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Spring 2006, Math 104
Solutions to the **First Midterm**

10 Feb., 2006
3:10-4:00 PM

1. (32 points, 8 points each.) Complete the following definitions. You may use, without defining them, terms or symbols that Rudin defines before he defines the word or symbol asked for. Your definitions do not have to have exactly the same wording as those in Rudin, but for full credit they should be clear, and mean the same thing as his.

(a) An *ordered field* is a field F given with an ordering $<$ which (in addition to the conditions defining "field" and "ordered set") satisfies

Answer: (i) *If* $x, y, z \in F$ *and* $y < z$, *then* $x+y < x+z$.
(ii) *If* $x, y \in F$ *with* $x > 0$ *and* $y > 0$, *then* $xy > 0$.

(b) In the field of complex numbers, regarded as consisting of pairs (a, b) of real numbers, the real numbers are identified with the subfield consisting of *Answer: the pairs of the form* $(a, 0)$.

(c) Two sets A and B are said to have the same *cardinality* (or in Rudin, to be *equivalent,* written $A \sim B$) if *Answer: there exists a one-to-one mapping of* A *onto* B. *(In place of "one-to-one mapping of* A *onto* B*" you can say "bijection from* A *to* B*" or "one-to-one correspondence between* A *and* B*". But "one-to-one mapping from* A *to* B*", without a specification that the mapping is onto, will not get full credit.)*

(d) If X is a metric space and E a subset of X, then a point $p \in E$ is said to be an *interior point* of E if *Answer: there is a neighborhood* N *of* p *in* X *which is contained in* E.

2. (32 points, 8 points each.) For each of the items listed below, either *give an example* with the properties stated, or give a brief reason why *no such example exists.*

If you give an example, you do *not* have to prove that it has the property stated; however, your examples should be specific; i.e., even if there are many objects of a given sort, you should name a particular one. If you give a reason why no example exists, don't worry about giving reasons for your reasons; a simple statement will suffice.

(a) An ordered field which does not have the least upper bound property.

Answer: The field Q *of rational numbers.*

(b) Two members of the set of extended real numbers whose sum is not defined.

Answer: $+\infty$ *and* $-\infty$.

(c) A bijection (one-to-one correspondence) between the field Q of rational numbers and the field R of real numbers. *Answer: Does not exist:* Q *is countable but* R *is uncountable.*

(d) A set of real numbers which is neither open nor closed in R.

Answer: Many possible examples; e.g., Q, *the half-open interval* $[0,1)$ ....

3. (18 points) Prove that $\sup\{x+y-z \mid x, y, z \in R,\ x < y < z < 0\} = 0$.

Answer: Since $y < z$, *the term* $y-z$ *is* < 0, *so* $x+y-z < x+0 = x < 0$; *so* 0 *is an upper bound for the set shown. Moreover, given any* $c < 0$, *if we take, say,* $x = c/2$, $y = c/4$, $z = c/8$, *we have* $x < y < z < 0$ *and* $x+y-z = 5/8\,c > c$, *so* c *is not an upper bound for the above set. Since* 0 *is an upper bound, but no* $c < 0$ *is,* 0 *is the least upper bound.*

4. (18 points) For any subset E of a metric space X, let us write $L(E)$ for the set of all limit points of E in X. Prove that if $E_1, E_2, \ldots, E_n, \ldots$ is a sequence of subsets of X, then $L(\bigcap_{n=1}^{\infty} E_n) \subseteq \bigcap_{n=1}^{\infty} L(E_n)$.

Answer: What we need to show is that for every $p \in X$, *if* (i) $p \in L(\bigcap_{n=1}^{\infty} E_n)$ *then* (ii) $p \in \bigcap_{n=1}^{\infty} L(E_n)$.

Statement (i) *means that every neighborhood* N *of* p *contains a point* q $\neq p$ *of* $\bigcap_{n=1}^{\infty} E_n$. *To say that* q *lies in that intersection says that it lies in* E_n *for every* n. *Hence for each* n, *each neighborhood* N *of* p *contains a point* q *of* E_n, *showing that* $p \in L(E_n)$. *Since this is true for each* n, *we have* $p \in \bigcap_{n=1}^{\infty} L(E_n)$, *i.e., statement* (ii), *as required.*

Reminder: The reading for Monday is #9