A Survey of Lurie's *A Survey of Elliptic Cohomology*

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Abstract

These rather skeletal notes are meant for readers who have some idea of the general story of elliptic cohomology. More than anything, they should probably be used as a roadmap when reading the original *Survey* itself. They grew out of a desire to completely understand the shape of the proof of the main theorem, and so I've postponed the material on equivariant theories until after it in order to make the route to the main theorem as direct as possible. Preorientations and orientations can seem rather mysterious at first; the original paper carries out two examples in great detail. I've also omitted everything from §5, which although fascinating is already quite sketchy in the first place.

Elliptic cohomology

We assume the reader is familiar with the story of generalized (first) Chern classes, formal group laws, and the Landweber exact functor theorem.

We recall that an *elliptic curve* is a group object $\mathcal{E}$ in some the overcategory $\mathtt{Sch}/X$ whose geometric fibers are elliptic curves in the usual sense (i.e. genus-1 curves with a commutative group structure). Given an elliptic curve $\mathcal{E} \to X$, taking the formal completion along the identity section $e : X \to \mathcal{E}$ gives us a formal group $\widehat{\mathcal{E}}$. When this is Landweber-exact, the resulting cohomology theory is called the *elliptic cohomology theory* associated to $\mathcal{E}$.

We'd like to find a "universal elliptic cohomology theory", but there's no universal elliptic curve so this doesn't quite make sense. That is, there's no moduli scheme that represents the functor taking a scheme to elliptic curves over it; elliptic curves can have nontrivial automorphisms, so for instance an elliptic curve $\mathcal{E} \to X$ might have that all its geometric fibers $\mathcal{E}_0$ are isomorphic without being a product. A universal elliptic curve would have to have exactly one geometric fiber isomorphic to $\mathcal{E}_0$, but then the map from X classifying $\mathcal{E}$ would have to be constant, which would imply that $\mathcal{E}$ is a product.

Nevertheless, we can consider the category of elliptic curves over a fixed base as a groupoid, and then we have a perfectly well-defined moduli *stack* of elliptic curves $\mathcal{M}_{1,1}$. This is a *Deligne-Mumford stack*, meaning that there are "enough" étale morphisms from schemes.[1] If a morphism $\phi : \text{Spec } R \to \mathcal{M}_{1,1}$ is étale (or even just flat), the resulting formal group $\widehat{\mathcal{E}}_\phi$ is Landweber-exact. We denote the associated elliptic cohomology theory A_ϕ.

The functor $\{\phi : \text{Spec } R \to \mathcal{M}_{1,1} \text{ étale}\} \rightsquigarrow A_\phi$ defines a presheaf $\overline{\mathcal{O}}$ of cohomology theories on the étale site of $\mathcal{M}_{1,1}$ (or at least on its restriction to affine schemes), which we will also denote by $\mathcal{M}_{1,1}$. We'd like to take global sections to get a universal elliptic cohomology theory, but the geometry of "cohomology theories" is rather unmanageable. Instead, we'd like to lift $\overline{\mathcal{O}}$ to a presheaf of spectra. In fact, this involves some very difficult obstruction theory, but Hopkins and Miller realized that it's actually easier to lift $\overline{\mathcal{O}}$ to a presheaf of E_∞*-ring spectra* (or simply E_∞*-rings*); these are very rigid, so although it's harder to write down maps between them, it's also harder to write down the wrong maps between them, and hence the obstruction theory simplifies. This allowed Goerss, Hopkins, and Miller to prove the following theorem.

Theorem 1. *There exists a commutative diagram*

$$\begin{array}{ccc} & & E_\infty\text{-}\mathtt{Rings} \\ & \overset{\mathcal{O}_{\mathcal{M}^{\mathrm{Der}}}}{\nearrow} & \big\downarrow \\ \mathcal{M}_{1,1} & \xrightarrow{\overline{\mathcal{O}}} & \mathtt{Cohomology\ Theories} \end{array}$$

such that $\pi_0 \mathcal{O}_{\mathcal{M}^{\mathrm{Der}}} \simeq \mathcal{O}_{\mathcal{M}_{1,1}}$.

[1] What's the right thing to say here?

This gives a *derived* version of the moduli stack: it has the same topos but a new structure sheaf. We can now finally define $\texttt{tmf}[\Delta^{-1}] = \text{holim}\ \mathcal{O}_{\mathcal{M}^{\text{Der}}}$. (Here, Δ is the *discriminant* of an elliptic curve, which is invertible iff the curve is smooth. To obtain $\texttt{tmf}$, we must replace $\mathcal{M}_{1,1}$ with its Deligne-Mumford compactification $\overline{\mathcal{M}_{1,1}}$.) However, since $\mathcal{M}_{1,1}$ isn't affine, this loses a lot of information; we will consider $\mathcal{O}_{\mathcal{M}^{\text{Der}}}$ as our primary interest, which we will rediscover in the next section when we study *equivariant* cohomology theories.

Equivariant cohomology and motivation for the derived perspective

There's a naive definition of equivariant cohomology, called *Borel-equivariant cohomology*: for any theory A, a compact Lie group G, and a G-space X, we define $A_G^{\text{Bor}}(X) = A((X \times EG)/G)$ (the space $X \times EG$ just gives the "freeification" of the G-action on X). But group actions are delicate and really ought to be studied on the nose instead of just up to homotopy, so this doesn't generally give us what we truly want. For instance, the *equivariant K-theory* of a G-space X, denoted $K_G(X)$, is defined via vector bundles $E \to X$ where E is a G-space and the projection is G-equivariant. The *Atiyah-Segal completion theorem* implies that the natural map

$$\text{Rep}(G) = K_G(*) \to K_G(* \times EG) = K(BG) = K_G^{\text{Bor}}(*)$$

identifies the target as the completion of the representation ring $\text{Rep}(G)$ at the augmentation ideal of virtual representations with virtual dimension 0.

Let's look at the simplest case, where $G = S^1$. We can readily compute that $\text{Rep}(S^1) \cong \mathbb{Z}[\chi^{\pm}]$, where $\chi : S^1 \hookrightarrow \mathbb{C}^{\times}$ is the standard inclusion, and also that $K(BS^1) = K(\mathbb{C}P^{\infty}) \cong \mathbb{Z}[[c]]$, where $c = [\mathcal{O}(1)] - 1$ is the generalized first Chern class. The map $\mathbb{Z}[\chi^{\pm}] \to \mathbb{Z}[[t]]$ is then given by $\chi \mapsto (t+1)$.

The crucial insight is that we should really by viewing $\text{Rep}(S^1) = \mathcal{O}(\mathbb{G}_m)$, $K(\mathbb{C}P^{\infty}) = \mathcal{O}(\widehat{\mathbb{G}}_m)$, and $\mathcal{O}(\mathbb{G}_m) \to \mathcal{O}(\widehat{\mathbb{G}}_m)$ as restriction of functions. This suggests that given any algebraic group $\mathbb{G}$ and cohomology theory A with $\widehat{\mathbb{G}} \cong \text{Spf}\ A(\mathbb{C}P^{\infty})$, we should expect that $A_{S^1}(*) = \mathcal{O}(\mathbb{G})$. This yields a "completion map" $A_{S^1}(*) \to A(\mathbb{C}P^{\infty}) = A_{S^1}^{\text{Bor}}(*)$, which again should just be restriction of functions.

This is a great idea, but unfortunately we don't get A_G from $A_G(*)$ (even when $G = \{e\}$; the Atiyah-Hirzebruch spectral sequence measures the extent to which a cohomology theory isn't determined by its coefficients). Rather, we want restriction of functions to give us a *derived* completion map $A_G \to A_G^{\text{Bor}}$ of cohomology theories, which returns the above completion map as a special case. This suggests that $\mathbb{G}$ should be an algebraic group with sheaf of functions taking values in E_∞-rings.

Derived algebraic geometry

We will assume the reader is familiar with the basic ideas surrounding E_∞-rings, which are just E_∞-algebras in the category of (naive) ring spectra. They are inherently ∞-categorical in nature, so instead of hom-sets they have hom-spaces. There are two points of view which we will employ.

1. There is an analogy

   $$E_\infty\text{-rings} : \text{commutative rings} :: \text{commutative rings} : \text{reduced commutative rings};$$

   we should think of the functor $A \rightsquigarrow \pi_0 A$ as analogous to the functor $R \rightsquigarrow R/\mathfrak{n}_R$, where $\mathfrak{n}_R \subset R$ is the ideal of nilpotents.

2. An E_∞-ring determines functors $X \rightsquigarrow A^X \rightsquigarrow A^n(X)$, where $A^n(X) = \pi_{-n}(A^X)$ for $n \in \mathbb{Z}$, and in particular $X \rightsquigarrow A(X) = A^0(X)$ becomes a cohomology theory with multiplicative structure and higher-order cohomology operations.

Now, let A be an E_∞-ring. We define the associated *affine derived scheme* $(\text{Spec}\ A, \mathcal{O}_{\text{Spec}\ A})$ following classical algebraic geometry very closely:

- The topological space $\text{Spec}\ A$ is just the topological space $\text{Spec}\ \pi_0 A$.
- Given any $f \in \pi_0 A$, on the distinguished open set $D(f)$ we put $\mathcal{O}_{\text{Spec}\ A}(D(f)) = A[f^{-1}]$.

Of course, $A[f^{-1}]$ is again an E_∞-ring; it is a localization $A \to A[f^{-1}]$, which is characterized either by the fact that $\pi_* A \to \pi_*(A[f^{-1}])$ identifies $\pi_*(A[f^{-1}])$ with $(\pi_* A)[f^{-1}]$, or the fact that for any E_∞-ring B, the induced map $\mathrm{Hom}(A[f^{-1}], B) \to \mathrm{Hom}(A, B)$ is a homotopy equivalence of the source onto the subspace of the target consisting of those maps $A \to B$ carrying f to an invertible element of $\pi_0 B$. (This is a collection of path components of $\mathrm{Hom}(A, B)$.) Of course, a *derived scheme* is simply a pair $(X, \mathcal{O}_X)$ of a topological space and a sheaf of E_∞-rings which is locally equivalent to an affine derived affine. These are also ∞-categorical; given derived schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, their hom-space is defined by

$$\mathrm{Hom}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)) = \coprod_{f : X \to Y} \mathrm{Hom}_0(\mathcal{O}_Y, f_* \mathcal{O}_X),$$

where $\mathrm{Hom}_0(\mathcal{O}_Y, f_*\mathcal{O}_X) \subset \mathrm{Hom}(\mathcal{O}_Y, f_*\mathcal{O}_X)$ denotes the subspace of *local* maps of sheaves of E_∞-rings, i.e. maps which induce local homomorphisms $\pi_0 \mathcal{O}_{Y, f(x)} \to \pi_0 \mathcal{O}_{X,x}$ for every $x \in X$.

Classical rings can be viewed as E_∞-rings (via the Eilenberg-Mac Lane functor), and so a classical scheme can be viewed as a derived scheme (although the structure sheaf needs to be re-sheafified in this new context). Conversely, given a derived scheme $(X, \mathcal{O}_X)$, we can define a presheaf $U \rightsquigarrow \pi_0(\mathcal{O}_X(U))$ of classical rings on X. Its sheafification yields a classical scheme, and we denote the result $(X, \mathcal{O}_X)_0$ (or just X_0). However, note that this functor is only right adjoint to the inclusion of classical schemes if we restrict to *connective* E_∞-rings, i.e. those which have no homotopy groups below dimension 0.

E_∞-rings admit a notion of *module* (*spectra*): an A-module M is just a spectrum with an action map $A \wedge M \to M$ satisfying the usual diagrammatic axioms.

Definition 1. Let A be an E_∞-ring and M be a A-module. We say that M is *flat* if:

1. $\pi_0 M$ is classically flat as a $\pi_0 A$-module.

2. For all $n \in \mathbb{Z}$, the map $\pi_n A \otimes_{\pi_0 A} \pi_0 M \to \pi_n M$ induced by the action $A \wedge M \to M$ is an isomorphism.

If $p : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a map of derived schemes, we say p is *flat* if for all affine opens $U \subset X$ and $V \subset Y$ such that $p(U) \subset V$, the induced map of E_∞-rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is flat.

If $p : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is flat, then $\pi_0 p : (X, \pi_0 \mathcal{O}_X) \to (Y, \pi_0 \mathcal{O}_Y)$ must be classically flat. Conversely, if $(Y, \mathcal{O}_Y)$ is a classical scheme, then for p to be flat, $(X, \mathcal{O}_X)$ must be a classical scheme as well (and $p = \pi_0 p$ must be classically flat).

Derived group schemes and orientations

Recall our dream of a derived completion map

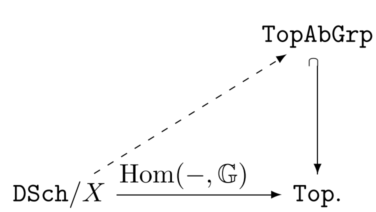
$$\mathcal{O}(\mathbb{G}) = A_{S^1}(*) \to A(\mathbb{C}P^\infty) = A^{\mathrm{Bor}}_{S^1}(*) = \mathcal{O}(\widehat{\mathbb{G}}).$$

We previously had the group scheme $\mathbb{G}$ defined over $A(*)$, i.e. we had a structure map $\mathbb{G} \to \mathrm{Spec}\ A(*)$. To improve this, we should instead take $\mathbb{G}$ to be a *derived* group scheme with structure map $\mathbb{G} \to \mathrm{Spec}\ A$.

Definition 2. Suppose X is a derived scheme. We say that a derived X-scheme $\mathbb{G}$ is a *commutative X-group* if:

1. The structure map $\mathbb{G} \to X$ is flat.

2. We have a specified lifting



We want some way of ensuring that our completion map is nontrivial. For this, we make the following definition.

Definition 3. If X is a derived scheme and $\mathbb{G}$ is a commutative X-group, then a *preorientation* of $\mathbb{G}$ is a map $\mathbb{C}P^\infty \to \mathbb{G}(X) = \mathrm{Hom}(X, \mathbb{G})$ of topological abelian groups. Since $\mathbb{C}P^\infty$ is the free topological abelian group on the based space S^2, this is equivalent to a based map $S^2 \to \mathrm{Hom}(X, \mathbb{G})$.

If $X = \mathrm{Spec}\ A$ (so $A = \mathcal{O}(X)$) and $\mathbb{G}$ is a commutative A-group, we can optimistically write $A_{S^1} = \mathcal{O}(\mathbb{G})$. Then we can compose our preorientation $\sigma : \mathbb{C}P^\infty \to \mathrm{Hom}(X, \mathbb{G})$ with the global sections map $\mathrm{Hom}(X, \mathbb{G}) \to \mathrm{Hom}(A_{S^1}, A)$ to obtain our desired derived completion map $A_{S^1} \to A^{\mathbb{C}P^\infty}$. The condition that the preorientation be a map of topological abelian groups ensures that the map $\mathrm{Spf}\ A^{\mathbb{C}P^\infty} \to \mathrm{Spec}\ A_{S^1}$ be compatible with the group structures.[2] Morally, a preorientation is an *orientation* if the derivative of this map is invertible. Fortunately, there is a way to make this precise without using formal derived algebraic geometry.

Recall that a preorientation is given by a based map $\sigma : S^2 \to \mathbb{G}(\mathrm{Spec}\ A) = \mathrm{Hom}(\mathrm{Spec}\ A, \mathbb{G})$. In classical algebraic geometry, the image of an affine is always contained in an affine; this holds in the derived setting too. Given any affine $\mathrm{Spec}\ B \subset \mathbb{G}$ containing the image $e(\mathrm{Spec}\ A)$ of the identity section, we necessarily have a factorization

$$\begin{array}{ccc} S^2 & \xrightarrow{\sigma} & \mathrm{Hom}(\mathrm{Spec}\ A, \mathbb{G}) \\ & \searrow & \uparrow \\ & & \mathrm{Hom}(\mathrm{Spec}\ A, \mathrm{Spec}\ B), \end{array}$$

and of course $\mathrm{Hom}(\mathrm{Spec}\ A, \mathrm{Spec}\ B) \cong \mathrm{Hom}(B, A)$. (After all, a derived scheme is just a topological space with a fancy sheaf on it; since the basepoint of S^2 lands in $\mathrm{Hom}(B, A) \subset \mathrm{Hom}(\mathrm{Spec}\ A, \mathbb{G})$, the rest of its points can only vary the map on E_∞-rings but cannot change the underlying map on classical schemes.) By the usual exponential adjunction, $\sigma : S^2 \to A^B$ corresponds to some $\hat{\sigma} : B \to A^{S^2} = A^{\Sigma^\infty_+ S^2}$ (although actually it lands in $A^{\Sigma^\infty S^2} \subset A^{\Sigma^\infty_+ S^2}$). We apply π_0, noting that

$$\pi_0 A^{S^2} = [\mathbb{S}, A^{\Sigma^\infty_+ S^2}] \cong [\mathbb{S} \wedge \Sigma^\infty_+ S^2, A] = [\Sigma^\infty_+ (S^0 \vee S^2), A] \cong \pi_0 A \oplus \pi_2 A,$$

which gives us a map $\tilde{\sigma} : \pi_0 B \to \pi_0 A \oplus \pi_2 A$ of $\pi_0 A$-algebras. The map $\pi_0 B \to \pi_0 A$ is a ring homomorphism coming from the identity section $e : \mathrm{Spec}\ A \to \mathrm{Spec}\ B \subset \mathbb{G}$, and so by definition the map $\pi_0 B \to \pi_2 A$ is a *$\pi_0 A$-algebra derivation* of $\pi_0 B$ into $\pi_2 A$.[3] We write $\omega = e^* \Omega_{\mathbb{G}_0 / \pi_0 A} = e^* \Omega_{\pi_0 B / \pi_0 A}$, which we consider as a $\pi_0 A$-module. Then our $\tilde{\sigma} : \pi_0 B \to \pi_0 A \oplus \pi_2 A$ corresponds to a map $\beta : \omega \to \pi_2 A$ of $\pi_0 A$-modules.

Definition 4. We say that a preorientation $\sigma : S^2 \to \mathbb{G}(A)$ of $\mathbb{G} \to \mathrm{Spec}\ A$ is an *orientation* if:

1. The induced map $\beta : \omega \to \pi_2 A$ yields isomorphisms $\pi_n A \otimes_{\pi_0 A} \omega \to \pi_n A \otimes_{\pi_0 A} \pi_2 A \to \pi_{n+2} A$ for all $n \in \mathbb{Z}$.
2. $\mathbb{G}_0 \to \mathrm{Spec}\ \pi_0 A$ is smooth of relative dimension 1.

More generally, if our derived group scheme is defined over an arbitrary derived scheme X, we say that a the preorientation is an *orientation* if it is an orientation when restricted to every open affine $\mathrm{Spec}\ A \subset X$.

If $\mathbb{G}$ is oriented then A must be *weakly periodic*, i.e. the natural map $A^2(*) \otimes_{A(*)} A^n(*) \to A^{n+2}(*)$ must be an isomorphism for all $n \in \mathbb{Z}$. (This implies that $A^2(*)$ is a projective $A(*)$-module of rank 1; A is called *periodic* if it is also free.) Conversely, when A is weakly periodic, then our preorientation is an orientation iff β is an isomorphism.

Oriented elliptic curves

Definition 5. derived elliptic curve

Definition 6. derived D-M stack

[2]But what's the difference between $\mathrm{Spec}\ A_{S^1} = \mathrm{Spec}\ \mathcal{O}(\mathbb{G})$ and $\mathbb{G}$ itself? This will come up later.

[3]Recall that for a ring R, an augmented R-algebra S, and an R-module M, an *R-algebra derivation of S into M* is by definition a function $S \to M$ giving a map of augmented R-algebras $S \to R \oplus M$, where the target has multiplication given by $(r_1 + m_1) \star (r_2 + m_2) = r_1 r_2 + r_1 \cdot m_2 + r_2 \cdot m_1$. These are classified by the *module of relative* (or *Kähler*) *differentials* (i.e., of sections of the dual of the space of vertical tangent vectors), denoted $\Omega_{S/R}$. More precisely, for any R-module M, $\mathrm{Hom}_{\mathtt{AugAlg}_R}(S, R \oplus M) \cong \mathrm{Hom}_{\mathtt{Mod}_R}(\Omega_{S/R}, M)$.

notation, notation, notation. (Weird yoga.)

DIAGRAM OF PROOF OF THEOREM 4.1

Theorem 4.1: $\mathrm{Hom}(\mathrm{Spec}\ A, \mathcal{M}^{\mathrm{Der}}) \simeq E(A)$

define E' and$(\mathcal{M}, \mathcal{O}')$; define $\mathcal{O}$, assume (1) and (2)

(1) for $n = 2k$,
$\omega^k \xrightarrow{\sim} \pi_n \mathcal{O}$ in $\mathtt{QCoh}(\mathcal{M}_{1,1})$

(2) for $n = 2k+1$,
$\pi_n \mathcal{O} = 0$

Proposition 4.1 (realization theorem)

Let k be a perfect field of characteristic p,
$\kappa : \mathrm{Spec}\ k \to \mathcal{M}_{1,1}$ be a closed point classifying $E_0 \to \mathrm{Spec}\ k$,
and $\mathcal{O}'_\kappa$ be the formal completion of $\mathcal{O}'$ at κ
(so $\pi_0 \mathcal{O}'_\kappa$ is the formal completion of $\mathcal{O}_{\mathcal{M}_{1,1}}$ at κ).
Then there is a universal preoriented deformation $E \to \mathrm{Spec}\ \mathcal{O}'_\kappa$
of $E_0 \to \mathrm{Spec}\ k$, whose preorientation gives $\beta \in \pi_2 \mathcal{O}'_\kappa$.
Define $\mathcal{O}_\kappa = \mathcal{O}'_\kappa[\beta^{-1}]$. Then $\mathcal{O}_\kappa$ is even and $\pi_0 \mathcal{O}_\kappa = \pi_0 \mathcal{O}'_\kappa$.

Equivariant cohomology revisited

Fix an E_∞-ring A, a commutative A-group $\mathbb{G}$, and a compact abelian Lie group T. In the case $T = S^1$ we want that $A_{S^1}(*) = \mathcal{O}(\mathbb{G})$, and we will have $A_{S^1}(X) = \Gamma(\mathbb{G}, \mathscr{F}_{S^1}(X))$ for some quasicoherent sheaf $\mathscr{F}_{S^1}(X)$ over $\mathbb{G}$. More generally, for any T we will construct a derived scheme M_T such that $A_T(*) = \mathcal{O}(M_T)$ and $A_T(X) = \Gamma(M_T, \mathscr{F}_T(X))$ for some quasicoherent sheaf $\mathscr{F}_T(X)$ over M_T.

Let us write $T^\vee = \mathrm{Hom}(T, S^1)$, the *character group* of T. Then, we obtain the derived A-scheme M_T by $M_T(B) = D\mathrm{Hom}_{\mathtt{Grp}}(T^\vee, \mathbb{G}(B))$ (where really the "derived" part of derived-hom only matters if T isn't connected).

We should expect that $\mathcal{O}(M_T) = A_T(*) = A(*//T)$; that is, this shouldn't depend on the chosen basepoint of the orbifold $*//T$. We can rephrase this as saying that we'd like a factorization

$$\begin{array}{ccc} \mathtt{CpctAbLieGrp} & \xrightarrow{T \rightsquigarrow M_T} & \mathtt{DSch}/A \\ & \searrow^{T \rightsquigarrow BT} & \uparrow{\scriptstyle \widetilde{M}} \\ & & \{\text{unbased spaces } BT \text{ (for } T \in \mathtt{CpctAbLieGrp})\}. \end{array}$$

Via the method of the universal example, one proves that such functors $\widetilde{M}$ are equivalent to preorientations of $\mathbb{G}$.

Definition 7. A T-space X is called *finite* if it admits a filtration $\emptyset = X_0 \subset X_1 \subset \ldots \subset X_n = X$ where X_{i+1} is obtained from X_i by equivariantly attaching a T-equivariant cell $(T/T_i) \times D^{k+1}$ for some closed subgroup $T_i \le T$, i.e.

$$X_{i+1} = X_i \coprod_{(T/T_i) \times S^k} (T/T_i) \times D^{k+1}.$$

We now have the following theorem, which functorially provides for equivariant cohomology theories with respect to all compact abelian Lie groups.

Theorem 2. *There exist essentially unique functors* $\mathscr{F}_T : \texttt{Finite } T\texttt{-Spaces}^{op} \to \texttt{QCoh}_{M_T}$ *for all compact abelian Lie groups* T *such that:*

1. *T-equivariant homotopy equivalences are taken to equivalences of sheaves.*
2. *For any finite diagram* $\{X_\alpha\}$ *in* $\texttt{Finite } T\texttt{-Spaces}$, $\mathscr{F}_T(\text{hocolim } X_\alpha) \simeq \text{holim } \mathscr{F}_T(X_\alpha)$.
3. $\mathscr{F}_T(*) = \mathcal{O}_{M_T}$.
4. *If* X *is a finite* T*-space,* $T \le T'$, $X' = (X \times T')/T$, *and* $f : M_T \to M_{T'}$ *is the morphism of derived schemes induced by the inclusion of groups, then* $\mathscr{F}_{T'}(X') = f_* \mathscr{F}_T(X)$.

(Actually we need some further naturality requirements, e.g. the isomorphisms in the final condition should behave well with respect to sequences of inclusions $T \le T' \le T''$*.)*

Then, we define $A_T(X) = \Gamma(M_T, \mathscr{F}_T(X))$, a module spectrum over $\Gamma(M_T, \mathcal{O}_{M_T}) = A_T(*)$, or if we wish we can even pass to classical algebra by defining $A_T^n(X) = \pi_{-n} A_T(X)$.

If T acts transitively on X, then there is a noncanonical isomorphism $X \cong T/T'$ for some closed subgroup $T' \le T$; a choice of isomorphism is equivalent to a choice of basepoint for X. So, the identification $\mathscr{F}_T(X) \cong f_* \mathscr{F}_{T'}(*) = f_* \mathcal{O}_{M_{T'}}$ is noncanonical. However, this is precisely accounted for by the fact that M_T only depends on the unbased space BT via the functor $\widetilde{M}$; a preorientation is precisely what we need for this all to work.

A good T-equivariant cohomology theory should be $\text{Rep}(T)$-graded instead of $\mathbb{Z}$-graded. By general representability theorems, A_T^0 is represented (T-equivariantly) by a T-space $Z(0)$, which is an infinite loopspace; more generally, A_T^n is represented by a T-space $Z(n)$ which is an n-fold delooping of $Z(0)$. Our new demand is equivalent to demanding deloopings of $Z(0)$ with respect to *representation spheres* $S^V = V \cup \{\infty\}$ (for V a T-representation). We carry this out presently.

If $X' \subset X$ is an inclusion of finite T-spaces, we define the "relative cohomology" by $\mathscr{F}_T(X, X') = \text{fib}(\mathscr{F}_T(X) \to \mathscr{F}_T(X'))$. Note that the projections $X \leftarrow X \times Y \to Y$ give maps $\mathscr{F}_T(X) \to \mathscr{F}_T(X \times Y) \leftarrow \mathscr{F}_T(Y)$, or equivalently a map $\mathscr{F}_T(X) \otimes \mathscr{F}_T(Y) \to \mathscr{F}_T(X \times Y)$; this extends to $\mathscr{F}_T(X, X') \otimes \mathscr{F}_T(Y) \to \mathscr{F}_T(X \times Y, X' \times Y)$ or even to $\mathscr{F}_T(X, X') \otimes \mathscr{F}_T(Y, Y') \to \mathscr{F}_T(X \times Y, X_0 \times Y \cup X \times Y_0)$.

Theorem 3. *Let* A *be an* E_∞*-ring,* $\mathbb{G}$ *an oriented* A*-group,* T *a compact abelian Lie group,* V *a finite-dimensional unitary* T*-representation, and denote by* $SV \subset BV$ *the unit sphere and ball of* V *respectively. Then* $\mathcal{L}_V = \mathscr{F}_T(BV, SV)$ *is a line bundle (i.e. it is invertible), and for all* T*-spaces* X*, the map* $\mathcal{L}_V \otimes \mathscr{F}_T(X) \to \mathscr{F}_T(X \times BV, X \times SV)$ *is an isomorphism.*

By this theorem, we get equivalences $\mathcal{L}_V \otimes \mathcal{L}_W \xrightarrow{\sim} \mathcal{L}_{V \oplus W}$, so we can actually extend the definition above to virtual representations. Now for any virtual T-representation V, we can define $A_T^V(X) = \pi_0 \Gamma(M_T, \mathscr{F}_T(X) \otimes \mathcal{L}_V^{-1})$. This is represented by a T-space $Z(V)$; if V is an honest T-representation, then $Z(V)$ is a S^V-delooping of $Z(0)$, i.e. $\text{Hom}^T(S^V, Z(V)) \simeq Z(0)$.[4]

For the case of a general compact Lie group G which isn't necessarily abelian, we extrapolate formally from the results above.

Theorem 4. *Let* A *be an* E_∞*-ring. There exist essentially unique functors* $A_G : G\texttt{-Spaces}^{op} \to \texttt{Spectra}$ *for all compact Lie groups* G *such that:*

1. *G-equivariant homotopy equivalences are taken to equivalences of spectra.*
2. *If* $H \le G$ *is an inclusion of compact Lie groups, then there exist natural equivalences* $A_H(X) \simeq A_G((X \times G)/H)$.
3. *For any diagram* $\{X_\alpha\}$ *of* G*-spaces,* $A_G(\text{hocolim } X_\alpha) \simeq \text{holim } A_G(X_\alpha)$.
4. *If* G *is abelian and* X *is a finite* G*-space, we recover the notions above.*
5. *Let* $E^{ab}G$ *be any* G*-space such that for all closed subgroups* $H \le G$*, the* H*-fixed points* $(E^{ab}G)^H$ *are empty if* H *is nonabelian and weakly contractible if* H *is abelian. Then for all* G*-spaces* X, $A_G(X) \xrightarrow{\sim} A_G(X \times E^{ab}G)$.

[4]Notation for equivariant maps?

It might seem more natural to replace the last two conditions with:

4'. If $G = \{e\}$, then $A_G(X) = A^X$.

5'. For any G-space X, $A_G(X) \xrightarrow{\sim} A_G(X \times EG)$.

However, it turns out that these new conditions actually characterize Borel-equivariant cohomology. On the other hand, if $A = K$ and $\mathbb{G} = \mathbb{G}_m$, then for any compact Lie group G, the conditions of the theorem recover equivariant K-theory. More generally, when G is connected we can actually mimic the previous discussion: we get something like derived schemes M_G and functors $\mathscr{F}_G$, and if the theorem concerning $\mathcal{L}_V$ still holds, then condition 5 above still holds too.