Math 54: Linear Algebra and Differential Equations Worksheets

7^{th} Edition

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Preface

This booklet contains the worksheets for Math 54, U.C. Berkeley's linear algebra course.

The introduction to each worksheet very briefly motivates the main ideas but is not intended as a substitute for the textbooks or lectures. The questions emphasize qualitative issues and the problems are more computationally intensive. The additional problems are more challenging and sometimes deal with technical details, or tangential concepts.

Typically, more problems were provided on each worksheet than can be completed during a discussion period. This was not a scheme to frustrate the student; rather, we aimed to provide a variety of problems that can reflect different aspects professors and G.S.I.'s may choose to emphasize.

William Stein coordinated the 5th edition in consultation with Tom Insel; then Michael Wu has reorganized the 2000 edition. Michael Hutchings made tiny changes in 2012 for the 7th edition.

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1. Introduction to Linear Systems

Introduction

A linear equation in n variables is an equation of the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b,$$

where $a_1, a_2, ..., a_n$ and b are real numbers (constants). Notice that a linear equation doesn't involve any roots, products, or powers greater than 1 of the variables, and that there are no logarithmic, exponential, or trigonometric functions of the variables. Solving a linear equation means finding numbers $r_1, r_2, ..., r_n$ such that the equation is satisfied when we make the substitution $x_1 = r_1, x_2 = r_2, \ldots, x_n = r_n$. In this course we will be concerned with solving *systems* of linear equations, that is, finding a sequence of numbers $r_1, r_2, \ldots, r_n$ which simultaneously satisfy a given set of equations in n variables. No doubt you have solved systems of equations before. In this course we will not only learn techniques for solving more complicated systems, but we will also be concerning ourselves with important properties of the solution sets of systems of equations.

Questions

1. (a) What does the graph of $x + 2y = 5$ look like?
   (b) What does the graph of $2x - 3y = -4$ look like?
   (c) Do the two graphs above intersect? If so, what does their intersection look like?
2. Write down a system of two linear equations in two unknowns which has no solution. Draw a picture of the situation.
3. Suppose you have a system of two linear equations in three unknowns. If a solution exists, how many are there? What might the set of solutions look like geometrically?

Problems

1. Solve the following system of equations and describe in words each step you use.

$$\begin{aligned} x + 3y - z &= 1 \\ 3x + 4y - 4z &= 7 \\ 3x + 6y + 2z &= -3 \end{aligned}$$

   How many solutions are there, and what does the solution set look like geometrically?

2. Find all solutions of the system

$$\begin{aligned} x + y - 3z &= -5 \\ -5x - 2y + 3z &= 7 \\ 3x + y - z &= -3 \end{aligned}$$

   Describe (but don't draw) the graphs of each of the three above equations and their intersection.

3. What condition on a, b, c, and d will guarantee that there will be exactly one solution to the following system?

$$\begin{aligned} ax + by &= 1 \\ cx + dy &= 0 \end{aligned}$$

4. Consider a system of four equations in three variables. Describe in geometric terms conditions that would correspond to a solution set that
   (a) is empty.
   (b) contains a unique point.
   (c) contains an infinite number of points.

Additional Problems

1. Set up a system of linear equations for the following problem and then solve it:

   The three-digit number N is equal to 15 times the sum of its digits. If you reverse the digits of N, the resulting number is larger by 396. Also, the units (ones) digit of N is one more than the sum of the other two digits. Find N.

2. Consider the system of equations

$$\begin{aligned} ax + by &= k \\ cx + dy &= l \\ ex + fy &= m \end{aligned}$$

   Show that if this system has a solution, then at least one equation can be thrown out without altering the solution set.

2. Matrices and Gaussian Elimination

Introduction

A linear system corresponds to an augmented matrix, and the operations we use on a linear system to solve it correspond to the elementary row operations we use to change a matrix into row echelon form. The process is called **Gaussian elimination**, and will come in handy for the rest of the semester.

Questions

1. *True* or *False*: The augmented matrix for the system

$$\begin{aligned} 2x_1 - 3x_2 + x_3 - x_4 &= 4 \\ 3x_1 + 2x_2 + x_3 - 3x_4 &= 1 \\ 5x_1 + x_2 - x_3 + x_4 &= 3 \\ 2x_1 - 5x_2 + 4x_3 - 6x_4 &= 6 \end{aligned} \quad \text{is} \quad \begin{bmatrix} 2 & -3 & 1 & -1 \\ 3 & 2 & 1 & -3 \\ 5 & 1 & -1 & 1 \\ 2 & -5 & 4 & -6 \end{bmatrix}.$$

2. (a) Identify the first pivot of the matrix

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ -2 & 3 & -5 & 0 \\ -1 & 2 & -1 & 0 \\ 1 & 0 & -1 & 3 \end{bmatrix}.$$

   (b) If that pivot in part (a) was not in the first row, interchange rows so that it is.

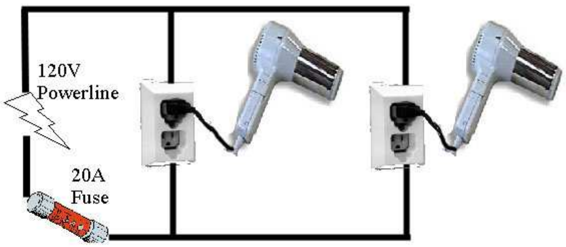
   (c) Now add suitable multiples of the first row to the other rows to make all other entries in the first column zero.

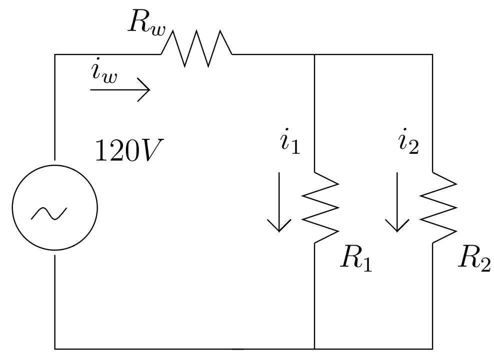
   (d) Ignoring the first row, find the next pivot and repeat steps (b) and (c) on the second column.

   (e) Continue until all of the rows that contain only zeros are at the bottom of the matrix and each pivot appears to the right of all the pivots above it.

Problems

1. You and a friend rent a room in an old house and find that if both of you are using your blow dryers, the 20 amp fuse for that circuit occasionally blows. Each blow dryer has a high and a low power setting, which has the effect of fixing the electrical resistance. The wire from the fuse box is as old as the house, and has an additional resistance which newer wiring would minimize.





Equations for the current flowing through each element of the circuit are obtained from Kirchhoff's laws. The first equation states that the current flowing through the wire goes into one or the other blow dryer, so that

$$i_w = i_1 + i_2$$

where w refers to the wire, while 1 and 2 refer to the blow dryers. Two additional equations result from the fact that for any loop of the circuit, the voltage dissipated by resistors must equal the source voltage. Thus,

$$\begin{aligned} R_w i_w + R_1 i_1 &= V \\ R_w i_w + R_2 i_2 &= V \end{aligned}$$

where R_w, R_1 and R_2 are the resistances and V is the line voltage (120 Volts).

(a) Write this system of three equations in matrix form $AX = B$, where X is a column vector whose entries are the three unknown currents.

(b) Solve this matrix equation for the currents when both blow dryers are in use. Consider three cases: both blow dryers operating at low power; both at high power; and one on low and one on high. Let the two power settings be 1000W and 1500W, for which the associated resistances are 15Ω and 10Ω, respectively. (Note that higher resistance reduces the power drawn by the dryer.) In all cases, let the wire resistance be $R_w = 0.5\Omega$. Can both blow dryers be used simultaneously under any conditions? Under what conditions will the fuse blow?

2. Write down the augmented matrix for the given system of equations and then reduce to row echelon form.

(a)
$$\begin{aligned} x_1 + 2x_2 \phantom{{}+ x_3} - x_4 &= -1 \\ -x_1 - 3x_2 + x_3 + 2x_4 &= 3 \\ x_1 - x_2 + 3x_3 + x_4 &= 1 \\ 2x_1 - 3x_2 + 7x_3 + 3x_4 &= 4 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 9 \\ 2x_1 - x_2 + x_3 &= 0 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 4x_1 - x_2 + x_3 &= 4 \end{aligned}$$

3. Solve the system in part (a) of problem 1.

4. Suppose you accept a software maintenance job in which you make \$80 a day for each day you show up to work, but are penalized \$20 per day that you don't go to work. After 60 days you find you've earned \$2200. How many days have you gone to work? (Assume that you were expected to work during each of the 60 days.) You may wish to set up a system of two linear equations and solve it.

5. Find a linear system in 3 variables, or show that none exists, which:

   (a) has the unique solution $x = 2$, $y = 3$, $z = 4$.

   (b) has infinitely many solutions, including $x = 2$, $y = 3$, $z = 4$.

6. As you know, two points determine a line. But what does this mean? The equation of a line is $ax + by + c = 0$. Use a linear system to find an equation of the line through the points $(-1, 1)$ and $(2, 0)$. Check your answer. How can two equations determine the three unknowns a, b, and c?

Additional Problems

1. For which values of λ does the system

$$\begin{aligned} (\lambda - 3)x + y &= 0 \\ x + (\lambda - 3)y &= 0 \end{aligned}$$

   have more than one solution?

2. Suppose that the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 0 \end{aligned}$$

   has only $x_1 = x_2 = x_3 = 0$ as a solution (the *trivial solution*). Then consider the system obtained from the given system by replacing the three zeros on the right with three 1's.

   (a) Must this new system have a solution or is it possible that the solution set of the new system is empty?

   (b) Might the new system have more than one solution?

3. The Algebra of Matrices

Introduction

You've seen how matrices are used in solving systems of equations, and how elementary row operations on a matrix can be useful. Multiplication of matrices is yet another tool for solving systems of equations. With an operation such as multiplication, we can analyze how certain matrices relate to each other, and how certain systems of equations can be simplified.

Questions

1. Answer the following with *True* or *False*. Explain your reasoning, or give a counterexample.

    (a) If A and B are any matrices, then $A + B$ is defined.

    (b) If A and B are both $n \times n$ matrices, then $A + B = B + A$.

    (c) If A and B are both $n \times n$ matrices, then $AB = BA$.

2. Suppose that Math 54W is being taught by two different professors. Prof. A's lecture is more popular than Prof. B's lecture. In fact, each week 90% of A's students remain in the lecture, while only 10% switch into B's lecture. On the other hand, 20% of B's students switch into A's lecture, with 80% remaining in B's section.

    This situation is described in the following table

	from A	from B
into A	90%	20%
into B	10%	80%

    which can be represented by the matrix $\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$. Supposing that at the start of the semester each professor had 200 students, use matrix multiplication to answer the following:

    (a) How many students are there in each professor's section after the 1'st week? (Hint: represent the number of students in each section by a 2×1 column matrix.)

    (b) How many students are there in each professor's section after the second week of classes?

Problems

1. (a) Write the following system as a matrix equation of the form $AX = B$.

$$\begin{aligned} 6x + 5y + 2z &= 11 \\ 5x + 4y + 2z &= 7 \\ -3x - 3y - z &= 4 \end{aligned}$$

   (b) Show that the same system of equations can also be written as a 1×3 matrix times a 3×3 matrix equaling a 1×3 matrix.

2. Let $A = \begin{bmatrix} -2 & 1 & -2 \\ 1 & 0 & 2 \\ 3 & -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \\ -1 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$.

   Which of the following matrix multiplications are defined? Compute those which are defined.

   (a) AB (b) BC (c) CA (d) ABC

3. Let A and B be $n \times n$ matrices. Under what conditions is it true that

$$(A + B)(A - B) = A^2 - B^2?$$

4. (a) What special properties does the matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ possess?

   (b) Given a 2×2 matrix A, can you always find another matrix B so that $AB{=}I$?

   (c) Given two 2×2 matrices A and B such that $AB = I$, is there anything noteworthy about BA?

5. Compute $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n$. What is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

6. Square matrices A and B are said to **commute** if $AB = BA$. Find all 2×2 matrices which commute with:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix};$$

$$E = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}; \quad F = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}; \quad G = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}.$$

   What patterns do you notice? For some of these it might help to notice that if A and B commute with M, then $A + B$ also commutes with M. What matrix always commutes with a square matrix M?

Additional Problems

1. Suppose x is a real number satisfying $x^2 = 1$. To solve for x, we factor $x^2 - 1 = (x-1)(x+1) = 0$, and conclude that $x = \pm 1$. What if X is a 2×2 matrix satisfying $X^2 = I$?

   (a) Show that $(X - I)(X + I) = 0$.

   (b) Are those the only solutions? What about $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$?

   (c) Let a be any real number, and let $A = \pm \begin{bmatrix} 1 & 0 \\ a & -1 \end{bmatrix}$. Show that $A^2 = I$.

   (d) Let b be any real number, and let $B = \begin{bmatrix} b & 1 - b^2 \\ 1 & -b \end{bmatrix}$. Show that $B^2 = I$.

   (e) Let $0 \leq \theta \leq 2\pi$, and let $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$. Show that $R^2 = I$.

   (f) Explain why there can be so many solutions. Where does the analogy between x and X break down?

2. Here is way to check your multiplication of an $m \times k$ matrix A with a $k \times n$ matrix B to form the $m \times n$ product matrix C.

   Make a new matrix $\mathcal{A}$ by adjoining to A an $(m+1)$th row that is the negative of the sum of the first m rows. Then make a new matrix $\mathcal{B}$ by adjoining to B an $(n+1)$th column that is the negative of the sum of the first n columns. Now multiply to get $\mathcal{C} = \mathcal{A}\mathcal{B}$. Deletion of the last row and the last column of $\mathcal{C}$ leaves $C = AB$. Moreover, the sum of the entries in any row or in any column of $\mathcal{C}$ should be 0; if it isn't, there is an error in that row or column.

3. Try this on a couple of examples. Explain why it works.

4. Inverses and Elementary Matrices

Introduction

Finding the inverse of an invertible matrix is an important step in many problems in linear algebra, but just knowing whether or not a particular matrix is invertible is by itself one of our best tools. For example, if you know that A is an invertible $n \times n$ matrix, and B is *any* $n \times 1$ matrix, then the equation $AX = B$ has a solution, namely $X = A^{-1}B$.

Questions

1. Answer the following *True* or *False*. Justify your answer.

   (a) If A and B are $n \times n$ matrices, and AB is not invertible, then either A is not invertible, or B is not invertible.

   (b) The diagonal matrix $\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 17 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is invertible.

   (c) If A and B are both invertible matrices, then $A + B$ is invertible.

2. Is $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ an elementary matrix? Why or why not?

Problems

1. (a) Use elementary matrices to compute the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$.

   (b) Use part (a) to solve the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad\;\; + 8x_3 &= 17 \end{aligned}$$

2. Without using pencil and paper, determine whether the following matrices are invertible.

(a) $\begin{bmatrix} 2 & 1 & -3 & 1 \\ 0 & 5 & 4 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 1 & 4 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

Hint: Consider the following associated systems

$$\begin{aligned} 2x_1 + x_2 - 3x_3 + x_4 &= 0 \\ 5x_2 + 4x_3 + 3x_4 &= 0 \\ x_3 + 2x_4 &= 0 \\ 3x_4 &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 5x_1 + x_2 + 4x_3 + x_4 &= 0 \\ 2x_3 - x_4 &= 0 \\ x_3 + x_4 &= 0 \\ 7x_4 &= 0 \end{aligned}$$

3. (a) Show that the equation $AX = X$ can be rewritten as $(A - I)X = 0$, where A is an $n \times n$ matrix, I is the $n \times n$ identity matrix, and X is an $n \times 1$ matrix.
   (b) Use part (a) to solve $AX = X$ for X, where

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & -1 \\ 2 & -2 & 1 \end{bmatrix} \qquad \text{and} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

   (c) Solve $AX = 4X$.

Additional Problems

1. Consider the following examples of matrix equations.
   (a) Suppose A is a matrix satisfying $A^2 - 3A + I = 0$. Show that $A^{-1} = 3I - A$.
   (b) Suppose A is a matrix satisfying $A^2 - 3A + 4I = 0$. Show that $A^{-1} = \frac{1}{4}(3I - A)$.
   (c) Suppose A is a matrix satisfying $A^2 - 3A = 0$. Give an example of such an A which is invertible, and an example of such an A which is not.

2. Let's compute the inverse of the matrix

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

   (a) Let J be the 5×5 matrix of all ones. Show that $J^2 = 5J$.
   (b) Notice that $M = J - I$.

(c) Using 2a and 2b show that $M^2 - 3M - 4I = 0$.

(d) Use Additional Problem 1 to compute M^{-1}.

3. Let $AX = 0$ be a system of n linear equations in n unknowns, and let B be an invertible $n \times n$ matrix. Show that the following two statements are equivalent.

   (a) $AX = 0$ has only the trivial solution $X = 0$.

   (b) $(BA)X = 0$ has only the trivial solution $X = 0$.

4. Prove: If A is invertible, then $A + B$ and $I + BA^{-1}$ are both invertible or both not invertible.

5. Transposes and Symmetry

Introduction

Symmetric matrices arise in many physical applications and have numerous nice properties. We will see later that symmetric matrices can be "diagonalized", a fact that is useful in finding solutions to linear differential equations. The transpose A^T of the matrix A is gotten by flipping A so that the rows of A become the columns of A^T, and the columns of A become the rows of A^T. The matrix A is called **symmetric** if it is equal to its transpose, i.e., $A = A^T$.

Questions

1. Let $A = \begin{bmatrix} 2 & 0 & 1 & -1 \\ 5 & 3 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 1 & -1 & -2 & 0 \end{bmatrix}$. What is A^T?

2. (a) Is the sum of symmetric matrices necessarily symmetric?
   (b) Is the product of symmetric matrices necessarily symmetric?
   (c) Is the inverse of an invertible symmetric matrix necessarily symmetric?

Problems

1. (a) Let A be a square matrix. Show that $A + A^T$ is symmetric. Why must A be square?
   (b) Let A be a matrix. Show that AA^T and A^TA are symmetric. Give an example which shows that A need not be square.

2. If A is a square matrix and n is a positive integer, is it true that $(A^n)^T = (A^T)^n$?

3. A **permutation matrix** is a square matrix with exactly one 1 in each row and exactly one 1 in each column, and 0's everywhere else. For example, $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is a permutation matrix, but $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ are *not*.

   (a) List all the 3×3 permutation matrices.
   (b) Let $A = \begin{bmatrix} 0 & 2 & 3 \\ -1 & -1 & 2 \\ 3 & 1 & 5 \end{bmatrix}$. Find a permutation matrix P such that PA is the result of interchanging the first two rows of A.

(c) Find a permutation matrix P such that PA is the result of sending the first row of A to the third row, the second row to the first, and the third row to the second.

(d) Pair up each 3×3 permutation matrix with its inverse.

(e) What is the relationship between the 3×3 permutation matrices and their inverses?

(f) Generalize part (b) for any 4×4 permutation matrix P and its inverse, and check your answer.

Additional Problems

1. Let $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   (a) What is $(A_\theta)^T$?

   (b) For what values of θ is A_θ invertible? (Hint: How many solutions does the equation $A_\theta X = 0$ have?)

   (c) Show that $(A_\theta)^{-1} = (A_\theta)^T$.

2. A nice property that symmetric matrices possess is **diagonalizability**. I.e., given a symmetric matrix A, there exists an invertible matrix S, and a diagonal matrix Λ (the capital Greek letter "lambda") so that

$$A = S\Lambda S^{-1}.$$

   (a) Find a simple formula for A^n.

   (b) Why does this formula simplify the computation of A^n?

3. Professors A and B teach different 54W lectures at Telebears University. At Telebears U. students switch classes as late into the semester as they please. Suppose A and B are equally popular. In fact, every week 75% remain in each professor's class, while 25% switch out to the other's lecture. This situation can be represented by the matrix

$$M = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

   (a) Show that M is diagonalizable. (Hint: Show that

$$M = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.)$$

   In Hill §5.3 you will learn a method which will let you do this problem without the hint.

(b) Find a simple formula for M^n. Compute the limit $\lim_{n\to\infty} M^n$. (Hint: Look at Additional Problem 2.)

(c) Suppose that at the beginning of the semester, A had 384 students, and B had 128 students. After 5 weeks, how many students will each professor have?

(d) Intuitively, how many students will be in each lecture at finals time (say 16 weeks after the start of the semester)? Is your intuition supported mathematically?

6. Vectors

Introduction

The idea of using ordered triples of numbers to represent points in 3-dimensional space can be extended to larger dimensions. It may be hard to visualize dimensions higher than 3, but we can still carry over many analytical properties of points and vectors. The usual notions of distance and angle, for example, have n-dimensional versions for any positive integer n.

Questions

1. (a) Given a vector $\mathbf{v}$ in $\mathbf{R}^2$, what is meant by $Span\{\mathbf{v}\}$?
   (b) Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^2$, what is meant by $Span\{\mathbf{u}, \mathbf{v}\}$?
   (c) If $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbf{R}^2$, under what conditions is $Span\{\mathbf{u}, \mathbf{v}\}$ all of $\mathbf{R}^2$?
2. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three vectors in $\mathbf{R}^3$, describe in geometric terms all of the possibilities for $Span\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.
3. (a) If $\mathbf{v} = (-1, 3, 2)$, then $\|\mathbf{v}\| =$ ____________ .
   (b) If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then $\|\mathbf{x}\| =$ ____________.
   (c) *True* or *False*: If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

Problems

1. (a) Express the vector $(3, 5)$ as a linear combination of the vectors $(1, 0)$ and $(0, 1)$. How many ways can this be done?
   (b) Is it possible to express the vector $(3, 5, -2)$ as a linear combination of the vectors $(1, 0, -1)$ and $(0, 1, 1)$? In how many ways?
   (c) Which vectors (x_1, x_2, x_3) in $\mathbf{R}^3$ can be expressed as a linear combination of the vectors $(1, 0, -1)$, $(0, 1, 1)$, and $(2, 2, 0)$? Give both an algebraic and a geometric answer.
2. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbf{R}^3$, and r and s are any real numbers, show that
   (a) $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$.
   (b) $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$.
3. Show that $A = (2, -1, 1)$, $B = (3, 2, -1)$, and $C = (7, 0, -2)$ are vertices of a right triangle. At which vertex is the right angle?

4. If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is a vector in $\mathbf{R}^n$, show that $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = d(\mathbf{u}, \mathbf{0})$

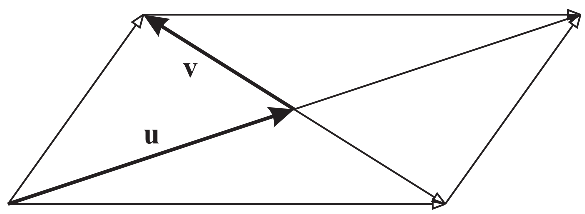
5. (a) Let L be the x-axis, i.e., $L = Span\{(1,0)\}$. Using Calculus, verify that the point of L closest to the point $P = (3,5)$ is just the projection of P onto $(1,0)$.

   (b) Now let L be the line spanned by $(1,-1,3,5)$ in $\mathbf{R}^4$. Find the point of L closest to $P = (-24, 5, -16, 1)$.

6. Establish the identity

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

for vectors in $\mathbf{R}^2$. Does the formula hold in $\mathbf{R}^n$? Label the parallelogram below with the missing vectors. Based on this picture, can you explain the identity geometrically?



Additional Problems

1. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbf{R}^3$, and let $k = \|\mathbf{u}\|$ and $l = \|\mathbf{v}\|$. Show that the vector

$$\mathbf{w} = l\mathbf{u} + k\mathbf{v}$$

bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

2. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbf{R}^n$. Show that if $\mathbf{u}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}$ is perpendicular to $2\mathbf{v} + 3\mathbf{w}$. Is it perpendicular to $k_1\mathbf{v} + k_2\mathbf{w}$ for all real numbers k_1 and k_2? Explain what is going on geometrically in $\mathbf{R}^3$.

7. General Vector Spaces

Introduction

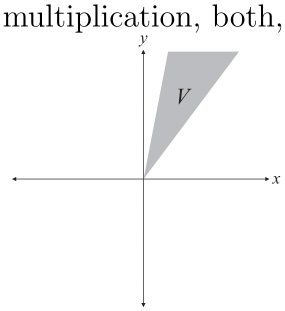
So far we have extended many of the properties of $\mathbf{R}^2$ and $\mathbf{R}^3$ to $\mathbf{R}^n$. We don't have to stop there. By taking the most fundamental properties of these "natural" sets of vectors, we can define a general vector space. Any set of objects equipped with an addition and scalar multiplication which satisfies the eight vector space axioms qualifies as a vector space.

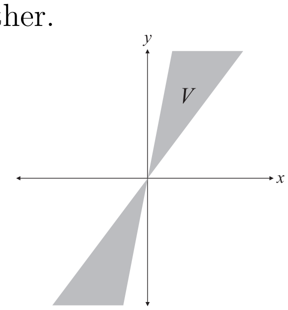
Questions

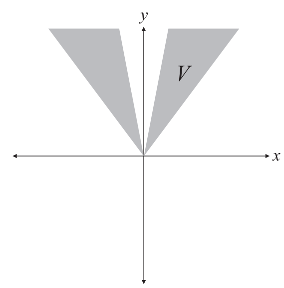
1. (a) Explain what it means for a set V to be *closed under addition*.

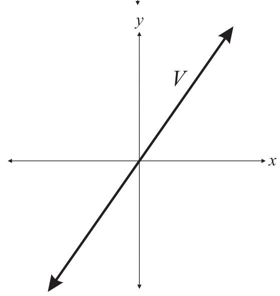
   (b) What does it mean for V to be *closed under scalar multiplication*?

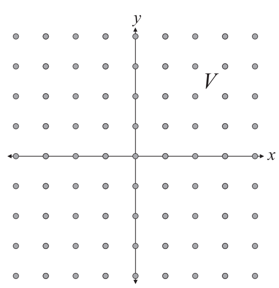
2. For each of the following shaded regions, determine whether it is closed under addition, scalar multiplication, both, or neither.

(a) 

(b) 

(c) 

(d) 

(e) 

3. Determine if V is closed under addition, scalar multiplication, both, or neither.

   (a) $V = \{$ odd integers $\}$

   (b) $V = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, b, c, d$ are negative $\}$

   (c) $V = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is nonsingular $\}$

   (d) $V = \{f \in C[1, 2] : f(x) \geq 0 \text{ for all } x \in [1, 2]\}$

4. What special properties does the zero vector of a vector space have? Are there other vectors with these same properties? You should try to prove your answer.

Problems

1. Let V be the set of all solutions to the differential equation $y'' - 4y' = 0$.

   (a) Is 0 in V?

   (b) If y is a solution to the differential equation, is $2y$ also a solution?

   (c) If y_1 and y_2 are solutions, is $2y_1 + 3y_2$ also a solution?

   (d) Verify that V is a vector space.

2. Let V be the set of all functions representable by a Maclaurin series convergent on $(-1, 1)$, with addition and scalar multiplication defined as follows. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, then

$$f + g = \sum_{n=0}^{\infty}(a_n + b_n)x^n \qquad \text{and} \qquad cf = \sum_{n=0}^{\infty} ca_n x^n.$$

   (a) Is $f(x) = \frac{1}{1-x} = 1 + x + x^2 + \ldots$ in V?

   (b) Is $g(x) = \frac{1}{1-2x} = 1 + 2x + 4x^2 + \ldots$ in V?

   (c) Is V closed under addition and scalar multiplication?

   (d) Is V a vector space?

3. Let V be the set of all 2×2 matrices in which each of the entries is a polynomial of degree less than 2, with the usual matrix addition and scalar multiplication. For example, the matrix $\begin{pmatrix} x & x^2-5x \\ -3 & 3x^2+1 \end{pmatrix}$ is a vector in V.

   (a) If V is a vector space, what is the zero vector?

   (b) Is V a vector space?

Additional Problems

1. Let V be the set of all 3×3 matrices, and suppose that the "sum" of two such matrices M and N is defined to be $M \oplus N = MN$ (the usual matrix product). Does V with this definition of addition and the usual definition of scalar multiplication form a vector space? If not, which properties fail?

2. Let V be a vector space, and suppose that L and M are two subsets of V which just happen to also be vector spaces. Is it true that $L \cup M$ is a vector space? How about $L \cap M$? Why?

8. Subspaces, Span, and Nullspaces

Introduction

A subspace of a vector space is a subset which is also a vector space. Fortunately, we don't have to check all eight vector space axioms to determine if a subset of a vector space is a subspace. Since most of the properties of the vector space automatically hold in any nonempty subset, there are only two axioms that need to be checked. Subspaces turn out to be very useful in analyzing linear systems of equations.

Questions

1. (a) If S is a nonempty subset of a vector space V, what are the two necessary and sufficient conditions for S to be a subspace of V?

   (b) If S is a nonempty subset of a vector space, and S satisfies the two conditions in (a), how do you know that S contains the zero vector?

   (c) If S is a nonempty subset of a vector space V, and S satisfies the two conditions in (a), and $\mathbf{v} \in S$, how do you know that $-\mathbf{v} \in S$?

   (d) Suppose S is empty. Does S still satisfy the two conditions in (a)? If so, why isn't S a subspace of V?

2. (a) Let V be a vector space. Is V a subspace of itself?

   (b) If $\mathbf{0}$ is the zero vector in V, is the set $\{\mathbf{0}\}$ a subspace of V?

3. *True* or *False*: If $\mathbf{w} \in Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, then $\mathbf{w}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in only one way.

4. Which regions in Question 2 of Worksheet 6 are subspaces of $\mathbf{R}^2$?

5. Suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spans the vector space V, and for each i, that $\mathbf{v}_i$ lies in $\text{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$. Show that $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ spans V.

   Note: This can be a very useful tool! Suppose you want to know if the space spanned by $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is the same as the space spanned by $\{\mathbf{w}_1, \ldots \mathbf{w}_m\}$. If you can show that each $\mathbf{v}_i$ is in $\text{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ and each $\mathbf{w}_i$ is in $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then the spans must be the same. Make sure you understand this concept before moving on.

Problems

1. Let V be the vector space of all 2×2 matrices, and let
$$S = Span\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}.$$
   (a) Is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ in S?
   (b) Is $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ in S?
   (c) Is S a subspace of V?
   (d) What does a typical vector in S look like?

2. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Show that the set $NS(A)$ of all vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbf{R}^3$.

3. (a) Describe the span of $p(x) = 2x^2 - 5x$ and $q(x) = 2x - 5$ in P_2.
   (b) Describe the span of $f(x) = \sin^2 x$ and $g(x) = \cos^2 x$ in $C(-\infty, \infty)$ (the vector space of functions which are continuous on the entire real line).

4. Each null space in the first column corresponds uniquely to which span in the second column? Justify.

nullspace of ...	span of ...
$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Additional Problems

1. (a) Let L and M be subspaces of a vector space V. Are the following sets also subspaces of V?

   i. $L \cup M = \{\text{all vectors belonging to } L \text{ or } M \text{ or both}\}$

   ii. $L \cap M = \{\text{all vectors belonging to both } L \text{ and } M\}$

   iii. $L + M = \left\{ \begin{array}{l} \text{all vectors } \mathbf{w} \text{ that can be written as a sum} \\ \text{of a vector } \mathbf{x} \text{ in } L \text{ and a vector } \mathbf{y} \text{ in } M \end{array} \right\}$

   (b) Let $L = \{(x_1, x_2, x_3) | x_1 + x_2 + x_3 = 0\}$ and let $M = \{(x_1, x_2, x_3) | x_1 = x_2 = x_3\}$. Both are subspaces of $\mathbf{R}^3$. Give geometric descriptions of L and M; then describe the sets $L \cup M$, $L \cap M$, and $L + M$.

2. Define the **differential operator** $D = \frac{d^2}{dx^2} + 1$ by the rule:

$$D(f) = \left(\frac{d^2}{dx^2} + 1 \right) f = \frac{d^2 f}{dx^2} + 1 \cdot f.$$

   To make this rigorous, one should think of D as a function whose domain and range is the vector space $\mathcal{A}$ of all functions which possess MacLaurin series which converge on $\mathbf{R}$. As with matrices, one can define the null space of D by

$$NS(D) = \{f \in \mathcal{A} : D(f) = 0\}.$$

   (a) Show that $NS(D)$ is a subspace of $\mathcal{A}$.

   (b) Find $NS(D)$. (Hint: Solve a differential equation.)

9. Linear Independence

Introduction

At this point, you probably understand intuitively what is meant by "dimension" in $\mathbf{R}^n$. Intuitively $\mathbf{R}^2$ is two dimensional and $\mathbf{R}^3$ is three dimensional. Before we can define dimension formally, we need to understand certain sets of vectors a little better. Before beginning this worksheet, make sure you know the definitions of *linear dependence* and *linear independence*.

Questions

1. Can the polynomial $x^2 + 3x + 2$ be written as a linear combination of $x^2 + x$ and $3x$? Explain.

2. By inspection, determine whether the polynomials $x^3 + 2$ and $3x^3 + 6$ are linearly independent in P_3.

3. In general, how can you tell if a set with two vectors is linearly independent?

4. *True* or *False*: If the set of vectors $S = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is linearly independent, then

$$c_1\mathbf{v}_1 + ... + c_k\mathbf{v}_k \neq \mathbf{0}$$

for all numbers c_1, ..., c_k.

Problems

1. (a) Show that the vectors $\mathbf{v}_1 = (1, 0, 2)$, $\mathbf{v}_2 = (0, 1, 2)$, and $\mathbf{v}_3 = (0, 3, 0)$ are linearly independent in $\mathbf{R}^3$.

   (b) What is the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

   (c) Can you find another linearly independent set of vectors in $\mathbf{R}^3$ with the same span as the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? (Make sure your vectors are *not* scalar multiples of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.) Prove that your set has the same span as $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

2. Are the vectors

$$\mathbf{v}_1 = (1, 1, 2, 4), \quad \mathbf{v}_2 = (2, -1, -5, 2),$$
$$\mathbf{v}_3 = (1, -1, -4, 0), \quad \mathbf{v}_4 = (2, 1, 1, 6)$$

linearly independent in $\mathbf{R}^4$?

3. Is the set $\{\sin x, x\}$ linearly independent in $C[0, 1]$? How about $\{\sin 2x, \cos x \cdot \sin x\}$? Justify.

4. Show that $\{x^3+x+1, 2x^3+x+1, x^3+3x+1, x^3+x+4\}$ is a linearly dependent set in P_3. (Hint: Find a 3 dimensional subspace that they all lie in.)

5. (a) Show that any two vectors chosen from a linearly independent set are linearly independent.
   (b) Show that a set which contains two linearly dependent vectors must be a linearly dependent set.
   (c) Find three vectors in $\mathbf{R}^3$ which are linearly dependent, and such that any two of them are linearly independent.

Additional Problems

1. Suppose that $S = \{v_1, \dots, v_k\}$ and $T = \{w_1, \dots, w_\ell\}$ are subsets of a vector space V. Furthermore, S and T are each linearly independent sets, and

$$Span(S) \cap Span(T) = \{\mathbf{0}\}.$$

   Show that $S \cup T$ is a linearly independent set.

2. Let W be a subspace of $\mathbf{R}^n$, and let $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ be a linearly independent subset of W.
   (a) Suppose that $\{\mathbf{w}_1, \dots, \mathbf{w}_k, \mathbf{w}\}$ is linearly dependent for each $\mathbf{w}$ in W. Show that $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ is a basis for W.
   (b) Suppose moreover that $\{\mathbf{w}_1, \dots, \mathbf{w}_k, \mathbf{v}\}$ is linearly dependent for each $\mathbf{v}$ in $\mathbf{R}^n$. Show that $k = n$.

3. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
   (a) Show that the vectors I, A, A^2 are linearly *dependent* in $M_{2,2}$. (Hint: Write A^2 as a linear combination of I and A.)
   (b) Deduce that A^n is in $Span\{I, A\}$ for all non-negative integers n. Show that if A is invertible, then A^n is an $Span\{I, A\}$ even if n is a negative integer.

10. Basis and Dimension

Introduction

A subset S of a vector space V is said to *span* V if $Span(S) = V$. If S spans V, and in addition, S is linearly independent, then S is called a **basis** for V. With this in mind, we are ready for a rigorous analog to the intuitive idea of dimension. The dimension of a vector space V is n (a nonnegative integer) if V has a basis consisting of n vectors. Although a vector space can have many bases, they all have the same number of elements, so that dimension is a well defined notion.

Questions

1. (a) What is the standard basis for P_3, the vector space of all polynomials of degree 3 or less?

   (b) Find another basis for P_3.

2. If V is a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for V, can we find another basis for V with four vectors?

3. Answer the following *True* or *False*. Justify your answers.

   (a) $\dim P_5 = 5$.

   (b) $C[0, 1]$ is infinite dimensional. (Hint: See Theorem 3.58, page 191 of Hill.)

   (c) If W is a subspace of V, and S is a basis for W, then we can add vectors to S to form a basis for V.

   (d) If W is a subspace of V, and S is a basis for V, then some subset of S is a basis for W.

Problems

1. Let $V = C[-\infty, \infty]$ be the space of continuous functions $f : \mathbf{R}^1 \to \mathbf{R}^1$.

   (a) Let W be the subspace spanned by $\sin^2 x$, $\cos^2 x$, $\sin 2x$, and $\cos 2x$. What is $\dim W$?

   (b) Show that $C[-\infty, \infty]$ is infinite dimensional. (Hint: For any n, find n linearly independent vectors. Why does this prove it?)

2. Let $W = \{(x_1, x_2, x_3) | x_1 + x_2 + x_3 = 0\}$.

(a) Find a matrix A such that $NS(A) = W$. [Hint: $NS(A)$ is the null space of A.]

(b) Use part (a) to find a basis for W and determine its dimension.

3. Let V be the vector space consisting of all polynomials $p(x)$ of degree 3 or less, satisfying $p(1) = 0$. What is $\dim V$? Give a basis for V.

4. (a) Find a basis for the vector space of all 3×3 matrices. What's the dimension?

   (b) Find a basis for the vector space of all 3×3 symmetric matrices. What is the dimension of this vector space?

   (c) What is the dimension of $M_{n,n}$? What is the dimension of the subspace of $M_{n,n}$ consisting of all symmetric matrices?

Additional Problems

1. Let $p_1(x) = \frac{1}{2}x(x-1)$, $p_2(x) = -x^2 + 1$, and $p_3(x) = \frac{1}{2}x(x+1)$.

   (a) Show that $\{p_1, p_2, p_3\}$ is linearly independent in P_2, and hence forms a basis. Why don't you have to show that the vectors span P_2?

   (b) Let $q(x) = c_1p_1(x) + c_2p_2(x) + c_3p_3(x)$. Find:

      i. $q(-1)$

      ii. $q(0)$

      iii. $q(1)$

   (c) Let $r(x) = 2x^2 - 3x + 7$. Use part (b) to write $r(x)$ as a linear combination of p_1, p_2, and p_3.

2. Let $V = P_5$, the vector space of polynomials of degree 5 or less. Let W be the set of polynomials from V which are even functions. Is W a subspace of V? If so, find a basis for W.

3. What is the dimension of the space of all functions which are solutions to the differential equation

$$y^{(n)} + c_{n-1}y^{(n-1)} + \cdots + c_1y' + c_0y = 0.$$

   Justify.

11. Fundamental Subspaces and Rank

Introduction

There are three vector spaces associated to every matrix: the *row space*, *column space*, and *null space*. We have already seen that the null space of a matrix A is the set of solutions to the equation $A\mathbf{x} = \mathbf{0}$. The dimensions of these vector spaces are related, and this provides us with a very useful relationship between the size of the solution set of a system of equations and the size of its coefficient matrix.

Questions

1. Let A be an $m \times n$ matrix. Review the definitions of $RS(A)$, $NS(A)$, and $CS(A)$. Which ones are subspaces of $\mathbf{R}^n$? Are any of them subspaces of $\mathbf{R}^m$?

2. Let A be an $m \times n$ matrix and let U be an $m \times n$ matrix in row echelon form which is obtained from A by row operations.

   Answer the following *true* or *false*. Explain your reasoning, or give a counterexample.

   (a) $RS(A) = RS(U)$

   (b) $CS(A) = CS(U)$

   (c) $\dim RS(A) = \dim RS(U)$

   (d) $\dim CS(A) = \dim CS(U)$

   (e) $\dim RS(A) = \dim CS(A)$

3. Suppose A is an invertible $n \times n$ matrix.

   (a) What is $rk(A)$?

   (b) What is $\dim NS(A)$? [Hint: $NS(A)$ is the null space of A.]

4. Show that if A is not square, then either the rows of A or the columns of A are linearly dependent.

Problems

1. Consider the linear transformation $B : \mathbf{R}^3 \to \mathbf{R}^2$ with coordinate matrix

$$B = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -2 & 2 \end{bmatrix}.$$

(a) Carefully sketch the null space of B. (Of what vector space is it a subspace?)

(b) Carefully sketch the column space of B. (Of what vector space is it a subspace?)

2. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

   (a) Show that the null space of A is the z-axis and the column space of A is the xy-plane.

   (b) Find a 3×3 matrix whose null space is the x-axis and whose column space is the yz-plane.

   (c) Find a matrix whose row space is spanned by $(1, 0, 1)$ and $(0, 1, 0)$ and whose null space is the span of $(1, 0, -1)$.

3. Find a basis for the subspace W of $\mathbf{R}^4$ spanned by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 6 \\ 15 \end{bmatrix}.$$

   Hint: Form the 3×4 *matrix* A *with* $\mathbf{v}_1$, $\mathbf{v}_2$, *and* $\mathbf{v}_3$ *as its rows.*
   What is $\dim W$?

4. Let A be a 4×5 matrix, and let $\mathbf{b}$ be a vector in $\mathbf{R}^4$.

   (a) What does it mean for the equation $A\mathbf{x} = \mathbf{b}$ to have a solution?

   (b) Show that $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in $CS(A)$.

5. Let A be the matrix formed by taking n vectors from $\mathbf{R}^m$ as its columns. Consider the possible relationships between m and n: $m \leq n$, $n \leq m$, or $m = n$. In each of the following situations, what is the relationship between m and n?

   (a) The n vectors are linearly independent.

   (b) The n vectors span $\mathbf{R}^m$.

   (c) The n vectors form a basis for $\mathbf{R}^m$.

   (d) A has rank m.

   (e) A is invertible.

6. Find the rank of the following matrices (it may depend on t):

$$\text{(a)} \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}; \qquad \text{(b)} \begin{bmatrix} t & -1 & 2 \\ t & t & 1 \\ t & t & t \end{bmatrix}.$$

7. Let A be an $n \times m$ matrix. Show that if the rank of A is 1 then A must be of the form $\mathbf{u}\mathbf{v}^T$, where $\mathbf{u}$ is a column vector in $\mathbf{R}^m$ and $\mathbf{v}$ is a column vector in $\mathbf{R}^n$.

Additional Problems

1. (a) Suppose the $\mathbf{u}$ is a non-zero $m \times 1$ matrix, and $\mathbf{v}$ is a non-zero $1 \times n$ matrix. Show that $A = \mathbf{uv}$ is an $m \times n$ matrix of rank 1.

   (b) Show that the converse is true also. That is, suppose that A is an $m \times n$ matrix of rank 1. Show that there is a column matrix $\mathbf{u}$ and a row matrix $\mathbf{v}$ so that $A = \mathbf{uv}$.

12. Error Correcting Codes

Introduction

Have you ever wondered how the Mars Rover could transmit information with high precision across the vast reaches of space? Error correcting codes are one of the key tools used in transmitting information accurately, and many of them are based on linear algebra. In this problem we will explore the Hamming code.

A digital message can be considered as a list of 0's and 1's. Such a list reminds us of vectors, and we are thus led to define a vector space, which we call $\mathbf{Z}_2^n$, consisting of column vectors having n entries, each of which is either 0 or 1. Vectors are added componentwise using the rules:

$$0+0=0, \quad 0+1=1+0=1, \quad 1+1=0.$$

In particular, note that $1+1=0$, not 2 as usual. (This new addition is like the "exclusive or" in computer science.) Multiplication is as usual:

$$0\times 0=0, \quad 0\times 1=1\times 0=0, \quad 1\times 1=1.$$

Note that numbers like "2" and "-1" never come up.

Questions

Compute each of the following sums

1. $\begin{bmatrix}1\\1\\0\end{bmatrix}+\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}\quad\end{bmatrix}, \qquad \begin{bmatrix}1\\0\\1\\0\end{bmatrix}+\begin{bmatrix}0\\0\\1\\1\end{bmatrix}=\begin{bmatrix}\quad\end{bmatrix}, \qquad \begin{bmatrix}1\\1\\1\\0\end{bmatrix}+\begin{bmatrix}1\\1\\1\\0\end{bmatrix}=\begin{bmatrix}\quad\end{bmatrix}$

   Note that adding a vector to itself *always* yields the zero vector.

Carrying the analogy further, we define matrices with entries 0 or 1. We can multiply them to obtain matrices whose entries are all 0 or 1. Compute the following products (keeping in mind that $1+1=0$, and that your answers should be matrices of 0's and 1's).

2. $\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1&1\\1&1\end{bmatrix}=\begin{bmatrix}\qquad\end{bmatrix}, \qquad \begin{bmatrix}1&1&1\\1&0&0\\0&1&1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}\quad\end{bmatrix}.$

Most of the usual notions and theorems from linear algebra carry over to this new context. For example, let W be the subspace of $\mathbf{Z}_2^3$ *spanned* by the two vectors

$$\begin{bmatrix}1\\0\\1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

3. W consists of exactly 4 vectors, list them.

Problems

Next let V be the subspace of $\mathbf{Z}_2^7$ spanned by

$$\mathbf{u_1} = \begin{bmatrix}1\\0\\0\\0\\0\\1\\1\end{bmatrix}, \quad \mathbf{u_2} = \begin{bmatrix}0\\1\\0\\0\\1\\0\\1\end{bmatrix}, \quad \mathbf{u_3} = \begin{bmatrix}0\\0\\1\\0\\1\\1\\0\end{bmatrix}, \quad \mathbf{u_4} = \begin{bmatrix}0\\0\\0\\1\\1\\1\\1\end{bmatrix}.$$

1. Show that $\mathbf{u_1}, \ldots, \mathbf{u_4}$ are linearly independent.
2. Describe how you could list the 16 vectors in V. (*Don't* actually list them.)
3. Let $H = \begin{bmatrix}0&0&0&1&1&1&1\\0&1&1&0&0&1&1\\1&0&1&0&1&0&1\end{bmatrix}$. (If you know about binary numbers note that the columns of H represent 1 through 7.) Show that V is the null space of H. [Hint: The usual theorems of linear algebra still apply. Thus it suffices to show that the null space of H contains V, has dimension four, and then apply problem 4 above.]

A message can be encoded as a sequence of 1's and 0's. By breaking the message up into blocks of four "bits", we see that it is enough to encode and decode blocks of length four. Such a block can be encoded as an element of V by taking the corresponding *linear combination* of $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$. Thus the four-bit message 0-1-1-1 is encoded as

$$\mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 = \begin{bmatrix}0\\1\\1\\1\\1\\0\\0\end{bmatrix}$$

and 1-0-0-1 is encoded as $\mathbf{u}_1 + \mathbf{u}_4$.

4. What is 1-0-1-0 encoded as? What about 1-1-0-1?

5. Given a vector which encodes a four digit message, how can you recover the message?

You can often understand speech even when you can't hear every word. This is partially because human language has built in redundancy. Error correcting codes are based on this principle.

To send a message first encode it as a vector $\mathbf{v} \in V$ and send $\mathbf{v}$. (Note that 7 digits must be sent instead of 4, there's the redundancy.) If all goes well $\mathbf{v}$ is received and the receiver recovers the message as the first four digits of $\mathbf{v}$. If exactly one digit is "garbled" (i.e., an error occurs), so that some vector $\mathbf{w}$ is received instead, then $\mathbf{v}$ can still be recovered because it is the *only* vector in V which can have one digit changed to give $\mathbf{w}$. (This is a special "redundancy" property of V which won't be explained here.) There is an algorithm for finding out which digit was garbled:

Decoding a message which contains an error:

Assume that $\mathbf{v}$ was sent, $\mathbf{w}$ was received, and exactly one error occured. The vector $H\mathbf{w}$ must be a column of H, say the ith (indeed, every nonzero vector is a column of H). Then the ith entry of $\mathbf{v}$ was transmitted incorrectly. Change the ith entry of $\mathbf{w}$ to recover $\mathbf{v}$.

6. In each case, determine the four bits which were sent (assume that at most one digit was garbled):

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

7. Pick a vector $\mathbf{v} \in V$, change one of its digits, and have another member of your group determine which digit you changed.

13. Linear Transformations

Introduction

After developing the theory of abstract vector spaces, it is natural to consider functions which are defined on vector spaces that in some way "preserve" the structure. These functions are called *linear transformations*, and they come up in both pure and applied mathematics. In calculus, the derivative and the definite integral are two of the most important examples of linear transformations. These examples allow us to reformulate many problems in differential and integral equations in terms of linear transformations on particular vector spaces.

Questions

1. Find the function $f(x_1, x_2)$ induced by the matrix $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

2. Answer the following *true* or *false*. Explain your reasoning, or give a counterexample.
   (a) The function $T(x, y) = (x^2, y)$ is a linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$.
   (b) The function $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ which rotates the xy-plane $20°$ is a linear transformation.
   (c) The function $f_A : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ induced by
   $$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$
   rotates the yz-plane $45°$ and reflects the x-axis.

3. Let $A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$.
   Describe geometrically what the induced function f_A does to the x-axis and y-axis.

Problem

1. Let $V = C[0, 1]$ and let $T : V \longrightarrow \mathbf{R}$ be defined by $T(f) = \int_0^1 f(x)\, dx$. Show that T is a linear transformation.

2. Consider the function $\frac{d}{dx}$ from P_n to P_{n-1}.
   (a) Show that $\frac{d}{dx}$ is a linear transformation.

(b) Write the matrix of the transformation in terms of the standard basis for P_n.

(c) What is the null space of $\frac{d}{dx}$? What is the image (i.e. the column space) of $\frac{d}{dx}$?

3. Let A be a *fixed* 2×3 matrix, and let X be a 2×2 matrix. Define the function $T : M_{22} \longrightarrow M_{23}$ by $T(X) = XA$. Is T linear? Explain.

4. Let $S = \{1, x, x^2\}$ be the standard basis for P_2, and suppose that $T : P_2 \longrightarrow P_2$ is a linear transformation such that $T(1) = 3x - 5$, $T(x) = x^2 + 1$, and $T(x^2) = 3$.

   (a) What is $T(2x^2 + 1)$?

   (b) What is $T(a_0 + a_1 x + a_2 x^2)$?

   (c) Find a matrix which induces T.
   Hint: Think of 1 the vector $(1, 0, 0)$, x as $(0, 1, 0)$, etc.

5. Show that if $T : V \longrightarrow W$ is a linear transformation, then $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ in V.

6. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

   (a) What are the induced functions $f_A(x, y)$ and $f_B(x, y, z)$?

   (b) What is $f_A(f_B(x, y, z))$?

   (c) Is the composition $f_A \circ f_B$ a linear transformation? Why or why not? If it is, find a matrix which induces $f_A \circ f_B$.

   (d) Is there a linear transformation $f_B \circ f_A$?

Additional Problems

1. A linear transformation $T : V \to W$ is said to be **1-1** (**one-to-one**) if, whenever $\mathbf{x}_1 \neq \mathbf{x}_2$, $T(\mathbf{x}_1) \neq T(\mathbf{x}_2)$. Prove the following:

   (a) The transformation which rotates the xy-plane by an angle of 90° is 1-1.

   (b) If T is 1-1, the null space of the matrix representation of T consists of $\mathbf{0}$ alone.

   (c) If the null space of the matrix representation of T consists of $\mathbf{0}$ alone, then T is 1-1.

   (d) If T is 1-1, then $\dim W \geq \dim V$.

2. A linear transformation $T : V \to W$ is said to be **onto** if for any vector $\mathbf{y}$ in W there is a vector $\mathbf{x}$ in V so that $T(\mathbf{x}) = \mathbf{y}$. Prove the following:

   (a) The transformation which rotates the xy-plane by an angle of 90° is onto.

   (b) If T is onto, the column space of the matrix representation of T has dimension equal to $\dim W$.

(c) If T is onto, then $\dim V \geq \dim W$.

3. A linear transformation $T : V \to W$ is said to be an **isomorphism** if it is invertible. I.e., there is a linear transformation $S : W \to V$ such that for all $\mathbf{x}$ in V, $S(T(\mathbf{x})) = \mathbf{x}$, and for all $\mathbf{y}$ in W, $T(S(\mathbf{y})) = \mathbf{y}$.

   (a) For each of the following matrices, decide if the induced linear transformation is an isomorphism:

      i. $\begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

      ii. $\begin{bmatrix} 3 & 0 & 2 \\ 11 & -8 & 26 \\ -1 & -2 & 4 \end{bmatrix}$

      iii. $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

   (b) Show that if T is an isomorphism, then it is 1-1 and onto.

   (c) Show that T is an isomorphism if and only if its matrix representation is invertible.

   (d) Prove the converse of (b). I.e., if T is 1-1 and onto, then T is an isomorphism.

4. If A is an $m \times n$ matrix, show that the dimension of the solution space of $A\mathbf{x} = 0$ is at least $n - m$.

14. Inner Products and Least Squares

Introduction

A vector space which is equipped with an inner product is called an inner product space. By generalizing the notion of a dot product, we are able to define useful notions of length, angle and distance in general vector spaces. Two important relationships involving inner products are the Cauchy-Schwartz inequality and the triangle inequality.

Questions

1. Answer the following *true* or *false*. Explain your reasoning, or give a counterexample.

   (a) If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbf{R}^4$ such that $\mathbf{x} \cdot \mathbf{y} = \mathbf{0}$, then either $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$.

   (b) If $\mathbf{u} \neq \mathbf{0}$ is a nonzero vector in $\mathbf{R}^n$ then the vector $\frac{\mathbf{u}}{||\mathbf{u}||}$ has norm 1.

   (c) If A is a 3×3 matrix, and $\mathbf{x}$ and $\mathbf{y}$ are column vectors in $\mathbf{R}^3$, then

   $$A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A\mathbf{y}.$$

   (d) If A is an $m \times n$ matrix, the equation $A\mathbf{x} = \mathbf{y}$ has a solution if and only if $\mathbf{y}$ is in the column space of A.

   (e) If A is an $m \times n$ matrix whose columns are linearly independent, then $A^T A$ is invertible.

Problems

1. Let

   $$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{w} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.$$

   (a) Compute $\|\mathbf{u} + \mathbf{v}\|$ and $\|\mathbf{u}\| + \|\mathbf{v}\|$. Which is larger?

   (b) Compute $|\mathbf{v} \cdot \mathbf{w}|$ and $\|\mathbf{v}\| \, \|\mathbf{w}\|$. Which is larger?

   (c) Compute $\mathbf{u}^T\mathbf{v}$, $\mathbf{v}^T\mathbf{u}$, and $\mathbf{u} \cdot \mathbf{v}$. Is $\mathbf{v}\mathbf{u}^T$ defined? If so, compute it. If not, explain.

2. Does the formula $p(x) \cdot q(x) = p(0)q(0)$ define an inner product on P_2?

3. Let $f \cdot g = \int_0^1 f(x)g(x)\, dx$ for functions $f, g \in C[0,1]$.

   (a) What is $\|\sin \pi x\|$ in this inner product space?

(b) As in $\mathbf{R}^n$, the distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space is $\|\mathbf{u}-\mathbf{v}\|$. Compute the distance between the functions $x+1$ and e^x in this inner product space.

(c) Show that the functions $\sin 2\pi x$ and $\cos 2\pi x$ are orthogonal (perpendicular) in this inner product space.

4. Find the equation of the line $y = mx + b$ that best fits the points $(-1,-1)$, $(1,0)$, and $(2,4)$ in the least-squares sense by following these steps:

(a) Write down the (inconsistent) system of three equations in two unknowns for this problem.

(b) Rewrite the system as a matrix equation $A\mathbf{x} = \mathbf{y}$, where $\mathbf{x} = \begin{bmatrix} m \\ b \end{bmatrix}$. What are A and $\mathbf{y}$?

(c) Find A^T and form the equation $A^TA\mathbf{x} = A^T\mathbf{y}$.

(d) Since the columns of A are linearly independent, A^TA is invertible (see Question 1e above). Find $(A^TA)^{-1}$ and use it to solve the equation in part (c).

(e) What is the equation of the line that best fits the given points?

5. Let W be a subspace of $\mathbf{R}^n$. Define the **orthogonal complement** of W to be the subspace

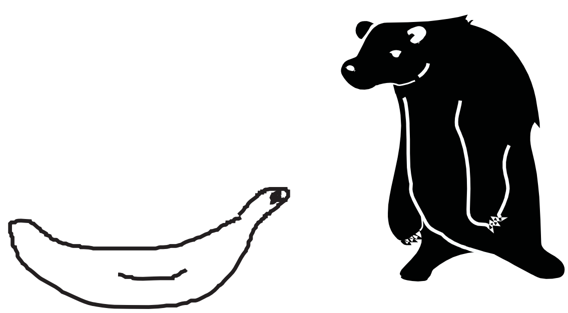
$$W^\perp = \{\mathbf{v} \in \mathbf{R}^n : \mathbf{w} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{w} \in W\}.$$

(a) Consider the subspace $W = \operatorname{span}\{(1,1,1),(2,0,-1)\}$ of $\mathbf{R}^3$. Find a vector which spans $W^\perp$.

(b) Express the vector $(2,1,-3)$ in the form $\mathbf{w} + \mathbf{w}^\perp$, where $\mathbf{w} \in W$ and $\mathbf{w}^\perp \in W^\perp$.

Additional Problems

1. The A.S.U.C. store is trying to decide at what price it should sell bananas.



Manager G. Bears notices the following obvious pattern: in general, the higher the price of bananas, the less bananas are sold. Manager Bears wants to maximize profits so embarks on the following experiment. During a single week manager Bears raises the price on bananas \$0.1 per day and records how many bananas were sold, comparing with the profit margin per banana:

	Profit Margin in \$	Bananas Sold
Monday	0.1	5
Tuesday	0.2	4
Wednesday	0.3	1
Thursday	0.4	1
Friday	0.5	-1

On Friday, no bananas were sold, but one student was so enraged at the banana prices that he squashed a banana at the register, resulting in a negative sale. (Yeah, yeah, I know. The "-1" messes the model up. If you insist, patch it up by assuming the base cost for a banana is \$0.5 so the profit on a squashed banana is $-\$0.5$.)

(a) Using the least squares method find the best linear estimate $b(p)$ for the number of bananas sold b as a function of the profit margin p.

(b) At what price should G. Bears set the price of bananas in order to maximize total profits. (Hint: maximize $p \cdot b(p)$)

2. In an inner product space V, the **unit sphere** is the set of all vectors $\mathbf{v}$ for which $||\mathbf{v}|| = 1$. Note that this "sphere" may not be the conventional sphere in dimensions different from 3. Describe the unit spheres of each of the following inner product spaces:

   (a) $V = \mathbf{R}^1$. Inner product is multiplication.

   (b) $V = \mathbf{R}^2$. Inner product is dot product.

   (c) $V = \mathbf{R}^3$. Inner product is dot product.

   (d) $V = P_3$. Inner product as in problem 3.

3. (a) Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$, with $\|\mathbf{u}\| = 1$ and $\|\mathbf{v}\| = 1$. Prove that

$$|\mathbf{u} \cdot \mathbf{v}| \leq \sum_{i=1}^{n} |u_i|\ |v_i| \leq \sum_{i=1}^{n} \frac{|u_i^2| + |v_i^2|}{2} = 1 = \|\mathbf{u}\|\ \|\mathbf{v}\|.$$

   (b) Use Additional Problem 1 to prove the Cauchy-Schwartz inequality:

$$|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}||\ ||\mathbf{v}||, \qquad \text{for any } \mathbf{u}, \mathbf{v} \in \mathbf{R}^n.$$

   (c) Now use the Cauchy-Schwartz inequality to prove the triangle inequality:

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||, \qquad \text{for any } \mathbf{u}, \mathbf{v} \in \mathbf{R}^n.$$

15. Orthonormal Bases

Introduction

We have already seen the important role of the standard basis of $\mathbf{R}^n$. Many of the important properties of this basis stem from it being an orthonormal set. Just as bases are building blocks of vector spaces, orthonormal bases are building blocks of inner product spaces.

Questions

1. Answer the following *true* or *false*. Explain your reasoning, or give a counterexample.
   (a) Any basis for P_n (the vector space of all polynomials of degree $\leq n$) must contain a polynomial of degree k for each $k = 0, 1, 2, ..., n$.
   (b) If A is a 3×5 matrix, then the column vectors of A are linearly independent.
   (c) If A is a 4×4 matrix whose column vectors form an orthonormal basis for $\mathbf{R}^4$, then A is invertible.
   (d) Every nonzero finite-dimensional inner product space has an orthonormal basis.

Problems

1. (a) Let θ be a real number. Show that the vectors $\mathbf{v}_1 = (\cos\theta, \sin\theta)$ and $\mathbf{v}_2 = (-\sin\theta, \cos\theta)$ form an orthonormal basis for $\mathbf{R}^2$.
   (b) Let $\mathbf{u}_1 = (1, 1)$ and $\mathbf{u}_2 = (0, -1)$. Find a 2×2 matrix A which induces a linear transformation T_A such that $T_A(\mathbf{u}_1) = \mathbf{v}_1$ and $T_A(\mathbf{u}_2) = \mathbf{v}_2$.

2. (a) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis for $\mathbf{R}^3$, where $\mathbf{v}_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, $\mathbf{v}_2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, and $\mathbf{v}_3 = (0, 0, 1)$.
   (b) Find the coordinates of $\mathbf{w} = (\sqrt{2}, 3\sqrt{2}, -4)$ with respect to this basis.
   (c) Let
   $$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$
   Show that $A\mathbf{v}_1 = 2\mathbf{v}_1$, $A\mathbf{v}_2 = 4\mathbf{v}_2$, and $A\mathbf{v}_3 = -\mathbf{v}_3$.
   (d) Check your answer to (b) by computing $A\mathbf{w}$ using the results of (c).

3. Find an orthonormal basis for the subspace of $\mathbf{R}^4$ spanned by $(1, 2, 2, 4)$ and $(2, 2, 3, 1)$.

4. Let $P_2[-1,1]$ be the space of polynomials on the interval $[-1,1]$ and having degree at most 2, and let $f \cdot g = \int_{-1}^{1} f(x)g(x)\, dx$.
   (a) Use the Gram-Schmidt process to construct an orthonormal basis for $P_2[-1,1]$ from the basis $\{1, x, x^2\}$.
   (b) Find $f \cdot g$, where $f(x) = 1 + x$ and $g(x) = x^2 - 2x$ in $P_2[-1,1]$. (Can you do the problem without *without integrating* ?)

5. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for the inner product space V, and let $\mathbf{u} \in V$.
   (a) Let $\alpha_i = \text{angle}(\mathbf{u}, \mathbf{v}_i)$. What does this mean?
   (b) Show that $\sum_i \cos^2 \alpha_i = 1$. (Hint: Multiply both sides by $||u||^2$.)
   (c) How does (b) relate to the Pythagorean theorem?

6. Let A be a 2×2 orthogonal matrix.
   (a) Show that A is of the form
   $$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix},$$
   for some value of θ in the interval $0 \le \theta < 2\pi$.
   (b) Geometrically, what does an orthonormal basis for $\mathbf{R}^2$ look like?
   (c) What does an orthonormal basis for $\mathbf{R}^3$ look like?

Additional Problems

1. In $C[0,\pi]$, let $f \cdot g = \int_0^{\pi} f(x)g(x)\, dx$.
   (a) Show that $\cos nx$ and $\cos mx$ are orthogonal in $C[0,\pi]$ if $m \ne n$.
   (b) What is $\|\cos nx\|$ in $C[0,\pi]$?
   (c) Is the set $\{1, \cos x, \cos 2x, \cos 3x, \ldots\}$ a linearly independent set? Justify.

2. Let A be a square matrix.
   (a) Show that $\|A\mathbf{x}\| = \|A^T\mathbf{x}\|$ if $AA^T = A^TA$.
   (b) Find a 2×2 matrix A such that $AA^T \ne A^TA$. For this A, find a vector $\mathbf{x} \in \mathbf{R}^2$ such that $\|A\mathbf{x}\| \ne \|A^T\mathbf{x}\|$.

3. A matrix is **orthogonal** if it is square and its columns are orthonormal.
   (a) Show that if the $n \times n$ matrix A is orthogonal, then the columns of A form an orthonormal basis for $\mathbf{R}^n$.
   (b) Show that a matrix P whose columns form an orthogonal basis for $\mathbf{R}^n$ has the nice property that P^TP is a diagonal matrix.

4. In light of the previous exercise, the definition of an orthogonal matrix may seem inconsistent. A better way to justify the definition is in terms of the properties of the corresponding linear transformation.

   (a) Let A be an orthogonal matrix, and let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be an orthogonal basis of $\mathbf{R}^n$. Show that $A\mathbf{v}_1, \ldots, A\mathbf{v}_n$ is also an orthogonal basis of $\mathbf{R}^n$.

   (b) Show that any matrix A which sends any orthogonal basis of $\mathbf{R}^n$ to an orthogonal basis satisfies $A^T A = \lambda I$. If furthermore A sends orthonormal bases to orthonormal bases, then λ equals 1. (Hint: Observe that $(A\mathbf{v}) \cdot (A\mathbf{u}) = \mathbf{v} \cdot (A^T A\mathbf{u})$ and show, for any orthogonal basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$, that $A^T A\mathbf{v}_i = \lambda_i \mathbf{v}_i$. Why must all λ_i be equal?)

   (c) Use part (b) to show that if A and B are orthogonal matrices, then so is AB.

   (d) Let A and B be the matrices

   $$A = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 1 & -4 \\ 3 & -1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & 1 & 1 \\ -1 & 2 & -2 \\ 2 & -1 & -3 \end{bmatrix}.$$

   Show that the column vectors of A and of B each form orthogonal bases of $\mathbf{R}^3$ but that the column vectors of AB are not orthogonal.

(Remark: Orthogonal matrices can also be characterized as *length preserving* transformations.)

16. Determinants

Introduction

At one time determinants played a major role in the study of linear algebra. Although our use of determinants will be mainly in connection with computing eigenvalues of square matrices, understanding why these computations work is at least as important as being able to carry them out.

Questions

1. Let A be a square matrix. List at least four different statements which are equivalent to the statement
$$\det A \neq 0.$$

2. Let A and B be $n \times n$ matrices. Answer the following *True* or *False*. If *False* give a counterexample.

   (a) $\det(AB) = \det(BA)$

   (b) $\det(AB) = \det A \det B$

   (c) $\det(A + B) = \det A + \det B$

   (d) If A and B are both invertible, then AB is invertible.

   (e) $\det A = \det A^T$

   (f) If A is invertible, then $\det(A^{-1}) = (\det A)^{-1}$.

   (g) If a is a scalar then $\det(aA) = a^n \det(A)$.

3. Suppose A is a 3×3 matrix with determinant 5. What is $\det(3A)$? $\det(A^{-1})$? $\det(2A^{-1})$? $\det((2A)^{-1})$?

Problems

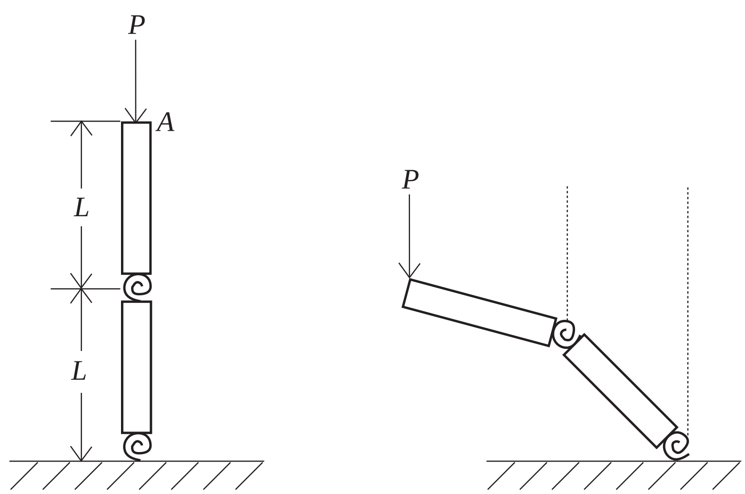
1. Linear equations and buckling of structures

The stability of structures is an important topic in civil and mechanical engineering, discussed in introductory courses such as CE 130 (Mechanics of Materials I). Structures like bridge supports, soda cans, and robot legs will buckle if they are subjected to excessive loads. We will show in this worksheet how to determine the load at which a simple structure will buckle. Mathematically, we will be studying systems of linear equations depending on parameters.

A structure with two degrees of freedom (able to bend at two places) is shown in the figure below. It consists of two rigid beams and two spring joints. A load (force) P is applied at point A. For certain values of P, called *buckling loads*, the structure will react by bending, as shown in the right hand part of the figure. To determine the buckling loads, one uses the equilibrium equations for moments (we won't do that here) and the small angle approximation $\sin\theta \approx \theta$ to get the system of linear equations for the deflection angles θ_i:

$$\begin{aligned} K(\theta_2 - \theta_1) - PL\theta_2 &= 0 \\ K\theta_1 - K(\theta_2 - \theta_1) - PL\theta_1 &= 0 \end{aligned}$$

where K is the spring constant and L is the length of a beam.



(a) Put the pair of equations above in matrix form ($AX = B$). The matrix A, which describes the structure's response to an applied force, is called the *stiffness matrix*.

(b) For any value of P, $X = 0$ is a solution of the equation which you just wrote. What does it mean in terms of the shape of the structure?

(c) For which values of P (expressed in terms of K and L) does the matrix equation have a nontrivial solution? For these values of P (the buckling loads), the nontrivial solutions, which are called the *buckling modes*, show how the structure bends. Find them. (**Hint:** The non-trivial solutions exist only when the determinant of the stiffness matrix vanishes.)

(d) Sketch the structure in each of the buckling modes if $L = 5$ meters and $K = 40$ newtons/meter. If the load is initially zero and increases gradually, which buckling mode do you expect to occur first?

(e) Suppose that there are three beams and joints instead of two. How do the results (number and magnitude of buckling loads, and geometry of buckling modes) change?

2. Prove that the determinant of a 2×2 matrix is 0 if and only if one row is a multiple of the other.

3. Compute the following:

$$\text{(a)} \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} \qquad \text{(b)} \begin{vmatrix} 1 & -1 & -2 \\ 3 & 0 & 1 \\ -1 & 1 & 1 \end{vmatrix} \qquad \text{(c)} \begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

4. By inspection, evaluate the following determinants:

$$\text{(a)} \begin{vmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 5 & 9 \end{vmatrix} \qquad \text{(b)} \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 9 \\ 0 & 16 & 25 & 36 \\ 49 & 64 & 81 & 100 \end{vmatrix} \qquad \text{(c)} \begin{vmatrix} 0 & \pi & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 22 \end{vmatrix}$$

5. (a) Let A be a square matrix. If $\det A \neq 0$, how many solutions does the equation $A\mathbf{x} = \mathbf{0}$ have?

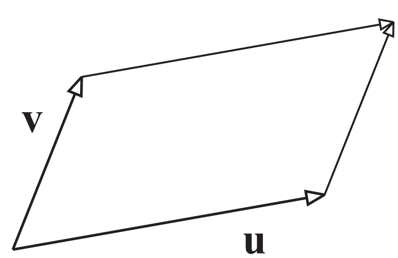
   (b) Let B be a 3×3 matrix such that the entries in each row of B add up to 0 (i.e., $b_{i1} + b_{i2} + b_{i3} = 0$ for $1 \leq i \leq 3$). Use part (a) to show that $\det B = 0$.

6. Show that if A is a square matrix with a row of zeros, then $\det A = 0$. What if A has a column of zeros?

Additional Problems

1. Prove that the area of the parallelogram spanned by the vectors $\mathbf{u}$ and $\mathbf{v}$ of $\mathbf{R}^2$ is given by the determinant of the 2×2 matrix formed by taking $\mathbf{u}$ as the first row, and $\mathbf{v}$ as the second row. That is:

$$\text{Area} = \begin{vmatrix} -\mathbf{u}- \\ -\mathbf{v}- \end{vmatrix}.$$



2. Prove that the rank of an $n \times n$ matrix A is the largest integer k for which there is a $k \times k$ sub-matrix of A that has a nonzero determinant.

3. Show that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ forms a basis for $\mathbf{R}^n$ if and only if the determinant of the coordinate matrix is not zero.

4. Suppose f, g, and h are vectors in $C^2(-\infty, \infty)$ (the vector space of functions which are twice differentiable on the entire real line). The function

$$W(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$

is called the *Wronskian* of f, g, and h. Prove that $\{f, g, h\}$ is a linearly independent set in $C^2(-\infty, \infty)$ if the Wronskian is not identically zero (i.e., there is at least one real number x_0 such that $W(x_0) \neq 0$).

17. Eigenvalues and Eigenvectors

Introduction

If A is an $n \times n$ matrix, there is often no obvious geometric relationship between a vector $\mathbf{x}$ and its image $A\mathbf{x}$ under multiplication by A. However, frequently there are some nonzero vectors that A maps into scalar multiples of themselves. These "eigenvectors" play an important role in the analysis of linear systems and arise naturally in many physical applications.

Questions

1. Answer the following *true* or *false*. Explain your reasoning, or give a counterexample.
   (a) If A is a 4×4 matrix, then $\det(\lambda I - A) = 0$ must have exactly four distinct roots.
   (b) The vector $\mathbf{0}$ is an eigenvector of any matrix. (*Careful*)
   (c) If a matrix has one eigenvector, then it has an infinite number of eigenvectors.
   (d) The sum of two eigenvalues of a matrix A is also an eigenvalue of A.
   (e) The sum of two eigenvectors of a matrix A is also an eigenvector of A.
   (f) There exists a square matrix with no real eigenvalues.
   (g) There exists an $n \times n$ matrix with $n + 1$ distinct eigenvalues.

2. For each of the following matrices, describe in geometric terms the eigenspaces (if any) and their associated eigenvalues. *Do not compute the matrices.*
   (a) The matrix induced by the linear transformation $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ which reflects each vector across the z-axis.
   (b) The matrix induced by the linear transformation $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ which rotates each vector by $\pi/4$ radians counterclockwise.

Problems

1. Let A be an $n \times n$ matrix. Show that A is invertible if and only if 0 is not an eigenvalue of A.

2. Find the eigenvalues, and bases for the associated eigenspaces, of

$$A = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & 2 \end{bmatrix}.$$

3. Give an example of a 2×2 matrix with two linearly independent eigenvectors, but only one eigenvalue.

4. Find a 3×3 matrix with eigenvalues $0, 1, -1$ and corresponding eigenvectors $(0, 1, 1)$, $(1, -1, 1)$, and $(0, 1, -1)$.

5. Two $n \times n$ matrices A and B are said to be *similar* if there is an invertible $n \times n$ matrix S such that $B = S^{-1}AS$. Show that similar matrices always have the same eigenvalues. Must they have the same eigenvectors?

Additional Problems

1. Let A be a 4×4 matrix.

   (a) If the eigenvalues of A are $1, -2, 3, -3$, can you find $\det(A)$?

   (b) What if the eigenvalues are $-1, 1, 2$? What about $-1, 0, 1$?

2. (a) Show that the eigenvalues of an upper triangular $n \times n$ matrix are the entries on the main diagonal.

   (b) Show that if λ is an eigenvalue of an $n \times n$ matrix A then λ^2 is an eigenvalue of A^2. More generally, show that λ^k is an eigenvalue of A^k if k is a positive integer.

   (c) Use (a) and (b) to find the eigenvalues of A^9, where

$$A = \begin{bmatrix} 1 & 3 & 7 & 11 \\ 0 & -1 & 3 & 8 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

3. Let A be an $n \times n$ matrix.

   (a) Prove that the polynomial $p(\lambda) = \det(\lambda I - A)$ has degree n.

   (b) Prove that the coefficient of λ^n in the polynomial $\det(\lambda I - A)$ is 1.

18. Diagonalization

Introduction

We now turn our attention to the so-called "diagonalization problem." Given an $n \times n$ matrix A, we will try to answer the following question: Does there exist a diagonal matrix Λ and an invertible $n \times n$ matrix S such that $S^{-1}AS = \Lambda$? Diagonalization can be useful in translating difficult computations into much simpler ones involving diagonal matrices.

Questions

1. Answer the following *true* or *false*. Explain your reasoning, or give a counterexample.
   (a) Any $n \times n$ matrix that has fewer than n real distinct eigenvalues is not diagonalizable.
   (b) Eigenvectors corresponding to the same eigenvalue are always linearly dependent.
   (c) If A is diagonalizable, then it has at least one eigenvalue.
2. Suppose A is a 2×2 matrix with characteristic polynomial $(\lambda - 2)^2$. What can you conclude about the diagonalizability of A? What can you conclude if B is a 4×4 matrix with characteristic polynomial $(\lambda - 2)^2(\lambda + 1)(\lambda - 1)$?

Problems

1. Let A be a 3×3 matrix with the following eigenvectors and corresponding eigenvalues:

   $(1, 1, 1)$ and $(1, -2, 0)$ are eigenvectors corresponding to the eigenvalue $\lambda = 3$.
   $(1, 1, -2)$ is an eigenvector corresponding to the eigenvalue $\lambda = -3$.

   (a) Find A by finding an invertible matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$.
   (b) Are your S and Λ unique? In other words, could you have used a different pair of matrices S and Λ to get the same A?
2. Let A be the matrix to the right. In this exercise we determine whether A is diagonalizable without doing any hard work.

   $$A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

   (a) Find the eigenvalues of A. Call them λ_1 and λ_2.
   (b) Write down the matrix $\lambda_1 I - A$.

(c) What is the rank of $\lambda_1 I - A$? What is its nullity? How many independent eigenvectors are there with eigenvalue λ_1? (Don't compute them.)

(d) What is the rank of $\lambda_2 I - A$? What is its nullity? How many independent eigenvectors are there with eigenvalue λ_2?

(e) Is A diagonalizable?

3. Show that if $b \neq 0$ then $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ is not diagonalizable.

4. Is $\begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$ diagonalizable?

5. (a) Suppose that $S^{-1}AS = \Lambda$, where Λ is diagonal. Show that $A^k = S\Lambda^k S^{-1}$ for any positive integer k.

   (b) Use part (a) to compute A^{10}, where $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$.

Additional Problems

1. Let A be a square matrix.

   (a) Show that $\det(A^T - cI) = \det(A - cI)$ for any constant c. (Hint: $(A - cI)^T = ?$)

   (b) Show that $\det(B^{-1}AB - cI) = \det(A - cI)$ for any constant c.

   (c) Argue that A, A^T, and $B^{-1}AB$ all have the same eigenvalues.

   (d) Show that if there exists a matrix P such that $P^{-1}AP$ is diagonal, then the diagonal entries of $P^{-1}AP$ are the eigenvalues of A.

   (e) Do A, A^T, and $B^{-1}AB$ all have the same eigenvectors?

2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show:

   (a) A is diagonalizable if $(a-d)^2 + 4bc > 0$.

   (b) A is not diagonalizable if $(a-d)^2 + 4bc < 0$.

   (c) Give examples to show that A may or may not be diagonalizable if $(a-d)^2 + 4bc = 0$.

3. Let A be an $n \times n$ matrix. Show that A is diagonalizable if and only if A^T is diagonalizable.

4. Pretend that you are head GSI for Math 54. For the first few days of the semester, you noticed that each day, 20% of the people in Math 54 switched to Math 54M, and 10% of the people in Math 54M switched into Math 54.

(a) Let $\mathbf{x}(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}$, where $x_1(n)$ is the number of people in Math 54 on day n and $x_2(n)$ is the number of people in Math 54M on day n. Show that (for the first few days)

$$\mathbf{x}(n+1) = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \mathbf{x}(n)$$

(b) You are trying to determine whether to open new discussion sections or not, so you'd like to guess how many people will end up in each class. In order to do this, let's assume that the pattern you have observed will continue. Under this assumption, show that $\mathbf{x}(n) = A^n\mathbf{x}(0)$, where A is the matrix above.

(c) Thus you would like to know what A^n looks like for large n. Toward that end, find the eigenvalues of A.

(d) Find a matrix P such that $P^{-1}AP$ is a diagonal matrix D.

(e) Show that $A^n = PD^nP^{-1}$. Compute D^n and PD^nP^{-1}.

(f) As n grows large show that A^n converges to $\begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$.

(g) Argue that *no matter how many people started in each class* that eventually 1/3 of all students will be in Math 54 and 2/3 will be in Math 54M.

19. Symmetric Matrices

Introduction

In our quest to understand which matrices can be diagonalized, it is useful to consider certain special cases. In the case of symmetric matrices, all of the theoretical and some of the important computational difficulties are removed. Not only is every symmetric matrix A diagonalizable, but it is *orthogonally* diagonalizable (i.e., there is a matrix S with orthonormal columns such that $S^{-1}AS$ is diagonal.)

Questions

1. Let A be an $n \times n$ matrix. Answer the following *true* or *false*. Explain your reasoning, or give a counterexample.
   (a) If A is orthogonally diagonalizable, then A is symmetric.
   (b) If A is not symmetric, then A has at least one non-real eigenvalue.
   (c) If A is symmetric with eigenvalue λ repeated 5 times then the eigenspace corresponding to λ has dimension 5.
2. Suppose that A is a 2×2 matrix with eigenvalues 0 and 1 and corresponding eigenvectors $(1, 3)$ and $(3, -1)$.
   (a) Is A symmetric?
   (b) Find A and check your answer to part (a).

Problems

1. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

   Find a matrix Q that orthogonally diagonalizes A, and determine $Q^{-1}AQ$.
2. Let
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$
   (a) What is $rk(A)$? If $NS(A)$ is the null space of A then what is $\dim NS(A)$?

(b) Why is 0 an eigenvalue of A, and what is the dimension of the eigenspace corresponding to 0?

(c) Does A have any other eigenvalues besides 0? Explain.

3. Prove that if there is an orthogonal matrix that diagonalizes A, then A is symmetric. (See Question 1a.)

4. Let A be a 4×4 matrix.

   (a) If the eigenvalues of A are 1, -2, 3, -3, can you figure out $\det(A)$?

   (b) What if the eigenvalues are -1, 1, 2?

   (c) What if the eigenvalues are -1, 0, 1?

Additional Problems

1. (a) Find the eigenvalues of $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

   (b) Diagonalize $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

   (c) Compute $\begin{bmatrix} a & b \\ b & a \end{bmatrix}^k$ for $k \geq 1$.

20. The Wronskian and Linear Independence

Introduction

In Math 1B, you solved second order linear homogeneous equations with constant coefficients by first finding two linearly independent solutions. You were told–most likely without proof–that any solution would be a linear combination of these two solutions. The Wronskian can be used to check that two solutions are linearly independent. These two solutions form a basis for the solution space.

This happens to be true of all second order linear homogeneous equations, and can be proved with the help of a determinant called the **Wronskian**. Also, the two functions used in the general solution of a linear homogeneous equation form a special set which is a basis for the vector space of solutions.

Questions

1. Suppose you are given a second order linear homogeneous differential equation.

   (a) What is meant by a *fundamental set of solutions* of the equation?
   Hint: what do you do with repeated roots?

   (b) Must a fundamental set of solutions exist? If so, is the set unique?

2. Give an example of a fundamental set of solutions for the equation $y'' - 2y' + y = 0$.

3. What would a fundamental set of solutions of a first order equation be? How many solutions must there be in the set?

4. Consider the differential equation $y'' + (x^2 + 1)y' - \frac{1}{x}y = 0$. Is there a solution that also satisfies $y(2) = -1$ and $y'(2) = 3$?

5. If the Wronskian of two functions is $W(t) = t\cos^2 t$, are the functions linearly independent or linearly dependent?

Problems

1. (a) Let V be a vector space. What does it mean for two vectors in V to be linearly independent? Show that $y_1 = e^{-x}$ and $y_2 = xe^{-x}$ are linearly independent as vectors in $C(-\infty, \infty)$.

(b) What does it mean for two functions to be linearly independent on the interval I? Use the Wronskian to show that $y_1 = e^{-x}$ and $y_2 = xe^{-x}$ are linearly independent on $(-\infty, \infty)$. Are there any infinite intervals I on which y_1 and y_2 are not linearly independent?

2. Find a fundamental set $\{y_1, y_2\}$ of solutions to the equation $y'' + 4y = 0$ such that $y_1(\frac{\pi}{2}) = 1$, $y_1'(\frac{\pi}{2}) = 0$, $y_2(\frac{\pi}{2}) = 0$, and $y_2'(\frac{\pi}{2}) = 1$. Is your set unique?

3. Show that the functions x^3 and $|x^3|$ are linearly independent, and *differentiable*, but that their Wronskian is identically zero. Why is this not a contradiction?

4. (a) Show that the functions x and x^2 are linearly independent on the interval $(-1, 1)$.
   (b) Show that $W(x, x^2)(0) = 0$.
   (c) What can you conclude about the possibility that x and x^2 are solutions of a differential equation $y'' + p(x)y' + q(x)y = 0$?
   (d) Verify that x and x^2 are solutions of the equation $x^2y'' - 2xy' + 2y = 0$.
   (e) Read Abel's theorem. Is there a contradiction here? Figure out what's going on.

5. (a) If a, b, and c are positive constants, show that all solutions of $ay'' + by' + cy = 0$ approach zero as $x \to \infty$.
   (b) If $a > 0$, $c > 0$, and $b = 0$, show that the result of part (a) is not true, but that all solutions are bounded as $x \to \infty$.
   (c) Now suppose that $a > 0$, $b > 0$, but that $c = 0$. Show that the result of part (a) is not true, but that all solutions approach a constant that depends on the initial conditions as $x \to \infty$. Determine this constant for the initial conditions $y(0) = y_0$, $y'(0) = y_0'$.

21. Higher Order Linear ODEs

Introduction

What you already know about solutions of second order linear equations can be generalized to higher order equations, and the methods for solving homogeneous equations with constant coefficients can also be generalized. As with second order equations, begin by assuming that $y = e^{rx}$ is a solution of $a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$ for some r, to obtain the characteristic equation $a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$. By solving for r and dealing with the cases that a given root is real and unique, real and repeated, or complex, one can then construct a general solution to the original ODE.

Questions

1. Without computing a determinant, find the Wronskian of the functions 2, $\cos 2t$, and $\sin^2 t$.

2. Answer the following *True* or *False*:

   (a) The differential equation

   $$(\tan x)y''' + (1-x)y'' + x^2 y' + y = 0$$

   has a solution on the interval $(0, 2)$.

   (b) The functions $y_1(t) = e^{\lambda t} \cos \mu t$ and $y_2(t) = e^{\lambda t} \sin \mu t$ are linearly independent.

   (c) The equation $r^4 - 2$ has precisely two distinct roots (over the complex numbers?, over the real numbers?).

Problems

1. Find the general solution to the differential equations (Hint: look for roots dividing the last coefficient.)

   (a) $y''' + 2y'' - y' - 2y = 0$

   (b) $y^{iv} + 5y''' - y'' - 17y' + 12y = 0$

2. Consider the second order differential equation $y'' + 3y' + 4y = 0$.

   (a) Let $x_1 = y$ and $x_2 = y'$. Rewrite the equation as a system of two first order equations in the two functions x_1 and x_2.

(b) Show that the system in (a) can be written as $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix}$.

(c) Show that the equation in problem 1(a) can also be written as $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}.$$

3. (a) Show that the general solution of $y^{iv} - y = 0$ can be written as

$$y = c_1 \cos t + c_2 \sin t + c_3 \cosh t + c_4 \sinh t.$$

(b) Determine the solution satisfying the initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$, and $y'''(0) = 1$.

4. (a) Find a homogeneous third order differential equation with constant coefficients that has

$$y(x) = 3e^{-x} - \cos 2x$$

as a solution.

(b) What is the general solution of the equation you found in part (a)?

Additional Problems

1. Recall the following existence and uniqueness theorem:

> If the functions $p_1, p_2, \ldots, p_n$, and g are continuous on the open interval I, then there exists exactly one solution $y = \phi(t)$ of the differential equation (2) that also satisfies the initial conditions (3). This solution exists throughout the interval I.

(a) Assume $G(t) = 0$ and each $p_i(t)$ is constant, so that the theorem is referring to a homogeneous nth order linear differential equation with constant coefficients. Show that the set of solutions forms a subspace of the vector space of all functions.

(b) Use the above theorem to show that the dimension of the solution space is exactly n.
[Hint: Suppose you have a basis $\{y_1, \ldots, y_k\}$. Use the existence part of the theorem to argue that $k \geq n$, and use the uniqueness part of the theorem to argue that $k \leq n$.]

2. Let L be a second order linear differential operator, and suppose that three solutions to the equation $Lx = g(t)$ are $x(t) = t$, $x(t) = t + e^{-t}$, and $x(t) = 1 + t + e^{-t}$.

(a) Find a solution to $Lx = g(t)$ satisfying the initial conditions $x(0) = 0$, $x'(0) = 0$.

(b) Find a solution to $Lx = g(t)$ satisfying the initial conditions $x(0) = 0$, $x'(0) = 1$.

(c) What is the solution to $Lx = g(t)$ satisfying the initial conditions $x(0) = x_0$, $x'(0) = x'_0$? Why must such a solution exist? (Hint: See the above existence and uniqueness theorem.)

22. Homogeneous Linear ODEs

Introduction

This worksheet provides a recap of some important concepts involved in solving linear homogeneous equations. A few of the problems here are taken from some recent Math 54 final exams.

Questions

1. Answer the following *true* or *false*. Explain your reasoning, or give a counterexample.

   (a) If the characteristic equation of a 5th order homogeneous equation with constant coefficients has complex roots, then the graph of any solution $x(t)$ will oscillate steadily as $t \to \infty$.

   (b) If the characteristic equation of a 5th order homogeneous equation with constant coefficients has a complex root $\lambda + \mu i$ which is repeated twice, then there are four linearly independent real-valued solutions corresponding to that root.

   (c) If f and g are two differentiable functions such that their Wronskian is given by $W(f,g)(t) = 0$ for all t, then f and g are linearly dependent.

   (d) If f and g are two differentiable functions such that their Wronskian is given by

   $$W(f,g)(t) = \begin{cases} 0, & \text{if} \quad t = 0 \\ t^2 + 1, & \text{if} \quad t \neq 0 \end{cases} \tag{1}$$

   then f and g cannot both be solutions to a differential equation of the form

   $$y'' + p(t)y' + q(t)y = 0.$$

   Hint: look carefully at equation 10.

Problems

1. Consider the differential equation $xy'' - (x+1)y' + y = 0$. Check that $y_1 = e^x$ is a solution and use Abel's theorem to find a second, linearly independent solution.

2. Find the Wronskian of two linearly independent solutions of $x^2y'' + xy' + (x^2 - 1)y = 0$ without solving the equation.

3. Consider the differential equation

$$(\sin x)y''' + (\cos x)y = (x-1)^{-1},$$

and suppose we are interested in a solution y_0 that satisfies

$$y_0(1.5) = 1,\ y_0'(1.5) = 1,\ \ldots,\ y_0^{(n)}(1.5) = 1$$

for as large an n as we can get.

   (a) What is the largest value of n for which we are assured that such a y_0 exists in some interval about $x = 1.5$?

   (b) What is the largest interval about $x = 1.5$ for which we know that such a solution exists?

Additional Problems

1. Find three functions $y_1(x)$, $y_2(x)$, and $y_3(x)$ defined on $(-\infty, \infty)$ whose Wronskian is given by

$$W(y_1, y_2, y_3)(x) = e^{4x}.$$

Are your functions linearly independent on $(-\infty, \infty)$?

2. (a) Show that if y_1 is a solution of

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0,$$

then the substitution $y = y_1(t)v(t)$ yields the following second order equation for v':

$$y_1 v''' + (3y_1' + p_1 y_1)v'' + (3y_1'' + 2p_1 y_1' + p_2 y_1)v' = 0.$$

   (b) Use part (a) to solve

$$(2-t)y''' + (2t-3)y'' - ty' + y = 0, \quad t < 2; \qquad y_1(t) = e^t.$$

23. Systems of First Order Linear Equations

Introduction

Systems of linear differential equations arise in many physical problems. Solving a system of first order linear homogeneous equations $\mathbf{x}' = A\mathbf{x}$ is analogous to solving a single first order equation of the form $y' = ky$. We assume that a solution of the form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ exists, and proceed by finding the vector $\boldsymbol{\xi}$ and the exponent r. The linear algebra tools we have acquired this semester will save the day.

Questions

1. (a) What does it mean for two vectors to be linearly independent?

   (b) What does it mean for two functions to be linearly independent?

   (c) What does it mean for two vector-valued functions

   $$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \text{and} \quad \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

   to be linearly independent?

   (d) What does it mean for two vector-valued functions $\mathbf{x}(t)$ and $\mathbf{y}(t)$ to be linearly independent on an interval?

   (e) Show that $\mathbf{x}(t) = \begin{bmatrix} e^t \\ te^t \end{bmatrix}$ and $\mathbf{y}(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}$ are linearly dependent for each *fixed* t in $[0, 1]$, yet they are linearly independent on the interval $[0, 1]$.

2. Discuss with your group the two types of Wronskians. Understand, among other things:

   (a) What is the Wronskian of a collection of (real-valued) functions?

   (b) What is the Wronskian of a family of n vector-valued functions, whose values are in $\mathbf{R}^n$?

   (c) An nth order linear homogeneous differential equation with constant coefficients has n linearly independent solutions. How does the Wronskian of these functions compare with the Wronskian of the n linearly independent vector-valued functions you obtain by solving the corresponding first order system of linear differential equations?

   (d) Do an example if you're the slightest bit confused. Pick an easy second order differential equation that you can solve, solve it and look at the Wronskian. Then solve the corresponding system of first order equations and look at the Wronskian of those solutions.

Problems

1. (a) Compute the Wronskian of the vector-valued functions

$$\mathbf{x}^{(1)} = \begin{bmatrix} t \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}^{(2)} = \begin{bmatrix} t^2 \\ 2t \end{bmatrix}.$$

   (b) On what intervals are $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ linearly independent?

   (c) What conclusion can you draw about the coefficients in a system of homogeneous differential equations satisfied by $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$?

   (d) Find a system of homogeneous differential equations satisfied by $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ and verify your conclusion in part (c).

2. Consider the system $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$.

   (a) Plot a direction field for the system.

   (b) Find the general solution of the system and describe the behavior of the solution as $t \to \infty$.

   (c) Sketch some trajectories of the system.

3. Part of a mixing device consists of two tanks connected by pipes. Suppose that initially, tank A contains 16 gallons of water with 8 pounds of salt dissolved in it, and tank B contains 16 gallons of pure water.

   (a) Suppose that the mixer pipes fluid from A to B at the rate of 1 gallon per minute, and another pipe takes fluid from B to A at the same rate. Set up a system of two first order equations and find the amount of salt in each tank as a function of time. Will tank B ever have more salt than tank A?

   (b) Now suppose that pure water is being added to tank A at the rate of 3 gallons per minute. The mixer is still running, pumping from A to B at 4 gallons per minute and from B to A at 1 gallon per minute. The solution in tank B is also draining at 3 gallons per minute. Find the amount of salt in each tank as a function of time.

   (c) At what time T will B switch over to become saltier than A ?

Additional Problems

For now, pretend you know nothing about how to solve a system of n first order linear homogeneous equations. Let's start from scratch with a very simple case.

1. Suppose A is a diagonalizable $n \times n$ matrix. Show that the system $\mathbf{x}' = A\mathbf{x}$ can be solved by setting $\mathbf{x} = P\mathbf{z}$ and solving the system

$$\mathbf{z}' = P^{-1}AP\mathbf{z},$$

   where P is an invertible $n \times n$ matrix which diagonalizes A.

2. Let $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$. Solve the system $\mathbf{x}' = A\mathbf{x}$ by following these steps:

   (a) Diagonalize A and find P.

   (b) Solve $\mathbf{z}' = P^{-1}AP\mathbf{z}$ for $\mathbf{z}$ and let $\mathbf{x} = P\mathbf{z}$.

3. Let $\mathbf{F}$ be the vector field defined by $\mathbf{F}(x, y) = (2x - y, 3x - 2y)$.

   (a) Sketch the vector field $\mathbf{F}$.

   (b) What is the direction field for the differential equation $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x}$?

   (c) Solve the differential equation in (b) and sketch some of the trajectories.

   (d) What are the eigenvectors of $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$?

24. Systems of First Order Equations–Continued

Introduction

We continue our investigation of the system $\mathbf{x}' = A\mathbf{x}$ by looking at two cases: When the $n \times n$ matrix A has fewer than n linearly independent eigenvectors, and when A has complex eigenvalues. Recall from Math 1B that when the auxiliary equation $ar^2 + br + c = 0$ had complex or repeated roots, the corresponding real-valued solutions of $ay'' + by' + cy = 0$ had a slightly different form. The method for systems of equations will be similar. Also, note that previously we only looked for real eigenvalues of real matrices. We now consider the possibility that some of the eigenvalues could be complex. Although complex eigenvalues will have complex corresponding eigenvectors, we can use them to find real-valued solutions of the system $\mathbf{x}' = A\mathbf{x}$. Furthermore, when λ is a *repeated* eigenvalue with only one eigenvector $\boldsymbol{\xi}$, there will be a vector $\boldsymbol{\eta}$, called a **generalized eigenvector**, which will have the following properties:

- $A\boldsymbol{\eta} = \lambda\boldsymbol{\eta} + \boldsymbol{\xi}$
- $\boldsymbol{\xi} t e^{\lambda t} + \boldsymbol{\eta} e^{\lambda t}$ is a solution to $\mathbf{x}' = A\mathbf{x}$.

Questions

1. Answer the following *true* or *false*. Explain your reasoning, or give a counterexample.

   (a) All of the eigenvalues of a real, symmetric matrix are real.

   (b) If a real, symmetric $n \times n$ matrix A has $n - 1$ distinct eigenvalues, then A does not have n linearly independent eigenvectors.

   (c) If A is a 2×2 real matrix with one eigenvalue ρ (of multiplicity 2), and only one corresponding eigenvector $\boldsymbol{\xi}$, then there is another vector $\boldsymbol{\eta}$ such that

   $$(A - \rho I)\boldsymbol{\eta} = \boldsymbol{\xi}$$

2. Show that $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$ solves $\mathbf{x}' = A\mathbf{x}$ if and only if r is an eigenvalue of A with eigenvector $\boldsymbol{\xi}$.

3. (a) Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$ be the general solution of the system $\mathbf{x}' = A\mathbf{x}$. Describe the behavior of the solutions in the plane as $t \to \infty$.

   (b) What is the function $x_1(t)$? Describe the graph of $x_1(t)$ versus t.

Problems

1. Suppose that the 2×2 matrix A has eigenvalues $r_1 = 1 + 2i$ and $r_2 = 1 - 2i$ with corresponding eigenvectors

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix} \quad \text{and} \quad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 1 + i \end{bmatrix}$$

   (a) Write down two corresponding complex-valued solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ to the system $\mathbf{x}' = A\mathbf{x}$.

   (b) Find the real and imaginary parts of $\mathbf{x}^{(1)}$ and use them to find two real-valued solutions $\mathbf{u}$ and $\mathbf{v}$.

   (c) Show that your vector-valued functions $\mathbf{u}$ and $\mathbf{v}$ from part (b) are linearly independent.

2. Let $A = \begin{bmatrix} 6 & -8 \\ 2 & -2 \end{bmatrix}$. Find an eigenvector and generalized eigenvector corresponding to the eigenvalue $r = 2$.

3. Find the general solution for each of the following systems and sketch a few trajectories.

   (a) $\mathbf{x}' = \begin{bmatrix} -6 & 4 \\ -8 & 2 \end{bmatrix} \mathbf{x}$

   (b) $\mathbf{x}' = \begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix} \mathbf{x}$

4. Verify that $\mathbf{x}$ satisfies the given differential equation.

   (a) $\mathbf{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{2t}; \quad \mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}.$

   (b) $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^{t}; \quad \mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} e^{t}.$

5. Verify that $\Psi(t)$ satisfies the matrix differential equation. $\Psi = \begin{bmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{bmatrix}; \quad \Psi' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \Psi.$

Additional Problems

1. Show that any solution to $\mathbf{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}$ approaches 0 as $t \to \infty$, provided that $a + d < 0$ and $ad - bc > 0$.

2. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix}$. Show that $\mathbf{x}(t) = \boldsymbol{\xi}\frac{t^2}{2!}e^{2t} + \boldsymbol{\eta}te^{2t} + \boldsymbol{\zeta}e^{2t}$ is a solution of $\mathbf{x}' = A\mathbf{x}$, where

$$\boldsymbol{\xi} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \qquad (A-2I)\boldsymbol{\eta} = \boldsymbol{\xi}, \qquad \text{and} \qquad (A-2I)\boldsymbol{\zeta} = \boldsymbol{\eta}\ .$$

3. Show that any nth order differential equation for $y(x)$ can be written as a system of n first order linear differential equations in the n unknowns

$$\mathbf{x}_1 = y(x), \quad \mathbf{x}_1 = y'(x), \quad \dots \quad , \quad \mathbf{x}_n = y^{(n-1)}(x).$$

   (a) If the equation is $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0$, then what is the system you get?

   (b) Write the system in matrix form.

4. Compute or recall the characteristic polynomial of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}.$$

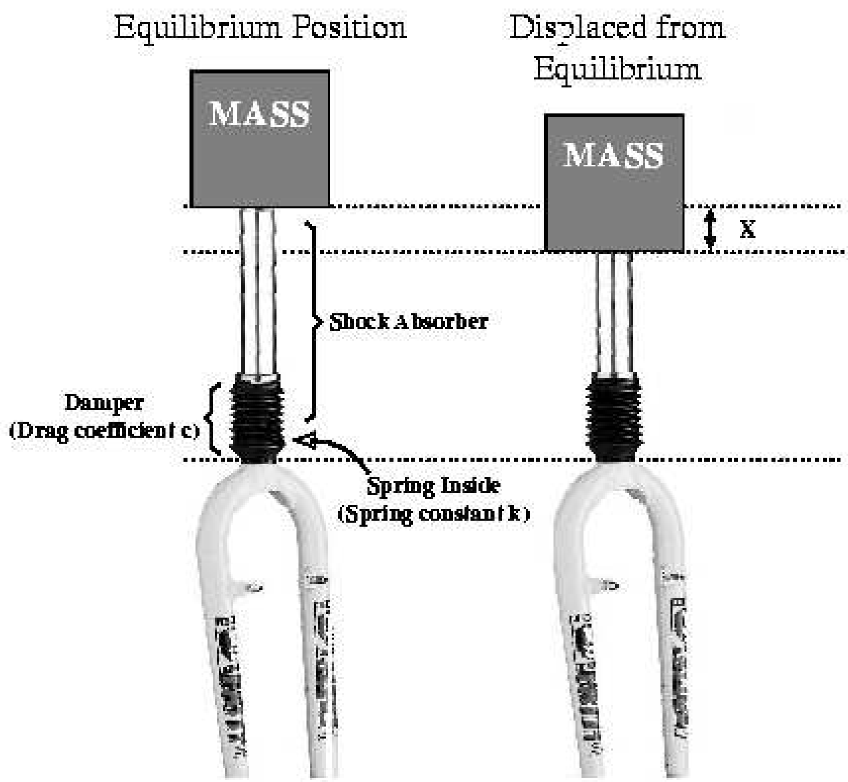
Use the results of the above exercises to describe the set of solutions to the equation $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0$. (Hint: Use Question 2 and Additional Problem 3. This is not the easiest way to do this, but it is good practice going back and forth between higher order linear equations and systems of first order linear equations.)

25. Oscillations of Shock Absorbers

Introduction

The differential equations for damped oscillations model the shock absorber found on any automobile, or even on a high-end mountain bike.

A shock absorber is essentially a spring and a damper. (See the figure below.) The spring cushions the shock and provides a restoring force $F_{\text{spring}} = -kx$ when it is stretched or squeezed by an amount x from its "neutral" position. (The proportionality coefficient k is called the *spring constant*.) The damper uses the viscosity of oil in a sealed container to produce a drag force which keeps the bike from bouncing up and down too much: $F_{\text{damper}} = -cv$, where c is the *drag coefficient* and $v = dx/dt$ is the (vertical) velocity of the effective mass m; i.e. the portion of the mass of the bike and rider supported by the front wheel.



Newton's law ($F_{\text{total}} = ma$) gives the equation:

$$-kx - cv = m\frac{dv}{dt},$$

which leads to the second order equation

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \tag{2}$$

where m and k are positive and c is non-negative. Equivalently, we can write a system of two first order equations in matrix form:

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}. \tag{3}$$

Questions

1. Derive Equation (2) from (3).

2. Show that the vector function $U(t) = \begin{bmatrix} a \\ b \end{bmatrix} e^{\lambda t}$, with a, b, and λ constants, satisfies the differential equation
$$\frac{d}{dt}U = \lambda U.$$

Problems

The following problems will familiarize you with the behavior of mass-damper-spring systems.

1. Equation (3) has solutions of the form $\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} e^{\lambda t}$. Find the possible values of λ by substituting the expression above into Equation (3) and using the result of Question 2 to get an eigenvalue problem. Find the eigenvalues λ_1 and λ_2.

2. Write the general solution of Equation (3) using the eigenvalues which you just found, and corresponding eigenvectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. (You do not need to find $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ explicitly yet.) The values of $\frac{k}{m}$ and $\frac{c}{m}$ will determine whether the the characteristic equation has real or complex eigenvalues. If you have complex eigenvalues, you should use Euler's formula: $e^{p+iq} = e^p(\cos q + i \sin q)$ to rewrite the complex exponentials in terms of sines and cosines.

3. Can the real part of any eigenvalue λ be positive, given that m, c and k are positive? (Consider both cases: real eigenvalues and complex eigenvalues.) Can x or v grow without bound as t increases? What is the meaning of this result in terms of a bouncing bicycle?

4. Illustrate the effect of the relationship between the stiffness k/m and the damping c/m by drawing a figure in the first quadrant of the $(c/m, k/m)$ plane which shows the types of solutions you get in different regions (e.g. "x and v oscillate but their magnitudes decrease with time"). Which values of c/m and k/m give solutions which do not oscillate?

Assume now that you and your bike have a mass of 100 kilograms, of which 40% is supported by the front wheel, and that your shock absorber has a rubber spring (k = 4,000 Newton/meter) and uses oil in the damper.

5. You forget to fill the damper with oil, so that $c = 0$, and you hit a bump that gives you a vertical velocity of 0.2 meters/second at a moment when $x = 0$. Find the particular solution of Equation (3) with these initial conditions, draw the phase portrait, and describe the vertical motion. (At this point, you *will* have to find the eigenvectors of the matrix in Equation (3), not just the eigenvalues.)

6. Now you put oil (with a drag coefficient $c = 500$ Newton-second/meter) in the damper, and you hit the same bump. Find and describe the motion, and draw the phase portrait. How does this differ from the undamped case? Is this amount of damping enough to prevent oscillations?

7. Why don't you want c to be too small or too large?

Additional Problems

1. Now you really want to tweak your bike. You look at some books on shock absorbers, and they say that the best you can do is to 'critically damp' the thing. This means that the damping is just enough to prevent oscillations; mathematically, critical damping occurs at the transition between real and complex eigenvalues–on the boundary between the regions in Problem 4.

   (a) Given the effective mass m and the spring constant k, find the drag coefficient c which you'll need to critically damp your shock. (Hint: look at the discriminant of the quadratic equation you used to find the eigenvalues λ_1 and λ_2.)

   (b) At critical damping there is only one distinct eigenvalue, i.e. $\lambda_1 = \lambda_2$, so there is another fundamental solution of the form

   $$\begin{bmatrix} x \\ v \end{bmatrix} = \boldsymbol{\xi} e^{\lambda_1 t} + \boldsymbol{\eta} t e^{\lambda_1 t}$$

   where $\boldsymbol{\xi}$ is an eigenvector and $\boldsymbol{\eta}$ is a generalized eigenvector. Use this to find the solution with the initial conditions produced by hitting a bump, as in Problems 5 and 6, draw the phase portrait, and describe the motion.

2. Here is another way to understand the effects of damping. Take the solution $\begin{bmatrix} x \\ v \end{bmatrix}$ which you found in Problems 5 and 6, and express the kinetic energy $K = \frac{1}{2}mv^2$ and potential energy $P = \frac{1}{2}kx^2$ as functions of time. Plot (K, P) as a parametric curve (like the phase portraits you drew before). Look at the difference between the plot for the undamped versus the damped shock absorber. What does the damper do to the total energy. Does the law of conservation of energy apply here?

26. Introduction to Partial Differential Equations

Introduction

In this worksheet we begin the study of partial differential equations and the methods by which we solve these equations.

Problems

1. Solve the following differential equations:
   (a) $y' = 3y$
   (b) $y' = xy$
   (c) $y' + (x - \sigma)y = 0$

2. Evaluate the following partial derivatives:
   (a) $\dfrac{\partial}{\partial x}(x^2 y + \cos y e^x - 2\sin x)$
   (b) $\dfrac{\partial}{\partial t}(t \cos tx)|_{t=0}$
   (c) $\dfrac{\partial}{\partial y}(x + 3)$
   (d) $\dfrac{\partial}{\partial y}(y^3 - 3\cos 2y)$

3. Suppose $f = f(x)$ and $g = g(y)$ are functions of the independent variables x and y. If $f(x) = g(y)$ for all x and y, what can you conclude about f and g?

4. Given a partial differential equation in two variables x and t, the idea of the method of *separation of variables* is to look for solutions which can be written as a product of a function of x and a function of t: $u(x,t) = X(x)T(t)$. The partial differential equation can then be reduced to two ordinary differential equations. Which of the following can be solved by this method? For those that can, write down the corresponding pair of ordinary differential equations.
   (a) $xu_{xx} = u_t$
   (b) $xu_{xx} + (x + t)u_t t = 0$
   (c) $tu_{xx} + xu_t = 0$
   (d) $[p(x)u_x]_x - r(x)u_{tt} = 0$

5. Now let's try to solve a partial differential equation:
   (a) Find all solutions of the form $u(x,t) = X(x)T(t)$ to the equation
   $$\frac{\partial u}{\partial t} + tu = \frac{\partial^2 u}{\partial x^2}$$
   on the intervals $0 < x < \pi$ and $t > 0$, with the boundary conditions
   $$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0.$$

(b) Find a solution satisfying the initial condition $u(x,0) = \sin^2 x$.

6. Let

$$\mathcal{S} = \left\{ \sqrt{\frac{2}{\pi}} \sin x, \sqrt{\frac{2}{\pi}} \sin 2x, \sqrt{\frac{2}{\pi}} \sin 3x, \dots \right\}, \text{ and}$$

$$\mathcal{C} = \left\{ \sqrt{\frac{1}{\pi}}, \sqrt{\frac{2}{\pi}} \cos x, \sqrt{\frac{2}{\pi}} \cos 2x, \dots \right\},$$

(a) Show that $\mathcal{S}$ is an orthonormal set in $C[0,\pi]$.

(b) Show that $\mathcal{C}$ is an orthonormal set in $C[0,\pi]$.

(c) Show that $\mathcal{S} \cup \mathcal{C}$ is not an orthonormal set in $C[0,\pi]$.

(d) Show that $\mathcal{S} \cup \mathcal{C}$ is an *orthogonal* set in $C[-\pi,\pi]$. What normalization makes $\mathcal{S} \cup \mathcal{C}$ *orthonormal* in $C[-\pi,\pi]$?

7. Solve the heat equation $100u_{xx} = u_t$ on the intervals $0 < x < 1$ and $t > 0$, subject to the conditions $u(0,t) = 0$ and $u(1,t) = 0$ for all t. Assume also that the initial temperature profile is given by $u(x,0) = \sin 2\pi x - 2\sin 5\pi x$ for $0 \le x \le 1$.

27. Partial Differential Equations and Fourier Series

Introduction

This worksheet serves as an introduction to the Fourier series which we will use to solve partial differential equations.

Problems

1. Compute the following values of sine and cosine. Here k is an integer.

   (a) $\sin k\pi$ (c) $\sin \frac{2k-1}{2}\pi$ (e) $\sin \frac{k\pi}{2}$

   (b) $\cos k\pi$ (d) $\cos \frac{2k-1}{2}\pi$ (f) $\cos \frac{k\pi}{2}$

2. Let $g(x) = l$ on $(0, l)$.

   (a) Draw the even extension of g to $(-l, l)$. Compute the cosine series for g.

   (b) Draw the odd extension of g to $(-l, l)$. Compute the sine series for g.

3. Let $f(x) = x$ on $[-l, l]$.

   (a) Compute the Fourier series of f.

   (b) What is the solution to the heat equation $u_{xx} = 4u_t$ on the intervals $-l < x < l$ and $t > 0$ with the boundary conditions $u(-l, t) = u(l, t) = 0$ and the initial condition $u(x, 0) = \sin \frac{n\pi x}{l}$?

   (c) Use the principle of superposition to solve the heat equation $u_{xx} = 4u_t$ on $-l < x < l$ and $t > 0$ with the boundary conditions $u(-l, t) = u(l, t) = 0$ and the initial condition $u(x, 0) = 0$.

4. Let $f(x) = \begin{cases} 0 & \text{if } -l < x < 0, \\ l/2 & \text{if } x = 0, \\ l & \text{if } 0 < x < l. \end{cases}$

   (a) Sketch f and its periodic extension.

   (b) Compute the Fourier series of f.

   (c) Compare this Fourier series to the one from Problem 2. Note that $f = \frac{l}{2} + h$ for some odd function h.

   (d) The Fourier theorem guarantees that the Fourier series you computed in (b) converges to f for each x in $(-l, l)$. What does this mean?

   (e) Evaluate the infinite series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. (Hint: Use part (d), and choose a good value to plug in for x.)

5. Assume that f has a Fourier sine series $f(x) = \sum_1^\infty b_n \sin(n\pi x/l)$ for $0 \leq x \leq l$.

   (a) Show that
   $$\frac{2}{l}\int_0^l f(x)^2 dx = \sum_1^\infty b_n^2.$$
   (Hint: Recall the inner product $\langle f, g\rangle = \int_0^l f(x)g(x)dx$. Think about $||f||$ in two ways.)

   (b) Apply part (a) to the series you computed in Problem 3 to show that
   $$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^\infty \frac{1}{n^2}.$$

28. Applications of PDE's

Introduction

In this worksheet we consider the heat and wave equations and conclude with a review of linear algebra.

Problems

1. Find the steady state temperature in a bar that is insulated at the end $x = 0$ and held at the constant temperature 0 at the end $x = l$. (How does the word "insulated" translate into a boundary condition?)

2. Find the steady state temperature in the bar of the previous exercise if the bar is insulated at the end $x = 0$ and held at the constant temperature T at the end $x = l$.

3. A steel wire 5 feet in length is stretched by a tensile force of 50 pounds. The wire weighs 0.026 pounds per linear foot.

   (a) Find the velocity of propagation of the transverse waves in the wire.

   (b) Find the natural frequencies of the vibration.

   (c) How do the natural frequencies change when the tension is changed?

4. This problem illustrates the method of variation of parameters. Suppose you are trying to solve the differential equation $\mathbf{x}' = A\mathbf{x} + \mathbf{b}(t)$, and you already know that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent solutions to the homogeneous ODE $\mathbf{x}' = A\mathbf{x}$. Form the matrix whose columns are $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\mathbf{\Phi} = \begin{bmatrix} | & | \\ \mathbf{x}_1 & \mathbf{x}_2 \\ | & | \end{bmatrix}$$

   In Math 1B, you learned a method of variation of parameters for solving inhomogeneous differential equations. Starting with solutions x_1 and x_2 to the corresponding homogeneous equation you guess that there exist *functions* u_1 and u_2 such that $u_1x_1 + u_2x_2$ is a solution to the inhomogeneous equation. The difficult step is finding u_1 and u_2.

   (a) Show that the general solution of $\mathbf{x}' = A\mathbf{x}$ is given by $\mathbf{x} = \mathbf{\Phi}\mathbf{c}$ where $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is an arbitrary *constant* vector.

(b) Here we mimic the variation of parameters procedure and guess that something of the form $\mathbf{\Phi u}$ solves the equation $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$, where $\mathbf{u} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$. Why is this analogous to what you did in 1B?

(c) Show that for any $\mathbf{u}$,

$$(\mathbf{\Phi u})' = A(\mathbf{\Phi u}) + \mathbf{\Phi u}'.$$

(d) Conclude, therefore, that $\mathbf{\Phi u}$ is what we are looking for exactly when $\mathbf{\Phi u}' = \mathbf{b}(t)$.

(e) Show that $\mathbf{\Phi u}$ is a solution of $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$, where $\mathbf{u} = \int [\mathbf{\Phi}(t)]^{-1}\mathbf{b}(t)\ dt$.

(f) Find a solution of

$$\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 4t \end{bmatrix}.$$