The Spectrum of the Laplacian in Riemannian Geometry

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1 Introduction

To any compact Riemannian manifold (M, g) (with or without boundary), we can associate a second-order partial differential operator, the Laplace operator Δ, defined by $\Delta(f) = -\text{div}(\text{grad}(f))$ for $f \in L^2(M, g)$. We will also sometimes write Δ_g for Δ if we want to emphasize which metric the Laplace operator is associated with. The set of eigenvalues of Δ (the spectrum of Δ, or of M), which we will write as $\text{spec}(\Delta)$ or $\text{spec}(M, g)$, then forms a discrete sequence $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$. For simplicity, we will assume that M is connected; this will for example imply that the smallest eigenvalue, λ_0, occurs with multiplicity 1.

Note that the Laplacian also acts on p-forms in addition to functions via the definition $\Delta = -(d\delta + \delta d)$, where δ is the adjoint of d with respect to the Riemannian structure on the manifold. This aspect of the Laplacian will not be treated in this paper, the focus being the ordinary Laplacian acting on functions.

With that in mind, there are two broad questions that are at the heart of spectral geometry:

(i) What can we say about the spectrum of M given the geometry?

(ii) What can we say about the geometry of M given the spectrum?

What we will attempt to cover in this survey are some selected aspects of those two questions. In section 2, we will discuss two aspects of the first question: what types of sequences can occur as spectra and what restrictions does geometry impose on the first nonzero eigenvalue of the Laplacian. In section 3, we will discuss some positive results relevant to the second question: what geometric information can we extract from the spectrum? Finally, in section 4, we will talk about some negative results relevant to the second question, focusing on examples of manifolds where the spectrum fails to allow us to distinguish between them at all. Good background references on the topics covered in this paper (and on other topics not covered) are the books [2] and [7], and the survey [9].

2 Estimates on the eigenvalues

Obviously, the geometry of a Riemannian manifold completely determines the spectrum: the metric determines the Laplace operator and hence the spectrum. On the other hand, there are only few examples of manifolds where the spectrum is known explicitly. In this section, we will examine the restrictions that geometry can impose on the spectrum.

2.1 What sequences can be spectra?

An interesting first question to ask is what types of sequences can occur as the spectra of manifolds. A version of this question has been answered: what finite sequences can occur as the initial part of the spectra of manifolds? Colin de Verdière showed that there are essentially no restrictions at all. If M is a closed connected manifold of dimension greater than or equal to 3, then any preassigned finite sequence $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k$ is the sequence of first $k+1$ eigenvalues of Δ_g for some choice of the metric g on M ([5], [6]). In particular, this means that for closed connected manifolds of dimension 3 or greater, there are no restrictions on the multiplicities of the eigenvalues λ_i for $i > 0$.

In dimension 2, there are some restrictions on the multiplicities of the eigenvalues. Let M be a closed connected 2-manifold with Euler characteristic $\chi(M)$, and let m_j be the multiplicity of the j-th eigenvalue, $j > 0$, of the Laplace operator associated to a metric on M. Then

(i) if M is the unit sphere, then $m_j \leq 2j + 1$;

(ii) if M is the real projective plane, then $m_j \leq 2j + 3$;

(iii) if M is the torus, then $m_j \leq 2j + 4$;

(iv) if M is the Klein bottle, then $m_j \leq 2j + 3$;

(v) if $\chi(M) < 0$, then $m_j \leq 2j - 2\chi(M) + 3$.

See [18] for details. For finite sequences $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k$ however, the result by Colin de Verdière holds even in dimension 2.

2.2 Estimates on the first eigenvalue

The geometry of a manifold affects more than just the multiplicities of the eigenvalues. Here we will focus on bounds on the first non-zero eigenvalue λ_1 imposed by the geometry.

The first lower bound is due to Lichnerowicz [16]:

Theorem 1 *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 2$ and let Ric be its Ricci tensor field. If*

$$Ric(X, X) \geq (n-1)k > 0$$

for some constant $k > 0$ and for all $X \in \mathcal{T}(M)$, then $\lambda_1 \geq nk$.

More recently, in 1979, Li and Yau [15] proved that if M is a closed Riemannian manifold whose Ricci curvature is nonnegative, then $\lambda_1 \geq \frac{\pi^2}{2D^2(M)}$, where $D(M)$ is the diameter of M. This was achieved via a gradient estimate on the first eigenfunction. Zhong and Yang [24], using a similar method, improved on this by showing that under the same conditions, $\lambda_1 \geq \frac{\pi^2}{D^2(M)}$.

Even more recently, Yang [23] generalized the previous result:

Theorem 2 *Let (M, g) be a closed Riemannian manifold. If*

$$Ric(X, X) \geq (n-1)k \geq 0$$

for some nonnegative constant k and for all $X \in \mathcal{T}(M)$, then

$$\lambda_1 \geq \frac{(n-1)k}{4} + \frac{\pi^2}{D^2(M)}.$$

It is in general much easier to give upper bounds on λ_1 than it is to give lower bounds. The basic result in this area is a comparison theorem due to Cheng [3]: in a complete Riemannian n-manifold whose Ricci curvature is $\geq (n-1)k$ (where k is some constant), λ_1 for a geodesic ball of radius r in M is less than or equal to λ_1 for a geodesic ball of the same radius in a space of constant curvature k. From this, we can obtain the bound:

Theorem 3 *If M is a compact n-manifold with Ricci curvature $\geq (n-1)(-k)$, $k > 0$, then*

$$\lambda_1 \leq \frac{(n-1)^2}{4}k + \frac{c_n}{D^2(M)},$$

where c_n is a positive constant depending only on n.

3 Geometric implications of the spectrum

The spectrum does not in general determine the geometry of a manifold. Nevertheless, some geometric information can be extracted from the spectrum. In what follows, we define a spectral invariant to be anything that is completely determined by the spectrum.

3.1 Invariants from the heat equation

Let M be a Riemannian manifold. A heat kernel, or alternatively, a fundamental solution to the heat equation, is a function $K : (0, \infty) \times M \times M \to M$ that satisfies

(i) $K(t, x, y)$ is C^1 in t and C^2 in x and y;

(ii) $\frac{\partial K}{\partial t} + \Delta_2(K) = 0$, where Δ_2 is the Laplacian with respect to the second variable (i.e., the first space variable);

(iii) $\lim_{t\to 0^+} \int_M K(t,x,y)f(y)\,dy = f(x)$ for any compactly supported function f on M.

The heat kernel exists and is unique for compact Riemannian manifolds. Its importance stems from the fact that the solution to the heat equation

$$\frac{\partial u}{\partial t} + \Delta(u) = 0, \quad u : [0,\infty) \times M \to \mathbb{R},$$

(where Δ is the Laplacian with respect to the second variable) with initial condition $u(0,x) = f(x)$ is given by

$$u(t,x) = \int_M K(t,x,y)f(y)\,dy.$$

If $\{\lambda_i\}$ is the spectrum of M and $\{\xi_i\}$ are the associated eigenfunctions (normalized so that they form an orthonormal basis of $L^2(M)$), then we can write

$$K(t,x,y) = \sum_i e^{-\lambda_i t}\xi_i(x)\xi_i(y).$$

From this, it is clear that the heat trace, $Z(t) = \int_M K(t,x,x)\,dx = \sum_i e^{-\lambda_i t}$, is a spectral invariant. The heat trace has an asymptotic expansion as $t \to 0^+$:

$$Z(t) = (4\pi t)^{\dim(M)/2} \sum_{j=1}^{\infty} a_j t^j,$$

where the a_j are integrals over M of universal homogeneous polynomials in the curvature and its covariant derivatives ([17], see [8] or [7] for details). The first few of these are

$$a_0 = \mathrm{vol(M)}, \quad a_1 = \frac{1}{6}\int_M S, \quad a_2 = \frac{1}{360}\int_M (5S^2 - 2|Ric|^2 - 10|Rm|^2),$$

where S is the scalar curvature, Ric is the Ricci tensor, and Rm is the curvature tensor. The dimension, the volume, and the total scalar curvature are thus completely determined by the spectrum. If M is a surface, then the Gauss-Bonnet theorem implies that the Euler characteristic of M is also a spectral invariant.

A more in depth study of the heat trace can yield more information. It is known for example that if M is a closed, connected Riemannian manifold of dimension $n \leq 6$, and if M has the same spectrum as the n-sphere S^n with the standard metric (resp. $\mathbb{R}P^n$), then M is in fact isometric to S^n (resp. $\mathbb{R}P^n$). More on this can be found in [7].

3.2 Other invariants

There are other invariants besides those mentioned above. For generic closed Riemannian manifolds for example, the geodesic length spectrum — the set of lengths of closed geodesics — is a spectral invariant [4].

As another way to get spectral invariants, we can try to study the fundamental solution to the wave equation $\frac{\partial^2 u}{\partial t^2} + \Delta(u) = 0$ and the associated wave trace $\sum_k e^{\mathrm{i}\lambda_k t}$. The asymptotics of the wave trace near $t = 0$ turns out to give the same information as the asymptotics of the heat trace. On the other hand, the singularities of the wave trace can in some cases yield new invariants such as the geodesic length spectrum mentioned above and the quantum Birkhoff normal form for the Poincaré map about certain geodesics (in fact, the singularities of the wave trace occur at the lengths of closed geodesics; for definitions and a better overview, see [1] and [9]).

4 Isospectral manifolds

As was alluded to earlier, geometry is not in general a spectral invariant. Two manifolds are said to be isospectral if they have the same spectrum. The first example of nonisometric isospectral manifolds was found in 1964 by John Milnor, who exhibited two distinct but isospectral 16-dimensional manifolds. This was followed by the construction in the 1980s and 1990s of many different examples of nonisometric but isospectral manifolds. Among these are discrete families of isospectral manifolds, continuous families of isospectral manifolds, isospectral plane domains, and even isospectral conformally equivalent manifolds.

In general, there are three known methods to construct or discover these examples of nonisometric isospectral manifolds. For a more complete overview of these methods see [9].

4.1 Direct computation of the spectrum

The first of these is straightforward: direct computation. It is only rarely possible to explicitly compute the spectrum of a manifold, but the first examples of isospectral manifolds were actually discovered via this method. Milnor's example mentioned above consists of two isospectral flat tori, flat tori — quotients of Euclidean space by lattices of full rank — being one of the few examples of Riemannian manifolds whose spectra can be computed explicitly. Spherical space forms — quotients of spheres by finite groups of orthogonal transformations acting without fixed points — form another class of examples of manifolds whose spectra is known. Some of the more remarkable examples of pairs of isospectral manifolds are the pairs constructed by Ikeda of lens spaces which are isospectral for the Laplacian acting on p-forms for $p \leq k$ but not for the Laplacian acting on p-forms for $p = k + 1$ (recall that a lens space is a spherical space form where the group is cyclic) [14].

4.2 Representation theoretic methods

The second method involves representation theoretic techniques. The two known variants of this method actually produce examples of manifolds that are strongly isospectral: any natural strongly elliptic operator on the manifolds (such as the

Laplace operator acting on p-forms) has the same spectrum. Let G be a Lie group. We call subgroup Γ cocompact if Γ/G is compact. Define $R_{\Gamma,a}$ to be the right translation operator on Γ/G, i.e., $R_{\Gamma,a}(\Gamma x) = (\Gamma xa)$. If we let $L^2(\Gamma/G)$ be the space of measurable functions that are square integrable with respect to the Haar measure on Γ/G induced by the bi-invariant Haar measure on G, and if we define $\rho_\Gamma(a)f = f \circ R_{\Gamma,a}$ for $f \in L^2(\Gamma/G)$ and $a \in G$, then ρ_Γ is a unitary representation of G. Two cocompact discrete subgroups Γ_1 and Γ_2 of G are said to be representation equivalent if there exists a unitary isomorphism $T : L^2(\Gamma_1/G) \to L^2(\Gamma_2/G)$ such that $T(\rho_{\Gamma_1}(x))T^{-1} = \rho_{\Gamma_2}(x)$ for all $x \in G$. Isospectral manifolds can then be constructed via

Theorem 4 *Let Γ_1 and Γ_2 be cocompact discrete subgroups of a Lie group G, and let g be a left-invariant metric on G. If Γ_1 and Γ_2 are representation equivalent, then*

$$\mathrm{spec}(\Gamma_1/G, g) = \mathrm{spec}(\Gamma_2/G, g).$$

The first known examples of isospectral manifolds with different fundamental groups were constructed via this method, as were the first examples of continuous families of isospectral manifolds (see [9] for details).

The other general method of constructing isospectral manifolds involving representation theoretic ideas was discovered by Sunada in 1985:

Theorem 5 *Let Γ_1 and Γ_2 be representation equivalent subgroups of a finite group G, which acts on a compact Riemannian manifold (M, g) on the left by isometries. If the nonidentity elements of Γ_1 and Γ_2 act as fixed point free isometries on M, then*

$$\mathrm{spec}(\Gamma_1/M, g) = \mathrm{spec}(\Gamma_2/M, g).$$

(See [7] or [9] for a proof.) The construction of isospectral plane domains by Gordon, Webb, and Wolpert [12] was achieved by a variant of this method; this answered the question asked by Marc Kac: "Can you hear the shape of a drum?" in the negative.

4.3 Riemannian submersions

The last known general technique for constructing isospectral manifolds involves Riemannian submersions. A submersion $\pi : M \to N$ of a Riemannian manifold M into another Riemannian manifold N is said to be a Riemannian submersion if the differential π_* maps the horizontal space at any $p \in M$ isometrically to $T_{\pi(p)}N$. The fibers (of the submersion) are said to be totally geodesic if any geodesic in M which starts tangent to a fiber stays in the fiber. The main result is

Theorem 6 *Let $\pi : M \to N$ be a Riemannian submersion with totally geodesic fibers. Then the Laplacians on M and N, Δ_M and Δ_N, satisfy $\pi^*\Delta_N = \Delta_M\pi^*$, so that in particular,* $\mathrm{spec}(\Delta_N) = \mathrm{spec}(\Delta_{M|\pi^*C^\infty(N)})$.

Note that this can be combined with the previous methods to construct new pairs of isospectral manifolds by starting from known pairs of isospectral manifolds in some circumstances.

The two previously discussed methods cannot produce examples of isospectral manifolds with different local geometry because they depend on looking at manifolds that have common Riemannian covers. The interest in this final method lies in the fact that until very recently, all known examples of isospectral manifolds with different local geometry arise from a theorem that is a consequence of the previous theorem:

Theorem 7 *Let T be an n-dimensional torus for $n > 1$. Assume M_1 and M_2 are principal T-bundles and assume that the fibers, with the induced Riemannian metrics, are totally geodesic flat tori. Further assume that the quotient manifolds M_1/S and M_2/S, with the induced metric, are isospectral for every subtorus S of T of codimension ≤ 1. Then M_1 and M_2 are isospectral.*

Among the examples of isospectral manifolds constructed using this theorem are continuous isospectral deformations of metrics on $S^{m-1} \times T^2$; in some of these, the maximum scalar curvature changes during the deformation [11]. Other examples are isospectral deformations of metrics on $S^3 \times S^3 \times S^5$ — these provide the first constructions of isospectral simply connected manifolds [19].

4.4 Recent constructions

A wealth of examples of isospectral manifolds are covered in the survey [9]. What we will attempt here is to briefly mention some of the more recent constructions that postdate that survey.

One of the more striking achievements in recent years has been the construction of isospectral metrics on spheres. By relaxing the conditions of the previous theorem, Gordon was able to construct isospectral metrics on spheres $S^{n\geq 8}$ and balls $B^{n\geq 9}$ [10]. This was achieved by reformulating the theorem so that it no longer requires the torus T to act freely on M_1 and M_2. A slight variation of this was used by Schüth to produce isospectral metrics on S^5 and B^6 [21].

A different approach by Szabó also produces isospectral metrics on spheres and balls [22]. Let $\mathbf{v}$ and $\mathbf{z}$ be Euclidean vector spaces. Then every linear map $j : \mathbf{z} \to \mathfrak{so}(\mathbf{v})$ defines a Lie bracket $[,] : \mathbf{v} \times \mathbf{v} \to \mathbf{z}$ by $\langle [X,Y], Z\rangle = \langle j(Z)X, Y\rangle$ for $X, Y \in \mathbf{v}$ and $Z \in \mathbf{z}$. This makes $\mathbf{v} \oplus \mathbf{z}$ a two-step nilpotent Lie algebra (extend $[,]$ to $\mathbf{v}\oplus\mathbf{z}$ by setting it to 0 if one of its arguments is in $\mathbf{z}$). Denote by N^j the corresponding simply connected Lie group, and let g^j be the left-invariant metric that arises from the given inner product on its Lie algebra $\mathbf{v} \oplus \mathbf{z}$. Two linear maps $j_1, j_2 : \mathbf{z} \to \mathfrak{so}(\mathbf{v})$ are said to be isospectral if for each $Z \in \mathbf{z}$, $j_1(Z)$ is conjugate to $j_2(Z)$. Gordon and Wilson used isospectral pairs of such maps to construct isospectral pairs of balls and tori [13]. To produce isospectral spheres and balls, Szabó showed that under certain conditions, one can obtain isospectral spheres and balls in the Lie groups (N^{j_1}, g^{j_1}) and (N^{j_2}, g^{j_2}) and in their 1-dimensional solvable extensions. What is required is that there exist an orthogonal endomorphism σ of $\mathbf{v}$ commuting with each $j(Z)$ such that $j_2(Z) =$

$\sigma j_1(Z)$ for all Z, and that there exist a nondegenerate anticommutator $A \in j(\mathbf{z})$ (i.e., $Aj(Z) = -j(Z)A$ for all $j(Z)$ orthogonal to A in $\mathfrak{so}(\mathbf{v})$). Among the more interesting examples produced by this method are pairs of isospectral metrics on spheres S^{4k-1}, $k \geq 3$, one of which is homogeneous, the other locally nonhomogeneous.

Along different lines, a variation of the theorem in section 4.3 allowed Schüth to construct

(i) non-locally isometric isospectral 4-manifolds,

(ii) isospectral left-invariant metrics on compact Lie groups, and isospectral simply connected irreducible manifolds,

(iii) non-locally isometric isospectral conformally equivalent manifoolds.

See [20] for details.

5 Conclusion

The study of the spectrum of the Laplacian, spectral geometry, remains a very active field of research. It is impossible to cover all of its aspects in such a short survey and so this paper attempts only to discuss a small selection of topics. The references mentioned in the introduction are good places to look for information on topics not covered here. While some fundamental questions have been answered (e.g., isospectral does not imply isometric), and some spectacular counterexamples constructed, there remain interesting unanswered questions. We will end with one such question: is there a sense in which for most manifolds, the spectrum does determine the geometry?

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