Pseudoholomorphic Curves and Mirror Symmetry

Santiago Canez

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Abstract

This survey article was written for Prof. Alan Weinstein's Symplectic Geometry (Math 242) course at UC Berkeley in Fall 2005. We review different constructions arising from the theory of pseudoholomorphic curves in symplectic manifolds and results concerning their use in giving a firm mathematical description of mirror symmetry. Specifically, we discuss the Homological Mirror Symmetry Conjecture.

1 Introduction

Mirror symmetry has arisen in the last decade and a half as an important subject in symplectic, complex, and algebraic geometry. It postulates a connection between pairs of Calabi-Yau manifolds and relates the complex geometry of one to the symplectic geometry of the other (the name "mirror symmetry" comes from a graphical realization of the relation between Hodge numbers of the mirror manifolds). These ideas first arose from considerations in string theory, where it is shown that mirror manifolds give rise to the same physical theory when used as the extra dimensions which are required by string theories. This helps to address one major difficulty in string theory — the fact that the theory cannot single out a unique manifold to model the "real world". If different manifolds give rise to the same physics, then this should not be too much of a concern. Having been first proposed by physicists, mirror symmetry is difficult to formulate mathematically. Current research in mirror symmetry seeks to give the mirror relation a firm mathematical foundation.

Numerous constructions used in attempts to define mirror symmetry mathematically are based on the theory of pseudoholomorphic curves. Pseudoholomorphic curves, introduced into symplectic geometry by Gromov [10] in 1985, are the natural generalizations of holomorphic curves to almost complex manifolds. Since their introduction, the study of pseudoholomorphic curves has revolutionized the subject of symplectic geometry.

The aim of this paper is to review some of the constructions that arise from the theory of pseudoholomorphic curves and discuss the progress that has been made in using these constructions to define the mirror symmetry relationship. Specifically, we give a quick review of Gromov-Witten invariants, quantum cohomology, and the Fukaya category (coming from ideas in Floer theory). After reviewing these and some basic concepts in mirror symmetry, we define the mirror symmetry relation in terms of Kontsevich's Homological Mirror Symmetry Conjecture and review progress that has been made in attempting to prove it.

2 Mirror Symmetry in Physics

To set up the context of mirror symmetry, we first recall some ideas from physics. In string theory, one replaces the classical notion of a point particle with that of a 1-dimensional vibrating string. Heuristic arguments show that, for this to lead to a solid physical theory, one should require that the universe be 10-dimensional. Four of these correspond to the usual dimensions we observe (three physical and one time), but the other six are more mysterious. String theorists assume that the universe is modelled by $\mathbb{R}^4 \times M$ where M is some compact 6-dimensional manifold. This M is responsible for the behaviour and properties of strings. Heuristic arguments then show that M should actually be a Calabi-Yau manifold, i.e. a Kähler manifold with vanishing first chern class. (A theorem of Yau says that Calabi-Yau manifolds can be equivalently defined as Ricci-flat Kähler manifolds.) As alluded to in the introduction, it turns out that there are many choices of Calabi-Yau manifolds that lead to reasonable physics, and the current theory cannot single out the correct one. This difficulty and the idea of doing computations on the mirror dual are perhaps where mirror symmetry is most physically important.

It is believed that, to each Calabi-Yau manifold, one can associate two $N = (2, 2)$ superconformal field theories — the so called A and B models. We will not review these constructions here, but refer instead to [28]. The symplectic geometry side of mirror symmetry corresponds to the A-model and the complex geometry side corresponds to the B-model. Mirror symmetry in this context can be roughly stated as follows:

Conjecture 2.1 (Mirror Symmetry Conjecture)**.** *If M and W are mirror Calabi-Yau manifolds, then the A-model on M is equivalent to the B-model on W, and vice-versa.*

Another formulation states that the moduli space of Kähler structures on one manifold, considered as symplectic structures, should be equivalent to the moduli space of complex structures on the mirror dual. Indeed, the dimensions of these spaces are equal to the Hodge numbers which mirror symmetry says should be the same.

One consequence of the physical formulation of mirror symmetry is a relation between the number of rational curves in a Calabi-Yau manifold and the variation of Hodge structures on its mirror manifold. The number of rational curves is given by correlation functions. Later we will see that mathematically these correlation functions are given exactly by Gromov-Witten invariants. A first major achievement of mirror symmetry was a prediction for the number of rational curves on certain manifolds given in [3]. This prediction has since proved to be accurate [8].

In the physics literature, mirror symmetry is most commonly stated solely as a relation between Calabi-Yau manifolds. However, even in this context, the mirror symmetry relation depends only on the symplectic structure of one manifold and the complex structure of the other. Hence it makes sense to try to define the mirror symmetry relationship more generally as a relation between symplectic and complex manifolds.

For an explicit, relatively simple example of mirror symmetry we refer the reader to the case of the quintic threefold summarized in [17]. A further introduction to mirror symmetry can be found in [19].

3 Pseudoholomorphic Curves

Recall that an almost complex structure on a smooth manifold M is a smooth $(1,1)$-tensor J such that the vector bundle isomorphism $J : TM \to TM$ satisfies $J^2 = -\operatorname{Id}$. In other words, J gives a smooth family of complex structures on the tangent spaces of M. A smooth manifold with an almost complex structure is called an almost complex manifold.

Definition 3.1. An almost complex structure J on (M, ω) is said to be ω-tamed if $\omega(u, Jv)$ is a positive definite bilinear form. Equivalently, J is ω-tamed if $\frac{1}{2}(\omega(u, Jv) + \omega(v, Ju))$ is a Riemannian metric. J is said to be ω-compatible if $\omega(u, Jv)$ is a Riemannian metric.

Definition 3.2. Let (M, J) be an almost complex manifold. A map $f : \Sigma \to M$ from a Riemann surface (Σ, j) into M is called a pseudoholomorphic (or J-holomorphic) curve if f satisfies the Cauchy-Riemann equation $\bar{\partial}_J f = 0$ where $\bar{\partial}_J f = \frac{1}{2}(df + J \circ df \circ j)$.

The condition $\bar{\partial}_J f = 0$ just says that df is complex linear. More generally, in the course of finding the invariants described below, one considers perturbed Cauchy-Riemann equations $\bar{\partial}_J f = g$.

Let (M, ω) be a symplectic manifold and let J be a tamed almost complex structure on M, which always exists (since in particular an ω-compatible almost complex structure always exists [23].) Let $\alpha \in H_2(M)$ be a homology class in M and fix a nonnegative integer g. We let $\mathcal{M}_{g,k}(\alpha, J)$ denote the moduli space of pseudoholormophic curves from Riemann surfaces (Σ, j) to M with k marked points representing the class α. This space can be compactified to give a space $\overline{\mathcal{M}}_{g,k}(\alpha, J)$ which in general will be an orbifold (a space which is locally homeomorphic to $\mathbb{R}^n$ mod the action of some finite group) — in the genus 0 case we actually get a smooth manifold. A nice exposition is given in [18]. The compactification was described by Gromov [10] in a special case, and also by Ruan [22]. The general notion of compactification uses the notion of stable maps.

Pseudoholomorphic curves arise in string theory in the following way. In string theory, a string sweeps out a 2-dimensional surface (the worldsheet) as it moves. One then wishes to compute path integrals over the space of such surfaces, which is difficult if not impossible to do since the spaces are infinite dimensional. However, in the A-model, the surfaces in question can be parameterized by pseudoholomorphic curves, and the path integrals become integrals over moduli spaces of these curves, which are finite dimensional. These latter integrals are then easier to compute and give information about the original objective.

We will also need the definition of a semipositive symplectic manifold, which we get from [21]. This is a $2n$-dimensional symplectic manifold (M, ω) such that for any tamed almost complex structure J on M there is no pseudoholomorphic curve C satisfying

$$3 - n \leq \int_C c_1(M) < 0.$$

As we will see, pseudoholomorphic curves are vital in the formulation of homological mirror symmetry. Counting pseudoholomorphic curves gives rise to new invariants, differentials used in different homology theories, and maps used in the construction of the Fukaya category.

4 Gromov-Witten Invariants

Invariants of symplectic manifolds arising from pseudoholomorphic curves were considered by Gromov in his fundamental paper [10]. More sophisticated invariants were introduced by Witten in [27] and then Ruan in [22], leading to the general notion of Gromov-Witten invariants. Intuitively, these invariants count the number of pseudoholomorphic curves in a symplectic manifold.

For our purposes, we will state a simple definition of Gromov-Witten invariants given in [16]. Recall that $\overline{\mathcal{M}}_{g,k}(\alpha, J)$ denotes the compactified moduli space of pseuoholomorphic curves in M from genus g surfaces with k marked points representing the class α. Then there is a natural evaluation map $ev : \overline{\mathcal{M}}_{g,k}(\alpha, J) \to M^k$ and a projection $\pi : \overline{\mathcal{M}}_{g,k}(\alpha, J) \to \overline{\mathcal{M}}_{g,k}$ to the moduli space of stable curves of genus g with k marked points.

Definition 4.1. The Gromov-Witten invariants are maps

$$GW^M_{g,k,\alpha} : H^*(M; \mathbb{Q})^{\otimes^k} \otimes H_*(\overline{\mathcal{M}}_{g,k}; \mathbb{Q}) \to \mathbb{Q}$$

defined by

$$GW^M_{g,k,\alpha}(a_1, \ldots, a_k; \beta) = \int_{\overline{\mathcal{M}}_{g,k}(\alpha,J)} ev_1^* a_1 \smile \cdots \smile ev_k^* a_k \smile \pi^* PD\beta.$$

where $PD\beta$ denotes the Poincaré dual of β.

The integrals involved in these definitions resemble the definitions of correlation functions that come up in physics. For example, the invariant $\widetilde{\Phi}$ introduced in [22] is exactly Witten's k-point correlation function [27]. The extrema of the action functional in Witten's topological sigma model [27] turn out to be pseudoholomorphic maps and this is the source of the connection between Gromov-Witten invariants and correlation functions.

An axiomatic approach to these invariants is given in [13], and we refer to Ruan's original paper [22] for his definition in the semipositive genus zero case. For a further development of Gromov-Witten invariants, we refer to [25] and [16].

5 Quantum Cohomology

Quantum cohomology was first introduced in physics by Vafa [26]. A mathematical theory later followed which uses Gromov-Witten invariants to define a new product in cohomology, giving rise to the quantum cohomology of a symplectic manifold [16],[21]. In the mirror conjecture, quantum cohomology helps to give a mathematical description of the A-model, and comes up in explaining the relation among Hodge numbers between mirror manifolds.

To define the new quantum product, we first need the notion of a quantum coefficient ring, taken from [16].

Definition 5.1. Let (M, ω) be a compact symplectic manifold and R a commutative ring with unit. A quantum coefficient ring over R for (M, ω) is a triple (Λ, ϕ, ι), where Λ is a $2\mathbb{Z}$-graded commutative ring with unit and also an R-module, $\phi : \Gamma(M, \omega) \to \Lambda$ is a ring homomorphism that preserves the R-module structure and the grading, and $\iota : \Lambda \to R$ is an R-module homomorphism that vanishes on elements of nonzero degree and satisfies $\iota(\phi(\lambda)) = \lambda(0)$.

Also, we need the notion of the effective cone.

Definition 5.2. The effective cone $K^{\text{eff}}(M)$ of a symplectic manifold (M, ω) with ω-tame almost complex structure J is

$$K^{\text{eff}}(M) = \{A \in H_2(M) \mid \text{there exists a pseudoholomorphic curve in class } A\}.$$

Definition 5.3. Let (M, ω) be a closed semipositive symplectic manifold and let $H^*(M)$ denote the quotient of $H^*(M, \mathbb{Z})$ by its torsion. Define the quantum cohomology of M with coefficients in Λ to be $QH^*(M, \Lambda) = H^*(M) \otimes_{\mathbb{Z}} \Lambda$.

We define a product $*$ on $QH^*(M, \Lambda)$ as follows. First a piece of notation: for $A \in K^{\text{eff}}$, let e^A denote the image under ϕ of the characteristic function of A. For each $a, b \in H^*(M)$ and $A \in K^{\text{eff}}$, let $(a * b)_A$ denote the element such that

$$\int_M (a * b)_A \smile c = GW^M_{0,3,A}(a, b, c)$$

for all $c \in H^*(M)$ where $\smile$ denotes the ordinary cup product. We define the quantum cup product $a * b$ by

$$a * b = \sum_{A \in K^{\text{eff}}} (a * b)_A \otimes e^A.$$

Note that Ruan and Tian [21] define the quantum product using Witten's k-point correlation function. Examples of quantum cohomology rings, such as $QH^*(\mathbb{C}P^n) = \mathbb{Z}[p, q]/\langle p^{n+1} - q \rangle$ can be found in [16].

One way in which this product arises in mirror symmetry is the following. For $h \in H^2(M)$, we have the following formula [16]:

$$\int_H h * h = 5 + \sum_{d=1}^{\infty} N_d d^3 \frac{q^d}{1 - q^d}$$

where $q = e^A$ and N_d is the number of rational curves of degree d in the class A. This formula is equivalent to

$$\sum_{d=1}^{\infty} \frac{1}{d^3} GW^M_{d,3,A}(h, h, h) q^d = \sum_{d=1}^{\infty} N_d \frac{q^d}{1 - q^d}.$$

Using mirror symmetry we can compute the left hand sides based on variations of Hodge structures and hence gain information about the numbers N_d. In general, one tries to compute the quantum cohomology ring in this manner since computations on the mirror manifold (the B-model) are usually simpler.

In physics, topoligical field theories, such as the A and B-modles, can be defined using the notion of a supercommutative Frobenius algebra. In particular, the algebra corresponding to the A-model on a Calabi-Yau manifold turns out to be the quantum cohomology ring of the manifold. For further discussion we refer to [12] and the references therein. It is in this way that quantum cohomology can be used to formulate the symplectic side of the mirror symmetry conjecture.

6 The Fukaya Category

Floer homology can be viewed as an infinite dimensional analog of Morse theory. In this paper we consider specifically the intersection Floer theory of two lagrangian submanifolds, whose study began with Floer's paper [5]. The chain complex of this homology theory is generated by intersection points of the lagrangian submanifolds in question, and the differential counts pseudoholomorphic discs with the two boundary arcs lying in the different submanifolds.

We do not give an account of Floer theory here, as we are mainly concerned with how it is used to define the Fukaya category. For a further development of Floer theory, we refer to [4] and [16].

It turns out that one can regard the lagrangian submanifolds of a symplectic manifold as the objects of a category and the Floer homology of two lagrangian submanifolds as the morphisms. Fukaya realized that what this defines is not actually a category, but what is known as an A_∞-category. We use the definition given in [20].

Definition 6.1. An A_∞-category is a collection of objects together with a $\mathbb{Z}$-graded space of morphisms $\mathrm{Hom}(X, Y)$ for each pair of objects X and Y and composition maps

$$m_k : \mathrm{Hom}(X_1, X_2) \otimes \cdots \otimes \mathrm{Hom}(X_k, X_{k+1}) \to \mathrm{Hom}(X_1, X_{k+1})$$

of degree $2 - k$ satisfying

$$\sum_{r=1}^{n} \sum_{s=1}^{n-r+1} (-1)^{\varepsilon} m_{n-r+1}(a_1 \otimes \cdots a_{s-1} \otimes m_r(a_s \otimes \cdots \otimes a_{s+r-1}) \otimes a_{s+r} \otimes \cdots \otimes a_n) = 0$$

for all $n \geq 1$ where $\varepsilon = (r+1)s + r(n + \sum_{j=1}^{s-1} \deg(a_j))$.

An A_∞-category is not a category in the usual sense since it does not necessarily satisfy the required associativity conditions, but instead satisfies the higher associativity laws given by the final condition in the definition. For the reader familiar with the notion of an A_∞-algebra, we note that an A_∞ algebra is just an A_∞-category with one object.

The Fukaya category of M is an A_∞-category $F(M)$ whose objects are pairs $(L, \mathcal{E})$ consisting of a Lagrangian submanifold $L \subset M$ and a flat complex line bundle $\mathcal{E}$ over

L, and whose morphisms are defined, only if L_1 and L_2 intersect transversally, by Floer homology: $Mor(L_1, L_2) = HF_*(L_1, L_2)$. The A_∞-structure comes from summing over pseudoholomorphic discs, as explained in [6]. More precisely, the map

$$m_k : \mathrm{Hom}(L_0, L_1) \otimes \cdots \otimes \mathrm{Hom}(L_{k-1}, L_k) \to \mathrm{Hom}(L_0, L_k)$$

is defined by counting pseudoholormophic curves from a disk mapping marked points to intersection points:

$$m_k(p_1, \ldots, p_k) := \sum_{\phi: D^2 \to M,\ q \in L_1 \cap L_{k+1}} \pm \exp\left(- \int_{D^2} \phi^* \omega \right) q.$$

The Fukaya category plays a critical role in the definition of homological mirror symmetry in that it describes the A-model of a Calabi-Yau manifold. The definition is slightly altered for each example we will discuss below, but the general idea remains the same.

7 Homological Mirror Symmetry

The first serious attempt at a mathematical foundation of mirror symmetry was the Homological Mirror Symmetry conjecture introduced by Kontsevich [14] in 1994. This conjecture roughly conjectures an equivalence between coherent analytic sheaves on a Calabi-Yau manifold (the complex side) and lagrangian submanifolds of its mirror (the symplectic side).

Remark 7.1. A coherent analytic sheaf on a complex manifold X is a holomorphic sheaf $\mathcal{F}$ that is the cokernel of some sheaf morphism $\mathcal{O}_X^s \to \mathcal{O}_X^r$. Every holomorphic vector bundle gives rise to such a sheaf by taking sections over open sets, but not every sheaf arises in this way. However, one can go a long way by just thinking of coherent analytic sheaves as holomorphic vector bundles.

Before stating the conjecture, we recall that the derived category of a category C is constructed using complexes in C. We refer to books on homological algebra for more details.

Conjecture 7.1 (Homological Mirror Symmetry Conjecture)**.** *Let (M, ω) be a symplectic manifold with vanishing first Chern class, and let W be a complex manifold which is mirror dual to M. Then the derived category constructed from the Fukaya category of M is equivalent to the bounded derived category of coherent analytic sheaves on W.*

The connection between this homological conjecture and the physical intuition given above comes from considering the homological objects as the D-branes of theory — the objects on which ends of open strings are allowed to move. Namely, a pair $(L, \mathcal{E})$ where L is a lagrangian submanifold and $\mathcal{E}$ is a flat complex line bundle is a D-brane of the A-model (an A-brane) and a coherent analytic sheaf is a D-brane of the B-model (a B-brane). For more precise explanations of the relation between branes and these homological objects, we refer to [28], [12], and the references therein.

It should be noted that Kontsevich proposed this framework before D-branes were studied seriously in string theory — the interpretation of the categories as D-branes came later. In [11] it is argued that in fact not all A-branes arise from lagrangian submanifolds, but this is beyond our current discussion. Homological Mirror Symmetry was first shown to be true for the case of elliptic curves in [20]; we discuss this in more detail later.

We note what the conjecture says for the Floer theory of lagrangian submanifolds, which is further developed in [7]. The equivalence described in the conjecture states that for pairs $(L, \mathcal{E})$ where L is a lagrangian submanifold and $\mathcal{E}$ is a flat complex line bundle over L, mirror symmetry induces an isomorphism

$$HF((L_1, \mathcal{E}_1), (L_2, \mathcal{E}_2)) \simeq \operatorname{Ext}(\mathcal{F}_1, \mathcal{F}_2)$$

between Floer homology on the left and Ext homology of the sheaves F_1 and F_2 corresponding to L_1 and L_2 resepectively under the mirror map.

Part of the difficulty in the homological mirror symmetry conjecture is that there is no explicit construction given for the complex mirror dual W. A basic idea for the construction is described in [15]. Following that exposition, let (M, ω) be a symplectic manifold with a submersion $p_M : M \to U$ whose fibers are lagrangian tori. Then M can be identified with T^*U/H where H is a lagrangian sublattice in T^*U. Let $H^t = \operatorname{Hom}(H, \mathbb{Z})$. Then $Y = TU/H^t$ should be the mirror dual of M. We note, as explained in [15], that Y no longer has a symplectic structure but now has a complex structure coming from the flat affine structure on U — which we expect from mirror symmetry.

In addition to homological mirror symmetry, there is another mathematical formulation of mirror symmetry due to Strominger, Yau, and Zaslow [24] that builds on the above idea. Here, Calabi-Yau manifolds are described in terms of torus fibrations, and mirror symmetry becomes the action of T-duality on each fiber. This has the advantage of giving a more geometric construction of the mirror dual of a manifold.

Remark 7.2. For the reader who is not familiar with T-duality, we recall the basic idea here. T-duality is a certain symmetry between string theories on spaces of differing radius. The idea is that replacing the radius R by $1/R$ and exchanging momentum and winding number does not change the physical theory. More details can be found in [19].

8 Examples of Homological Mirror Symmetry

8.1 Elliptic Curves

As mentioned above, the case of elliptic curves (i.e. 1-dimensional complex manifolds, or tori) was the first example where homological mirror symmetry was shown to be true. Here we discuss some details of this example, following [20]. The mirror dual of an elliptic curve (complex manifold) is a 2-torus (symplectic manifold). Note that as 1-dimensional complex manifolds, elliptic curves and tori are basically the same objects, but we make the distinction here to emphasize that we are only considering the complex structure of the elliptic curve and the symplectic structure of the torus.

Let M be a 2-torus with Kähler form ω. The objects of the Fukaya category here, special lagrangian submanifolds with flat bundles which we denote by $U_i = (L_i, \mathcal{E}_i)$, are simply geodesics since any 1-dimensional submanifold is trivially lagrangian and the minimality condition implies that the submanifolds must then be geodesics. The morphisms are given by $\mathrm{Hom}(U_i, U_j) = \mathbb{C}^{|L_i \cap L_j|} \otimes \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j)$. To define the A_∞-structure, we denote elements of $\mathrm{Hom}(U_j, U_{j+1})$ by (a_j, t_j). Then the map m_k is defined by

$$m_k(u_1 \otimes \cdots \otimes u_k) = \sum_{a_{k+1} \in L_1 \cap L_{k+1}} (a_{k+1}, C(u_1, \ldots, u_k, a_{k+1}))$$

where

$$C(u_1, \ldots, u_k, a_{k+1}) = \sum_{\phi} \pm e^{2\pi i \int \phi^* \omega} \cdot e^{\int \phi^* \beta}$$

and β is the connection of the flat bundle. The sum is over all pseudoholomorphic (holomorphic in this case) maps $\phi : D^2 \to M$ with conditions on the boundary.

Now, suppose that W is the mirror elliptic curve. Then we can write

$$W = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$$

where τ defines the complex structure of W. If $\rho = \int_M \omega$, then the map which gives rises to mirror symmetry exchanges ρ and τ. The Homological Mirror Symmetry equivalence is constructed by first giving the map between line bundles on W and geodesics on M, and then using this to define the equivalence between the derived categories. We refer to [20] for the details.

8.2 Fano Varieties

Homological mirror symmetry was shown to be true for certain Fano varieties in [1], building on work of Givental [9], and for Del Pezzo surfaces in [2]. Here, the B-model is described by sigma models with target spaces Fano varieties and the A-model is described by what are known as Landau-Ginzburg models. For further information about these models, the reader can consult [1] and the references therein. The definition of the Fukaya category in this setting is altered to consider so called lagrangian vanishing cycles.

The main result of [1] is that Homological Mirror Symmetry holds for the mirror pair consisting of the weighted projective plane $W = \mathbb{C}P^2(a, b, c)$ and the hypersurface $M = \{x^a y^b z^c = 1\} \subset (\mathbb{C}^*)^3$. Building on this, the same authors then prove Homological Mirror Symmetry for certain Del Pezzo surfaces and blowups of $\mathbb{C}P^2$ in [2]. These examples show that the mirror symmetry phenomenom can hold in situations other than the case of Calabi-Yau manifolds.

9 Conclusion

As we have seen, mirror symmetry is an exciting area of study in symplectic and complex geometry. Defining it mathematically requires a study of many different topics, especially the theory of pseudoholomorphic curves. While we have in no way given a full treatment of the concept of mirror symmetry, we hope that we have given enough of an introduction to motivate further study. We hope the reader will consult the references mentioned to learn more about Gromov-Witten invariants, quantum cohomology, Floer theory, and their use in mirror symmetry. Homological Mirror Symmetry, and the Strominger-Yau-Zaslow conjecture [24], seem to be the most fruitful attempts at giving mirror symmetry a rigorous mathematical foundation; we hope that future research makes further gains in these areas.

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