Introduction to Floer Homology and its relation with TQFT

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ABSTRACT. Floer theory is one of the most active fields in mathematical physics. In this expository paper, we will discuss where this theory comes from and what it is as well as its relation with TQFT.

§1 Foundation of Symplectic Geometry and Morse Homology

Historically, Eliashberg, Conley-Zehnder, Gromov respectively proved the Arnold conjecture for Riemann surfaces, $2n$-torus, the existence of at least one fixed point under $\pi_2(M) = 0$. Then Floer [F1,F2,F3,F4] made a breakthrough toward the Arnold conjecture. He first established the Arnold conjecture for Lagrangian intersections and symplectic fixed points (still under $\pi_2(M) = 0$). Then he used the variational method of Conley and Zehnder as well as the elliptic techniques of Gromov and Morse-Smale-Witten complex to develop his infinite dimensional approach to the Morse theory. Now this method is widely recognized as Floer theory.

We first give the background of the Arnold conjecture.

Let (M, ω) be a compact symplectic manifold. The symplectic form ω determines an isomorphism between T^*M and TM. Thus we can get a $Hamiltonian\ vector\ field$ $X_H : M \to TM$ from the exact form $dH : M \to T^*M$, where $H : M \to R$ is called a $Hamiltonian\ function$. We can write the above relation explicitly as $\iota(X_H)\omega = dH$. Then we extend the Hamiltonian function to a smooth time dependent 1-periodic family of Hamiltonian functions $H_{t+1} = H_t : M \to R$, for $t \in R$.

Consider the Hamiltonian differential equation

(1) $\dot{x}(t) = X_{H_t}(x(t))$

The solution of (1) generates a family of symplectomorphisms $\psi_t : M \to M$, s.t.

$\dfrac{d}{dt}\psi_t = X_t \circ \psi_t$, $\psi_0 = id$.

We find the fixed points of the map ψ_1 are in 1-1 correspondence with the 1-periodic solutions of (1) and denote such kind of solutions by $\mathcal{P}(H) = \{x : R/Z \to M | \dot{x}(t) = X_{H_t}(x(t))\}$

Definition 1.1 A periodic solution x is called $non-degenerate$ if the following identity holds.

(2) $\det(I - d\psi_1(x(0))) \neq 0$.

Now we can state the Arnold conjecture.

Conjecture 1.2 (Arnold) Let (M, ω) be a compact symplectic manifold and $H_t = H_{t+1} : M \to R$ be a smooth time dependent 1-periodic Hamiltonian function. Suppose that the 1-periodic solutions of (1) are all non-degenerate. Then

$\#P(H) \geq \sum_{i=0}^{2n} \dim H_i(M,Q)$, where $H_i(M,Q)$ denotes the singular homology of M with rational coefficients.

Remark: If H_t is independent of t, then the Arnold conjecture reduces to Morse theory. Because in that case, nondegeneracy condition of the 1-periodic condition implies the nondegeneracy of critical points for Hamiltonian function H, where it becomes Morse function.

It is a fact that the space of all compatible almost complex structures is nonempty and contractible. Then the first Chern class $c_1 = c_1(TM, J) \in H^2(M, Z)$ is independent of the choice of the compatible almost complex structure.

The Arnold conjecture can be proved under the assumption

(4) $\int_{S^2} v^* c_1 = \tau \int_{S^2} v^* \omega$ for every smooth map $v : S^2 \to M$ and some constant $\tau \in R$.

It is divided into three cases:

a) $\tau > 0$ positive curvature (2-spheres) the symplectic manifolds with this property is called *monotone*, which is the case treated by Floer [F5] and it led him to define the so called Floer homology.

b) $\tau = 0$ zero curvature (2-torus) Especially the case $c_1 = 0$ was treated by Hofer-Salamon [FHS, HS1], which can be considered as an extension of Floer's original work with the group coefficients in Novikov ring.

c) $\tau < 0$ negative curvature (high genus surface) The case of compact symplectic manifold was proved by Fukaya-Ono [FO], Liu-Tian [LT], and Hofer-Salamon.

In the following discussion, we need to make a further assumption that $\int_{S^2} v^* \omega \in Z$ for all smooth map $v : S^2 \to M$.

As Floer theory is actually a theory of extending the Morse theory to the infinite dimensional version, we must first familiar with the Morse theory, which have two approaches-classical one and newer one. We will mainly focus on the second approach, which leads to the Floer thoery.

Let M be a compact smooth Riemannian manifold and $f : M \to R$ be a Morse funciton. Now we need to introduce some symbol for further discussion.

$Crit(f) = \{x \in M | df(x) = 0\}$, $Hessian\ d^2 f(x) : T_x M \times T_x M \to R$

The Morse condition says that the Hessian for every critical points are nondegenerate.

Let $\varphi^s : M \to M$ denote the flow of the following (negative) gradient flow

(5) $\dot{u} = -\nabla f(u)$

Definition 1.3 The stable and unstable manifolds are defined as follows

$W^s(x; f) = \{y \in M | \lim_{s \to \infty} \varphi^s(y) = x\}$, $W^u(x; f) = \{y \in M | \lim_{s \to -\infty} \varphi^s(y) = x\}$

Definition 1.4 The Morse index of a critical point is the number of negative eigenvalues of the Hessian.

Here are some results from Morse theory.

Property 1.5 1) The stable and unstable manifolds are smooth submanifolds of M for every critical point x of f under the Morse condition.

2) The Morse index $ind_f(x) = \dim W^u(x; f)$.

Definition 1.6 The gradient flow is called a $Morse-Smale\ system$, if for any pair of critical points x, y of f, the stable and unstable manifolds intersect transversally. Denote $W^s(x; f) \cap W^u(y; f)$ by $\mathcal{M}(y, x; f)$.

Remark: The set $\mathcal{M}(y, x; f)$ is the space containing all the trajectories connecting y to x.

Property 1.7 For a Morse-Smale system, we have

1) The set $\mathcal{M}(y, x; f)$ is a smooth submanifold of M.

2) $\dim \mathcal{M}(y, x; f) = ind_f(y) - ind_f(x)$

3) Cosider the space of all the gradient flow lines running from y to x. Then $\widehat{\mathcal{M}}(y, x; f) = \mathcal{M}(y, x; f)/R$ is a manifold of dimension $ind_f(y) - ind_f(x) - 1$, where R acts on $\mathcal{M}(y, x; f)$ by translation.

Remark:

1) Now Morse-Smale condition tells us that $ind_f(y) - ind_f(x) > 0$ whenever there is a connecting orbit from y to x.

2) If $\dim \widehat{\mathcal{M}}(y, x; f) = 0$, then $\widehat{\mathcal{M}}(y, x; f)$ is a finite set.

In order to define the Morse homology, we need to discuss the orientation first.

Fix an orientation of the unstable manifold $W^u(y; f)$ for every critical point x of f, which gives rise to a natural orientation for every connecting orbit. Then there exists a vector space isomorphism

$T_z W^u(y; f) \cap \nabla f(z)^{\perp} \to T_x W^u(x; f)$ determined by $d\varphi^t(z)$ for large t.

Then we can define $\epsilon(z) = \pm 1$ respectively for the case of orientation preserving or reversing.

Definition 1.8 Assume the gradient flow of f is a Morse-Smale system and fix an orientation of $W^u(x)$ for every critical point x.

The free abelian group was defined as $CM_k(f) = \bigoplus\limits_{\substack{df(x)=0 \\ ind_f(x)=k}} Z\langle x \rangle$ and the boundary operator $\partial^M : CM_k(f) \to CM_{k-1}(f)$ is defined as

$\partial^M \langle y \rangle = \sum\limits_{\substack{df(x)=0 \\ ind_f(x)=k-1}} \sum\limits_{[u] \in \widehat{\mathcal{M}}(y,x;f)} \epsilon(u) \langle x \rangle$ for $y \in crit(f)$ and $ind_f(y) = k$.

The above complex is called the $Morse-Smale-Witten\ complex$.

Theorem 1.9 If the gradient flow of f is a Morse-Smale system, then the corresponding operator ∂^M satisfies the condition $\partial^M \circ \partial^M = 0$. There exists a natural isomorphism between this new homology and the singular homology.

Remark: Proving this theorem in Morse homology involves compactification by broken flow lines [H].

Now we have necessary foundation, which will be used quite often in the further discussion.

Then we return to the monotone case of the Hamiltonian differential equations.

Our target is to develop the corresponding homology theory as an extension of Morse homology.

Roughly speaking, at the very beginning, we need to interpret 1-periodic solutions of (1) as some critical "points" of some functional on the loop space $\mathcal{L}M$.

The loop space is defined as $\mathcal{L}M \triangleq \{x \in C^\infty(R \to M) | x(t+1) = x(t)$ $\forall t \in R\}$.

A tangent vector to loop space $\mathcal{L}M$ at a point (i.e. a loop in M) is a vector field along x.

Explicitly, we can write as

$\forall \xi \in T_x\mathcal{L}M \Rightarrow \xi : R \to TM$ satisfies $\xi(t) \in T_{x(t)}M$ and $\xi(t+1) = \xi(t)$ $\forall t \in R$

Thus we have $T_x\mathcal{L}M = C^\infty(R/Z, x^*TM)$, where x^*TM is the pullback bundle on R/Z.

For every 1-periodic Hamiltonian, there is a natural 1-form Ψ_H on the loop space $\mathcal{L}M$, defined as

$\Psi_H : T\mathcal{L}M \to R$, $\Psi_H(x;\xi) = \int_0^1 \omega(\dot{x}(t) - X_{H_t}(x(t)), \xi(t))dt \ \forall \xi \in T_x\mathcal{L}M$

Remark: This 1-form is actually a closed form.

1-form Ψ_H can be expressed as the differential of a functional $\alpha_H : \mathcal{L}M \to R/Z$ defined as

$\alpha_H(x, u) = -\int_B u^*\omega - \int_0^1 H_t(x(t))dt \ \forall x \in \mathcal{L}M$

where $u : B = \{z \in C||z| \le 1\} \to M$ is a smooth map such that $u(e^{2\pi i t}) = x(t) \ \forall t \in R$.

Remark:

1) Generally speaking, Ψ_H is not an exact form. Because such map u exist only when x is a contractible loop.

2)The former assumption $\displaystyle\int_{S^2} \upsilon^*\omega \in Z$ $\forall$ smooth map $\upsilon : S^2 \to M$ assures the image of α_H is well-defined in R/Z.

Floer's idea is to use this functional α_H taking over the role of the Morse function in Morse homology.

Thus we need to make another assumption that the 1-periodic solutions $x : R/Z \to M$ must be non-degenerate as the analogy in the Morse function. So now our task is to understand the gradient flow lines of this functional $\alpha_H : \mathcal{L}M \to R/Z$.

First step is to define a metric on the loop space as follows

$< \xi, \eta >= \int_0^1 < \xi(t), \eta(t) >_t dt$, where $< \xi(t), \eta(t) >_t = \omega(\xi(t), J_t\eta(t))$

Remark: 1) Here is the canonical way to define the Riemannian metric from symplectic form ω and compatible almost complex structure J_t.

2) Here we use a 1-periodic family of almost complex structures $J_t = J_{t+1} \in \mathcal{J}(M, \omega)$.

Then the gradient of this functional with respect to this metric can be computed explicitly as

$\text{grad}\alpha_H(x)(t) = J_t(x(t))\dot{x}(t) - \nabla H_t(x(t))$

Remark: $\text{grad}\alpha_H(x) \in T_x\mathcal{L}M$ and $\text{grad}\alpha_H$ is a vector field on $\mathcal{L}M$.

Just as in Morse homology, we need to introduce the gradient flow line of α_H located in $\mathcal{L}M$ not in M!

So we need another variable other than time t, then the equation of gradient flow line can be written as

$\dfrac{\partial u(s,\cdot)}{\partial s} + grad\alpha_H(u(s,\cdot)) = 0$, where $u : R \to \mathcal{L}M$ i.e. $u : R \times R/Z \to M$ and $u(s, t+1) = u(s)$

Combined with the expression of $grad\alpha_H$, we get the following equation for flow u

(7) $\dfrac{\partial u(s,t)}{\partial s} + J_t(u(s,t))\dfrac{\partial u(s,t)}{\partial t} - \nabla H_t(u(s,t)) = 0$

Remark:

1) If J, H and u are independent of t, this is the upward gradient flow of $H = H_t$.

2) If $u(s,t) \equiv x(t)$, (7) reduce to the Hamiltonian equation (1).

3) If $H_t \equiv 0$ and $J_t \equiv J$, then (7) become $\dfrac{\partial u(s,t)}{\partial s} + J(u(s,t))\dfrac{\partial u(s,t)}{\partial t} = 0$, which is actually the equation for $J-$holomorphic curves.

Now we introduce the energy of a solution of (7) in the following way.

$E(u) = \dfrac{1}{2}\displaystyle\int_0^1\int_{-\infty}^{\infty}\left(\left|\frac{\partial u(s,t)}{\partial s}\right|^2 + \left|\frac{\partial u(s,t)}{\partial t} - X_{H_t}(u(s,t))\right|^2\right) dsdt$

There are some important result for solutions of (7).

Proposition 1.10 Let $u : R \times R/Z \to M$ be a solution of (7). Then the following are equivalent.

a) $E(u) < \infty$

b) There exist $x^{\pm} \in \mathcal{P}(H)$ s.t.

(8) $\lim\limits_{s\to\pm\infty} u(s,t) = x^{\pm}(t)$

c) There exist constants $\delta > 0$ and $c > 0$ s.t. $|\partial_s u(s,t)| \leq ce^{-\delta|s|}$ $\forall s, t \in R$

Denote the space of all solutions of (7) and (8) by $\mathcal{M}(x^-, x^+) = \mathcal{M}(x^-, x^+; H, J)$.

Theorem 1.11 [S] There exist a subset $\mathcal{H}_{reg} = \mathcal{H}_{reg}(J) \subset C^{\infty}(M \times R/Z)$ of the second categorie in the sense of Baire s.t. the 1-periodic solutions of (1) are all non-degenerate, and the moduli space $\mathcal{M}(x^-, x^+; H, J)$ is a finite dimensional smooth manifold for all $x^{\pm} \in \mathcal{P}(H)$ and all $H \in \mathcal{H}_{reg}$. Furthermore, if (4) holds then there is a function $\eta_H : \mathcal{P}(H) \to R$ s.t. for $\forall u \in \mathcal{M}(x^-, x^+; H, J)$ the dimension of the moduli space is given by

(10) $\dim_u \mathcal{M}(x^-, x^+; H, J) = \eta_H(x^-) - \eta_H(x^+) + 2\tau E(u)$ locally near u.

To prove b) implying c) in Proposition 1.10 and Theorem 1.11 involves a lot of analysis. We will introduce it in the next section.

§2 Fredholm Theory, Conley-Zehnder index and spectral flow

We first give the outline of Fredholm theory and see how it works for our problem.

Definition 2.1 Let X and Y be two Banach spaces. A bounded linear operator $D : X \to Y$ is called a *Fredholm operator* if it has a closed range and the kernel and cokernel of D are both finite dimensional. The *index* of a Fredholm operator is defined as the difference of the dimensions of kernel and cokernel:

$index D = \dim \ker D - \dim co \ker D$.

Remark: The Fredholm property and the index are are stable under perturbations. If there is another compact linear operator $K : X \to Y$, then $D + K$ is again a Fredholm operator and it has the same index as D.

Definition 2.2 A smooth map $f : X \to Y$ between Banach spaces is called a *Fredholm map* if its differential $df(x) : X \to Y$ is a linear Fredholm operator for every $x \in X$. In this case it follows from the stability of the Fredholm index that the index of $df(x)$ is independent of the choice of x and we identify $index(f)$ with $index(df(x))$. A vector $y \in Y$ is called a *regular value* of f if $df(x) : X \to Y$ is onto for every $x \in f^{-1}(y)$. If y is a regular value, then $\mathcal{M} = f^{-1}(y)$ is a smooth finite dimensional manifold.

We must express such kind of moduli spaces $\mathcal{M}(x^-, x^+; H, J)$ as zero sets of functions between suitable Banach spaces in order to prove these moduli spaces are smooth finite dimensional manifolds.

Denote $\dfrac{\partial u}{\partial s} + J_t(u)\dfrac{\partial u}{\partial t} - \nabla H_t(u)$ by $\overline{\partial}_{H,J}(u)$, which is a vector field along u, then we fix an element $u \in \mathcal{M}(x^-, x^+)$and consider a vector space $\mathcal{X}_u \subset C^\infty(R \times R/Z, u^*TM)$ of all vector fields ξ along u which satisfy a suitable exponential decay condition as $s \to \pm\infty$.

A function near u which also satisfy the limit condition (8) can be expressed as $u' = \exp_u(\xi)$ for some $\xi \in \mathcal{X}_u$. Then the set of solutions of (7) and (8) can be expressed as the zero set of a function $\mathcal{F}_u : \mathcal{X}_u \to \mathcal{X}_u$, defined by $\mathcal{F}_u(\xi) = \Phi_u(\xi)^{-1}\overline{\partial}_{H,J}(\exp_u(\xi))\ \forall \xi \in \mathcal{X}_u$ and $\Phi_u(\xi) : T_uM \to T_{\exp_u(\xi)}M$ denotes the parallel transport along the geodesic $\tau \to \exp_u(\tau\xi)$.

Remark: Here $\exp_u(\xi)$ means the exponential map, explicitly it can be written as $u'(s,t) = \exp_{u(s,t)}(\xi(s,t)) \in M$ in the common sense.

One can compute the following identity by using the relation between the parallel tranport and connection

(11) $D_u\xi \triangleq d\mathcal{F}_u(0)\xi = \nabla_s\xi + J(u)\nabla_t\xi + \nabla_\xi J(u)\partial_t u - \nabla_\xi \nabla H_t(u)$

Then D_u is a Fredholm operator between L^p and $W^{1,p}$, which are Sobolev completions of $\mathcal{X}_u$. We pick a unitary trivialization $\Phi(s,t) : R^{2n} \to T_{u(s,t)}M$ which identifies the standard ω_0 and J_0 on R^{2n} with the corresponding ω and J on TM. Thus D_u has the form

(12) $D\xi = \partial_s \xi + J_0 \partial_t \xi + S\xi \ \forall \xi : R \times S^1 \to R^{2n}$

where the matrices $S(s,t) \in R^{2n \times 2n}$ are defined by $S = \Phi^{-1} D_u \Phi$.

Associated to this symmetric matrix valued function is a symplectic matrix valued function $\Psi : R \times R \to Sp(2n)$ given by

(13) $J_0 \partial_t \Psi + S\Psi = 0$, $\Psi(s,0) = 1$

Now we have the following theorem.

Theorem 2.3 [S] Suppose that $\det(1 - \Psi^{\pm}(1)) \neq 0$. Then the operator $D : W^{1,p}(R \times S^1; R^{2n}) \to L^p(R \times S^1; R^{2n})$ given by (12) is Fredholm for $1 < p < \infty$. Its Fredholm index is given by the difference of the Conley-Zehnder indices:

(14) $indexD = \mu_{CZ}(\Psi^+) - \mu_{CZ}(\Psi^-)$

Of course, we need to define the Conley-Sehnder index to make sense of Theorem 2.3.

Denote by $Sp^*(2n)$ the open and dense set of all symplectic matrices, which do not have 1 as an eigenvalue.

Denote by $SP(n)$ the space of paths $\Psi : [0,1] \to Sp(2n)$ with $\Psi(0) = 1$ and $\Psi(1) = Sp^*(2n)$. Any such path admits an extension $\Psi : [0,2] \to Sp(2n)$, s.t. $\Psi(s) \in Sp^*(2n)$ for $s \geq 1$ and $\Psi(2)$ is one of the matrices $W^+ = -1$ and $W^- = diag(2, -1, ..., -1, 1/2, -1, ..., -1)$.

Then $\rho^2 \circ \Psi : [0,2] \to S^1$ is a loop ($\rho(W^{\pm} = \pm 1)$), where $\rho : Sp(2n) \to S^1$ is a continuous extension of the determinant map $\det : U(n) = Sp(2n) \cap O(2n) \to S^1$.

Definition 2.4 The $Conley-Zehnder\ index$ of Ψ is defined as $\mu_{CZ}(\Psi) = \deg(\rho^2 \circ \Psi)$.

We don't want to prove the theorem here, just provide the background knowledge, because these are the important ideas to understand the Floer homology. Next task is to discuss the spectral flow.

We rewrite the operator $D = \partial_s + J_0 \partial_t + S$ as $D = \partial_s + A(s)$

where $A(s) : W^{1,2}(S^1; R^{2n}) \to L^2(S^1; R^{2n})$, $A(s) = J_0 \partial_t + S(s, \cdot)$.

This a smooth family of unbounded self-adjoint operators on the Hilbert space $H = L^2(S^1; R^{2n})$, which implies the limit operators $A^{\pm} = \lim_{s \to \pm\infty} A(s)$ are both invertible. In this case the Fredholm index of D is given by the spectral flow of A [APS], which we now define as follows.

Definition 2.5 A number $s \in R$ is called a *crossing* if $\ker A(s) \neq \{0\}$.

Definition 2.6 If s is a crossing then the $crossing\ form$ is the quadratic form $\Gamma(A,s) : \ker A(s) \to R$ defined as

$\Gamma(A,s)\xi = \left\langle \xi, \dot{A}(s)\xi \right\rangle_H \ \forall \xi \in \ker A(s).$

Definition 2.7 A crossing s is called regular if the crossing form is non-degenerate.

Definition 2.8 [RS] Consider a smooth family $A(s)$ with only regular crossings the *spectral flow* is defined by

$\mu^{spec}(A) = \sum_{s \text{ is a crossing}} sign\Gamma(A,s)$

Remark: Roughly speaking, the spectral flow is the number of eigenvalues of $A(s)$ crossing zero from negative to positive as s moves from $-\infty$ to $+\infty$.

§3 Floer Homology

In this part, we will define the Floer homology group for monotone case. Consider the following chain complex

$CF_k(H) = \bigoplus_{\substack{x\in\mathcal{P}(H) \\ \mu_{CZ}(x;H)=k (\mathrm{mod}\ 2N)}} F\langle x\rangle$

where F is a PID (Principal ideal domain)

Remark: We have a formula for the Conley-Zehnder index,

$\mu_{CZ}(x, A\sharp u) = \mu_{CZ}(x,u) - 2c_1(A)$, $A \in \pi_2(M)$.

Without specifying the map $u : B \to M$, the Conley-Zehnder index of a periodic solution $x \in \mathcal{P}(H)$ is only well-defined modulo $2N$.

In order to define the boundary operator on this chain complex, we need to first do some research on the orientation just like the situation in Morse theory. But now the situation is more complicated. Roughly speaking, we need to prove the moduli spaces $\mathcal{M}(x^-, x^+)$ are all orientable, and then to choose a system of so called "coherent orientations" under which Floer gluing maps are orientation preserving[FH]. Actually these orientation are not unique. We can define a number $\epsilon(u) \in \{1, -1\}$ for each $u \in \mathcal{M}^1(x^-, x^+; H, J)$ by comparing this coherent orientation of $\mathcal{M}^1$ with the flow orientation.

Now the Floer boundary operator is defined by

$\partial^F\langle y\rangle = \sum_{\substack{x\in\mathcal{P}(H) \\ \mu_{CZ}(x;H)=k-1 (\mathrm{mod}\ 2N)}} \sum_{[u]\in\widehat{\mathcal{M}}^1(x^-,x^+;H,J)} \epsilon(u)\langle x\rangle$

where y is a periodic orbit $y \in \mathcal{P}(H)$ with $\mu_{CZ}(y;H) = k(\mathrm{mod}\ 2N)$.

Remark:

1) There are a lot of technical details here. It is impossible to discuss all the stuff in this paper, so we only present the most fundamental idea and its relation with Morse homology.

2) The Conley-zehnder index $\mu_{CZ}(y;H)$ takes over the role of the Morse index in the Morse theory.

3) The quotient space $\widehat{\mathcal{M}}^1(x^-, x^+; H, J) = \mathcal{M}^1(x^-, x^+; H, J)/R$ is a finite set for every pair $x^\pm \in \mathcal{P}(H)$.

The proof of this statement involves a bubbling technique, which also appears in Morse theory and some analysis of the energy of a solution.

Theorem 3.1(Floer) If (M, ω) is monotone and $H \in \mathcal{H}_{reg}$ then $\partial^F \circ \partial^F = 0$.
Remark: We need to use Floer's gluing theorem to prove this theorem.

Now we can define the Floer homology groups of a a pair (H, J) as the homology of the chain complex $(CF_*(H), \partial^F)$ in the following way

$$HF_*(M, \omega, H, J; F) = \frac{\ker \partial^F}{\operatorname{Im} \partial^F}$$

Remark:

1) (Floer) Actually,these groups are invariant of almost complex structure and the Hamiltonian.

2) (Floer) These group are naturally isomorphic to the ordinary homology groups of M in the following sense

$$HF_k(M, \omega, H, J; F) \overset{\simeq}{\to} \bigoplus_{j=k(\text{mod } 2N)} H_j(M; F)$$

Now the Arnold conjecture for monotone symplectic manifolds follows directly from here.

§4 Relation with TQFT

The Floer's idea fitted in quite well with the "instanton invariants" of 4-dimensional manifolds. Roughly speaking, Floer homology is the data required to extend this theory from closed 4-manifolds to manifolds with boundary, which indicate that something relate to Quantum Field Theory. Anyway, one of the Floer's starting points can be traced back to the paper of Witten [W], which provides a link between Quantum Mechanics and Morse theory. The former properties of the Floer groups as well as their relation with the invariants of 4-manifolds are parts of TQFT (Topological Quantum Field Theories), which was developed by Segal, Atiyah, Witten and others. Then one may ask what a topological field theory is. Now we give the definition of it.

Definition 4.1 A topological field theory of $d + 1$ dimensions consists of two functors on manifolds. The first assigns to each closed, oriented, d-manifold Y a vector space $\mathcal{H}(Y)$ (over complex numbers). The second assigns to each compact, oriented $(d + 1)$-dimensional manifold X with boundary Y a vector $Z(X) \in \mathcal{H}(Y)$ and satisfy the following three axioms:

1) The vector space assigns to a disjoint union $Y_1 \cup Y_2$ is the tnsor product $\mathcal{H}(Y_1 \cup Y_2) = \mathcal{H}(Y_1) \bigotimes \mathcal{H}(Y_2)$.

2) $\mathcal{H}(\overline{Y}) = \mathcal{H}(Y)^*$, where $\overline{Y}$ is Y with the reverse orientation.

3) Suppose X is a $(d + 1)$-manifold with boundary, and that X contains Y and $\overline{Y}$ as two of its boundary components. Let $X^\#$ be the oriented manifold obtained from X by identifying these two boundary components. Then we require that $Z(X^\#) = c(Z(X))$, where the contraction $c : \mathcal{H}(\partial X) \to \mathcal{H}(\partial X^\#)$ is induced from the dual pairing $\mathcal{H}(Y) \bigotimes \mathcal{H}(\overline{Y}) \to \mathcal{C}$ and the decomposition $\mathcal{H}(\partial \mathcal{X}) = \mathcal{H}(Y) \bigotimes \mathcal{H}(\overline{Y}) \bigotimes \mathcal{H}(\partial X^\#)$.

Remark:

1) $\mathcal{H}(\emptyset) \simeq C$ (canonically)

2) Suppose that a $(d+1)$-manifold U is cobordism from Y_1 to Y_2, so the oriented boundary of U is a disjoint union $\overline{Y_1} \cup Y_2$. Then, by Axiom 1 and 2, $Z(U)$ is an element of $\mathcal{H}(Y_1)^* \bigotimes \mathcal{H}(Y_2)$ and hence gives a linear map

$\zeta_U : \mathcal{H}(Y_1) \to \mathcal{H}(Y_2)$.

If V is a cobordism from Y_2 to Y_3 then Axiom 3 states that $\zeta_{V \circ U} = \zeta_V \circ \zeta_U : \mathcal{H}(Y_1) \to \mathcal{H}(Y_3)$. Thus we get a functor from category of d-manifolds, with morphisms defined by cobordisms, to the category of vector spaces and linear maps.

3) In a typical physical set-up the correspondent space $\mathcal{H}(Y)$ would be an infinite-dimensional Hilbert space defined by associating to Y a space of "fields" $\mathcal{C}(Y)$,and then letting $\mathcal{H}(Y)$ be a space of L^2 functions on $\mathcal{C}(Y)$.

The Yang-Mills invariants, and Floer groups, fit into this general scheme, with $d = 3$. For a 3-manifold, we can take Floer groups as

$\mathcal{H}(Y) = HF_*(Y)$. For a closed 4-manifold X the Yang-Mills instantons define a numerical invariant $Z(X)$, and for a 4-manifold with boundary we obtain invariants with values in the Floer homology of the boundary. (refer to [D])

In recent years, Khovanov and Seidel [KS] built a nice relation in quiver, Floer cohomology and braid group.

They consider the derived categories of modules over certain family A_m of graded rings and the Floer cohomology of Lagrangian intersections in the symplectic manifolds(Milnor fibres of simple singularities of type A_m). These two very different objects actually encode the topology of curves on a $(m+1)-$punctured disc. They also proved that the braid group B_m+1 acts faithfully on the derived category of A_m-modules, and it injects into the symplectic mapping class group of the Milnor fibers.

Now we need to explain what the Milnor fibre is.

Let $f(z_0, ..., z_n)$ be a non-constant polynomial function of $n+1$ complex variables $z_0, ..., z_n$ such that $f(0, ..., 0) = 0$, so that the set Vf of all complex $(n+1)-$vectors $(z_0, ..., z_n)$ with $f(0, ..., 0) = 0$ is a complex hypersurface of complex dimension n containing the origin of complex $(n+1)-$space. (For instance, if $n = 1$ then V_f is a complex plane curve containing $(0, 0)$.) The argument of f is the function $\frac{f}{|f|}$ mapping the complement of V_f in complex $(n+1)-$space to the unit circle S^1 in C. For any real radius $r > 0$, the restriction of the argument of f to the complement of V_f in the real $(2n+1)-$sphere with center at the origin and radius r is the Milnor map of f at radius r.

Theorem 4.2 Milnor's Fibration Theorem states that, for every f such that the origin is an isolated singular point of the hypersurface V_f, the Milnor map of f at any sufficiently small radius is a fibration over S^1. Each fiber is a non-compact differentiable manifold of real dimension $2n$, and the closure of each fiber is a compact manifold with boundary bounded by the intersection K_f of K_f with the $(2n+1)-$sphere of sufficiently small radius. Furthermore, this compact manifold with boundary, which is known as the Milnor fiber (of the isolated singular point of V_f at the origin), is diffeomorphic to the intersection of the $(2n+2)-$ball with the hypersurface V_g (non-singular) where $g = f - e$ and e is any sufficiently small non-zero complex number. This small piece of hypersurface is also called a $Milnor\ fiber$.

Milnor maps are named in honor of John Milnor, who introduced them to topology and algebraic geometry in his book [M] and earlier lectures.

Milnor maps at other radii are not always fibrations, but they still have many interesting properties. For most (but not all) polynomials, the Milnor map at infinity (that is, at any sufficiently large radius) is again a fibration.

Example 4.3 The Milnor map of $f(z,w) = z^2 + w^3$ at any radius is a fibration; this construction gives the trefoil knot its structure as a fibered knot.

Khovanov and Seidel [KS] mainly discuss on the connection between symplectic geometry and those parts of representation theory.

The connection with symplectic geometry is based on an idea of Donaldson. He associate to a compact symplectic manifold (M^{2n}, ω) a category $Lag(M, \omega)$ whose objects are Lagrangian submanifolds $L \subset M$, and whose morphisms are the Floer cohomology groups $HF(L_0, L_1)$. The composition of morphisms would be given by products $HF(L_1, L_2) \times HF(L_0, L_1) \to HF(L_0, L_2)$.

The remaining stuff refers to [KS].

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