MATH 185 – FINAL EXAM Spr 02

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INSTRUCTIONS: Answer each question on a separate sheet of paper, and write your name on each page. Each problem counts equally. Good luck.

Problem #1. Calculate the residue of

$$\frac{z^5}{z^4+16}$$

at $z_0 = \sqrt{2}(1+i)$.

Problem #2. Show that

$$\left|\frac{2z-i}{2+iz}\right| = 1$$

if $|z| = 1$.

Problem #3. Compute the first 4 terms of the Laurent expansion for

$$\frac{1}{e^z-1}$$

around $z_0 = 0$.

Problem #4. Find complex numbers a, b, c, d so that the linear fractional transformation

$$T(z) = \frac{az+b}{cz+d}$$

satisfies

$$T(-1) = \infty, \quad T(0) = 1, \quad T(i) = i.$$

Problem #5. Compute

$$\int_0^{2\pi} \frac{1}{3+\cos\theta}\, d\theta.$$

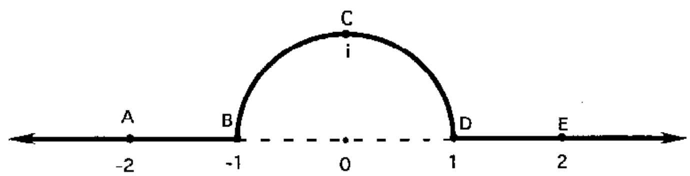
Problem #6. Assume $a, b > 0$ and calculate

$$\int_{-\infty}^{\infty} \frac{\cos bx - \cos ax}{x^2}\, dx.$$

Problem #7. (i) Find the image of the curve drawn in the picture under the mapping

$$f(z) = z + \frac{1}{z}.$$

In particular, plot the images A', B', C', D', E' of the points A, B, C, D, E.



(ii) What is $f'(z)$ at the points $z = B, D$? Is the mapping conformal at these points? Explain your answer.

Problem #8. (i) How many zeros (counting multiplicity) does the polynomial

$$z^8 + 3z^5 + 8z^2 + z + 1$$

have within the region $|z| < 1$?

(ii) How many zeros does this polynomial have within the region $|z| < 2$?

Explain carefully how you reached your answers.

Problem #9. Let C be a simple closed curve, which is the boundary of a region R. Assume f is analytic within R and f' is continuous on $R \cup C$.

Use Green's Theorem to prove *Cauchy's Theorem:*

$$\int_C f\,dz = 0.$$

Problem #10. Assume that f is analytic within the ring $R_1 < |z| < R_2$. Start with Cauchy's integral formula and derive the *Laurent expansion:*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n}.$$

Write out integral formulas for the coefficients a_n, b_n.

Problem #11. Suppose that f is analytic within a domain D, and $|f(z)| = 1$ for all $z \in D$.

Prove that f is constant.

Problem #12. Prove the *Casorati–Weierstrass Theorem*:

Assume f is analytic for $0 < |z-z_0| < 1$, and suppose that z_0 is an essential singularity of f. Let w_0 be any complex number. Then for each $\epsilon > 0$ and each $\delta > 0$, the inequality

$$|f(z) - w_0| < \epsilon$$

is satisfied for some point z in the deleted neighborhood $0 < |z - z_0| < \delta$.

(Hint: You may quote any relevant theorem on removable singularities.)