with left-invariant sections of $\ker Tr \subseteq TG$; a_r is right-invariant "extension" of A to left-invariant volume element along r-fibres.

Problem: How do we know that this depends only on $a^{-1} b \in \Gamma(Q_A)$.

Solution: Go back to discrete case; try to imitate the original Baez-Dolan construction. It turns out that we need to assume $\lambda = a^{-1}b \in \Gamma(Q_A)$ is an invariant section.

We can then do the following:

Use exact sequence:

$$0 \to \ker\rho \to A\xrightarrow{\rho} TG_0 \to \operatorname{cok}\rho \to 0$$

$$\ldots\ldots$$

to get $Q_A \simeq \boxed{\Lambda^{top}\ker\rho \otimes \Lambda^{top}(\operatorname{cok}\rho)^*.}$

Now work over regular part of G, G_0.

The rest is ignorable thanks to Zung's linearization theorem (and slice theorem --)

Write $a^{-1}b = \alpha^{-1}\beta$, where

$\alpha \in \Gamma(\Lambda^{top} \ker \rho^*)$ and $\beta = \Gamma(\Lambda^{top} \operatorname{cok} \rho^*)$

are separately invariant. (Can do this because

G compact $\Rightarrow G/G_0$ proper, and vanishing

theorem [Tu? Crainic?].)

Now, recalling

$$\#(G/\!/G_0) = \sum_{\theta \in G/G_0} \#(G_\theta)^{-1} = \sum_{y \in G_0} \#(r^{-1}(y))^{-1}$$

prove

$$\int_{\theta \in G/G_0} \left[\int_{G_\theta} \alpha\right]^{-1} \beta = \int_{y \in G_0} \left[\int_{r^{-1}(y)} a_\ell\right]^{-1} b \,.$$

The expression on the left does not depend on α & β except through $\alpha^{-1}\beta = a^{-1}b$, so we get a well-defined $\mathrm{vol}_\lambda(G_0/\!/G)$, even for proper (not neccessarily compact) groupoids.

Interpretation of the relative class

$$\begin{array}{ccc} G & \xrightarrow{\Phi} & G' \\ \downdownarrows & & \downdownarrows \\ G_0 & & G_0' \end{array}$$ generalized morphism

induces map of stacks $G_0/\!/G \xrightarrow{\Phi} G_0'/\!/G'$.

Modular class mod Φ is obstruction to <u>integration along fibres</u> of $\underline{\Phi}$.

<u>Example:</u>

$$\begin{array}{ccc} K & \subseteq & H \\ \downarrow & & \downarrow \\ pt & & pt \end{array}$$

leads to map of stacks

$$pt/\!/K \to pt/\!/H,$$

or $BK \to BH$.

If we think topologically, $BH = EH/H$, BK can be EH/K, and we have fibration $H/K \to \underbrace{BK \to BH}$,

so measure along fibres is "really" measure on H/K, which takes us back to A. Weil's criterion.

The Poisson case

For Poisson manifold P, $A = T^*P$ is naturally associated Lie algebroid and $Q_A = (\wedge^{top} T^*P)^{\otimes 2}$.

($\Rightarrow$ Modular class of P is $2 \times \mathrm{Mod}\, A$.)

$G \rightrightarrows P$ is symplectic groupoid. What is $\mathrm{vol}(P/\!/G)$ (i.e. volume of symplectic leaf space)?

For symplectic P, a natural choice of λ is square of Liouville measure. For fundamental groupoid,

$\mathrm{vol}_\lambda(P/\!/\pi_1(P)) = 1/\#(\pi_1(P))$

(for P connected, otherwise sum over components).

For pair groupoid, $\mathrm{vol}_\lambda = 1$ (or $\#\pi_0(P)$).

Other invariant λ's are (locally constant) multiples of Liouville measure and give rise to <u>different</u> volumes.

A simple Poisson example

$M = \Sigma \times \mathbb{R}$, where Σ is a compact symplectic surface, symplectic structure varying with t so that, in dual coordinate θ on isotropy, length of isotropy is $K(t)$ (proportional to rate of change of symplectic volume of Σ with respect to t).

Choose $\lambda = [\omega_t \wedge \underline{f(t)\,dt}]^2$, where ω_t is symplectic form. Then the induced measure on the leaf space turns out to be

$$\frac{f(t)^2}{K(t)}\,dt.$$

On the other hand, the "quotient measure" of $\sqrt{\lambda}$ by symplectic leaf measure is just $\underline{f(t)\,dt}$; these agree when $f(t) = K(t)$, thus producing a "natural measure" on the symplectic leaf space.

I think that this is the measure associated to the natural $GL(n;\mathbb{Z})$ structure on the quotient space of an integrable system.

SOME QUESTIONS

For G acting on G by conjugation, there is a canonical trivialization of Q_A for the action groupoid, hence a canonical measure on $\boxed{G//G}$, and its push-forward to $\boxed{G/G}$. What is it? How is it related to canonical measures on conjugacy classes?

same for $\mathfrak{g}//G$.

How about $\mathfrak{g}^*//G$? Relation to Plancherel measure?

What else can we do with these ideas? (Relation to duality theories in algebraic geometry, D-modules?)