Math 1B: Calculus Worksheets

7^{th} Edition

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Preface

This booklet contains the worksheets for Math 1B, U.C. Berkeley's second semester calculus course.

The introduction of each worksheet briefly motivates the main ideas but is not intended as a substitute for the textbook or lectures. The questions emphasize qualitative issues and the problems are more computationally intensive. The additional problems are more challenging and sometimes deal with technical details or tangential concepts.

About the worksheets

This booklet contains the worksheets that you will be using in the discussion section of your course. Each worksheet contains Questions, and most also have Problems and Additional Problems. The Questions emphasize qualitative issues and answers for them may vary. The Problems tend to be computationally intensive. The Additional Problems are sometimes more challenging and concern technical details or topics related to the Questions and Problems.

Some worksheets contain more problems than can be done during one discussion section. Do not despair! You are not intended to do every problem of every worksheet. Please email any comments to `calclab@math.berkeley.edu`.

Why worksheets?

There are several reasons to use worksheets:

- *Communicating to learn.* You learn from the explanations and questions of the students in your class as well as from lectures. Explaining to others enhances your understanding and allows you to correct misunderstandings.

- *Learning to communicate.* Research in fields such as engineering and experimental science is often done in groups. Research results are often described in talks and lectures. Being able to communicate about science is an important skill in many careers.

- *Learning to work in groups.* Industry wants graduates who can communicate *and* work with others.

The 5th and 6th editions have been revised by Cathy Kessel and Michael Wu. The third and fourth editions were prepared by Zeph Grunschlag and William Stein, and introduced a relatively small number of changes to Christine Heitsch's second edition. Michael Hutchings made tiny changes for the 7th edition.

The authors of the fourth edition would like to thank Roland Dreier for his helpful comments.

Contributors to this workbook include: Aaron Abrams, Zeph Grunschlag, Christine Heitsch, Tom Insel, George Johnson, David Jones, Reese Jones, Cathy Kessel, Julie Mitchell, Bob Pratt, Fraydoun Rezakhanlou, William Stein, Alan Weinstein, and Michael Wu.

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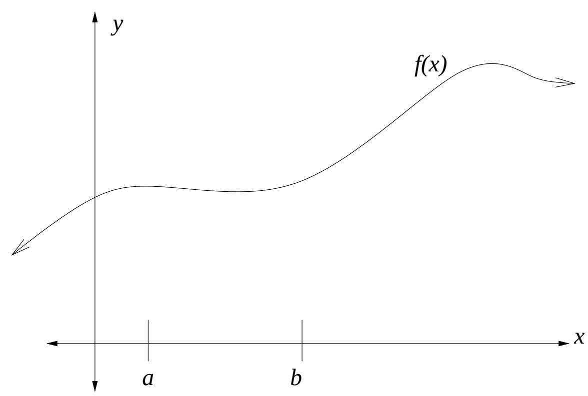
Appendix

1. Integration by Parts

Questions

1. Write an expression for the area under this curve between a and b.

2. Write an equation for the line tangent to the graph of f at $(a, f(a))$.



Problems

1. (a) Write down the derivative of $f(x)g(x)$.

   (b) If $\int h(x)dx = H(x)$, then $H'(x) = \ldots$?

   (c) Suppose you know that $H'(x) = f'(x)g(x) + f(x)g'(x)$. Can you write down a formula for $H(x)$?
   [Hint: Don't worry about the integration constant!]

   (d) Rearrange your equation(s) for $H(x)$ to get a formula for $\int f'(x)g(x)dx$.

   (e) Rearrange your equation(s) for $H(x)$ to get a formula for $\int g'(x)f(x)dx$.

   (f) Your book calls one of these two formulas the integration by parts formula. Why doesn't it call the other one the integration by parts formula?

2. This table will be helpful for Problem 3.

	antiderivative	derivative
x^n when $n \neq -1$		
$1/x$		
e^x		
e^{2x}		
$\cos x$		
$\sin 2x$		

3. Find the following integrals. The table above and the integration by parts formula will be helpful.
   (a) $\int x \cos x \, dx$
   (b) $\int \ln x \, dx$
   (c) $\int x^2 e^{2x} \, dx$
   (d) $\int e^x \sin 2x \, dx$
   (e) $\displaystyle\int \frac{\ln x}{x} \, dx$

Additional Problems

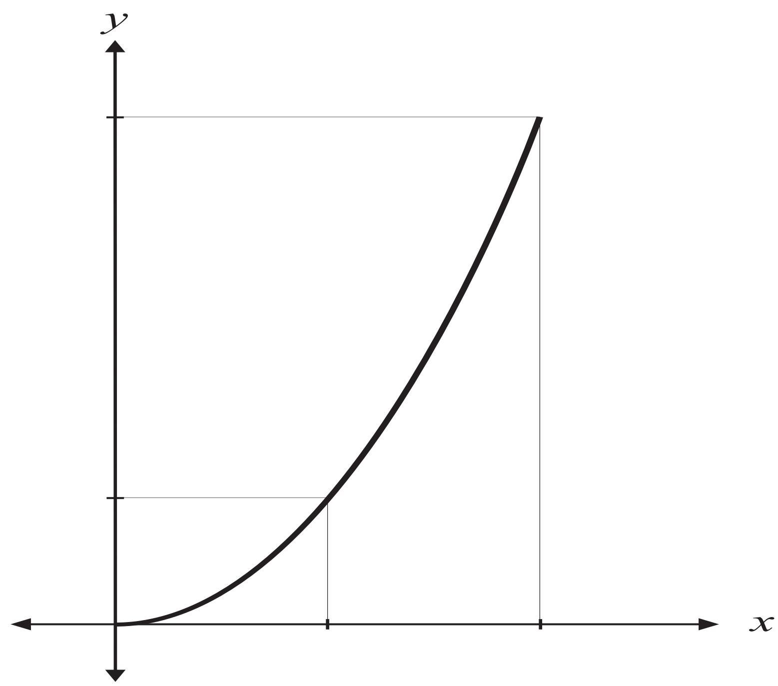
1. (a) Use integration by parts to prove the reduction formula
$$\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx$$
   (b) Evaluate $\int (\ln x)^3 \, dx$
2. (a) If f is any function, show that $(f \cdot e^x)' = (f + f')e^x$ and that $\int (f \cdot e^x) \, dx = f \cdot e^x - \int (f' \cdot e^x) \, dx$.
   (b) Find similar formulas for $(f \cdot e^x)''$ and $\int f' \cdot e^x \, dx$.
   (c) Can you use the formulas in part b to compute the third derivative of $e^x(x^3 + 5x - 2)$? How about $\int e^x(x^3 + 5x - 2) \, dx$?
   (d) If f is a polynomial of degree n, then which derivatives of f are *not* identically zero?
   [Hint: The mth derivative of f is identically zero if $f^{(m)}(x) = 0$ for all possible values of x.]
   (e) Show that if f is a polynomial of degree n, then
$$\int (f \cdot e^x) \, dx = (f - f' + f'' - f''' + \cdots + (-1)^? f^{(n)})e^x$$
   Should "?" be n or $n + 1$?

3. Work through the following questions to prove that if $f(x)$ is a function with inverse $g(x)$ and $f'(x)$ is continuous, then

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)dy.$$

   (a) i. First, let $f(x) = x^2$, $g(x) = \sqrt{x}$, $a = 1$, and $b = 2$. Fill in the graph below with the following labels:

$$y = f(x), \quad x = g(y), \quad a, \quad b, \quad f(a), \quad f(b).$$



   ii. Shade in the area representing $\int_a^b f(x)dx$.

   iii. Use your picture to explain the formula

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)dy.$$

   (b) Now, use integration by parts to show that $\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$.

   (c) Finally, use part b and the substitution $y = f(x)$ to obtain the formula for $\int_a^b f(x)\,dx$. Remember that f and g are inverses of each other!

   (d) Use what you have proven to evaluate $\int_1^e \ln x\,dx$.

4. Find reduction formulas for $\int x^n e^x\,dx$ and $\int x^n \sin x\,dx$.

5. Try to generalize Additional Problem 2. Can you find formulas for the derivatives and/or integrals of functions like $f \cdot e^{-x}$, $f \cdot e^{2x}$, etc.?
[Hint: You might want to do many cases at once by thinking more generally about the function $f \cdot e^{kx}$, where k is some constant.]

2. Trigonometric Integrals

Questions

1. (a) Use the Pythagorean Theorem to show that $\sin^2 x$ can be expressed in terms of $\cos x$. How do you express $\sin^4 x$, $\sin^6 x$, etc. in terms of $\cos x$?
   (b) Use part a to show that $\tan^2 x$ can be expressed in terms of $\sec x$.

Problems

1. (a) Find at least three different ways to integrate $\int \sin x \cos x \, dx$. [Hint: Use the double angle formula.] Compare your three answers. Are they the same? Should they be the same?
   (b) Find $\int \sin^2 x \, dx$.
   (c) Find $\int \sin^3 x \, dx$. There are at least two ways to do this.
   (d) Find $\int \sin^4 x \, dx$.
   (e) Find $\int \sin^5 x \, dx$.
   (f) Let n be a positive integer. Find $\int \sin^n x \, dx$.
   (g) This problem will help you do parts h, i, and j. Let k be an integer. Complete this table:

$2k-1$	$2k$	$2k+1$	$4k-3$	$6k$	$6k+2$	$6k+3$
	even					

   (h) Let m and k be non-negative integers. Find $\int \sin^m x \cos^{2k+1} x \, dx$.
   (i) Let m and k be non-negative integers. Find $\int \sin^{2k+1} x \cos^m x \, dx$.
   (j) Let m and k be non-negative integers. Find $\int \sin^{2k} x \cos^{2m} x \, dx$.

2. What is the volume of the solid generated by rotating $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$ around the y-axis?

3. (a) Use the fact that

$$\tan x = \frac{1}{\sec x} \cdot \sec x \tan x$$

   to find $\int \tan x \, dx$.
   (b) Evaluate $\int \tan^2 x \, dx$, $\int \tan^3 x \, dx$, $\int \tan^4 x \, dx$, and $\int \tan^5 x \, dx$. Which trig identities did you use?
   (c) If n is an integer, $n \geq 2$, show that

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx.$$

Additional Problems

1. Use the derivative of $\sec x \tan x$ to help you find $\int \sec^3 x\, dx$.

2. Write the identities for $\sin(a+b)$, $\cos(a+b)$, $\sin(a-b)$, and $\cos(a-b)$. Using these, find formulas for:

   (a) $\int \sin mx \cos nx\, dx$

   (b) $\int \sin mx \sin nx\, dx$

   (c) $\int \cos mx \cos nx\, dx$

   [Hint: Try adding and subtracting combinations of the trig identities. For example, what is $\sin(a+b)$? What is $\sin(a-b)$? Can you seen how to put together these two identities to get $\sin a \cos b$? Do the identities change if you let $a = mx$ and $b = nx$, or vice versa?]

3. Trigonometric Substitutions

Questions

1. Write some substitutions or strategies that would work for the following integrals. If you were to use substitutions to integrate, what would replace dx? Don't evaluate the integrals!

   (a) $\displaystyle\int \frac{x^2}{\sqrt{1-x}}\,dx$

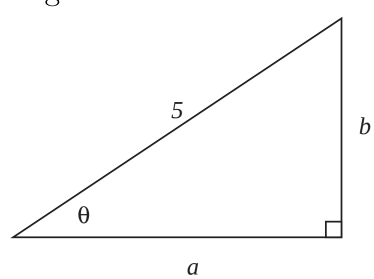
   (b) $\displaystyle\int \sqrt{1-x^2}\,dx$

   (c) $\int \sqrt{x^2+1}\,dx$

   (d) $\int x\sqrt{x^2+1}\,dx$

2. (a) Using the triangle below, express the following in terms of a and b.

$\sin\theta$	$5\cot\theta$	$\csc\theta$
$5\cos\theta$	$\tan\theta$	$5\sec\theta$

   (b) Use part a to help you decide which trig substitutions to use for the following integrals.

   i. $\displaystyle\int \frac{dx}{\sqrt{25-x^2}}$

   ii. $\displaystyle\int \frac{dx}{\sqrt{x^2+25}}$

   iii. $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-25}}$

3. Solve each equation by *completing the square*; do *not* use the quadratic formula.

   (a) $x^2 - 9 = 0$

   (b) $x^2 = 4x + 5$

   (c) $x^2 + 3x + 5 = 0$

4. Evaluate the following integrals:

   (a) $\int \sqrt{9-e^{2t}}\,dt$

(b) $\int \frac{dx}{\sqrt{x^2-4x-5}}$

(c) $\int \frac{dx}{x+x^3}$

Remember that your answer needs to be expressed in terms of the original variable (in this case x or t).

5. Using integration, show that the area of a circle with radius a is πa^2.
[Hint: Before setting up the integral, first sketch a circle of radius a centered at the origin.]

Problems

1. Let a be a positive real number. Let $f(x)$ be $\sqrt{a^2 - x^2}$.

   (a) For which values of x is $f(x)$ defined? Sketch the domain of f on a number line.

   (b) Draw a right triangle and decide which edges best represent x and $f(x)$. Label all three edges with an appropriate value. Express $\sin\theta$, $\tan\theta$, and $\sec\theta$ (where θ is an acute angle of your triangle) in terms of the values written on the edges.

   (c) Write x as a function of θ, $x = j(\theta)$. What is the domain and range of the function j?

   (d) Does the function $f(j(\theta))$ have the same domain and range as the function $f(x)$?

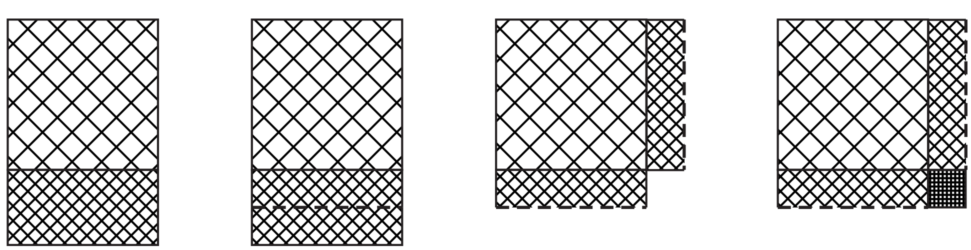
   (e) Now integrate $\int \frac{dx}{f(x)}$ using the substitution $x = j(\theta)$.

2. Use your work on problem 1 to find $\int \sqrt{a^2 + x^2}$ and $\int \sqrt{x^2 - a^2}$, by replacing $f(x) = \sqrt{a^2 - x^2}$ by $g(x) = \sqrt{a^2 + x^2}$ *or* by $h(x) = \sqrt{x^2 - a^2}$.

Additional Problems

1. Evaluate $\displaystyle\int \frac{x^2\,dx}{(x^2 + a^2)^{3/2}}$ first by trigonometric substitution and then by the hyperbolic substitution $x = a\sinh t$.

2. Completing the square also has a geometric meaning. Explain why all of these drawings represent the same area.



4. Integration of Rational Functions by Partial Fractions

Questions

1. (a) What is a rational function? Are polynomials rational functions? Give an example of a rational function, and another example of something which is *not* one.
   (b) When is a rational function "proper"?
   (c) Write $\frac{x^4}{1-x^2}$ as the sum of a polynomial and a proper fraction.
   (d) Write $\frac{2}{1-x^2}$ as the sum of two fractions with irreducible denominators.

2. Assume that A and B are real numbers and n is a positive integer. Find the following integrals:

   (a) $\displaystyle\int \frac{A}{x^2+1}\,dx$

   (b) $\displaystyle\int \frac{Ax+B}{x^2+1}\,dx$

   (c) $\displaystyle\int \frac{A}{x^2+x+1}\,dx$

   (d) $\displaystyle\int (a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0)\,dx$

Problems

1. What is the *structure* of the partial fractions decomposition for each of the following integrals? Don't bother finding the actual decomposition, leave the coefficients undetermined. For example,

$$\int \frac{2}{1-x^2}\,dx = \int \left(\frac{A}{1-x} + \frac{B}{1+x}\right)\,dx.$$

   (a) $\displaystyle\int \frac{x^3}{(x+1)(x+2)}\,dx$

   (b) $\displaystyle\int \frac{dx}{(x+2)(x+3)^3}$

   (c) $\displaystyle\int \frac{dx}{(x^2+2)^4}$

(d) $\displaystyle\int \frac{x^5}{(x^2-4)(x^2+3)^2}\,dx$

(e) $\displaystyle\int \frac{dx}{(x^3+2x^2+4x+8}$

2. Evaluate $\displaystyle\int \frac{2x^2-2x-2}{x^3+2x^2+2x}\,dx$ and $\displaystyle\int \frac{3e^{2t}}{e^{2t}-e^t-6}\,dt$.

Additional Problems

1. Let $f(x)$ be ax^2+bx+c where $a \neq 0$.
   (a) Sketch possible graphs of $f(x)$ for the following cases: f has no real roots, f has precisely one real root, f has two distinct real roots.
   (b) Based on the quadratic formula, what can you say about the sign of b^2-4ac in each case?
   (c) In which case(s) is $f(x)$ an irreducible quadratic?
   (d) How do you integrate $\frac{1}{f(x)}$ in each case?

2. (a) If a, b, and c are given, and $ax^2+bx+c=0$, then what do you know about x? How many x make the equation true? Which x in particular?
   (b) If a, b, and c are unknown, and $ax^2+bx+c=0$ for all x, then what do you know about a, b, and c? How many triples (a,b,c) make the equation true for all x? Which combination(s) in particular?
   (c) What is the difference between the number 0 and the function 0? Begin your answer by graphing each one. Which one appears on the right hand side of the equation in (a) and which appears on the right hand side of the equation in (b)?
   (d) Explain the difference between "$ax^2+bx+c=0$ for some x" and "$ax^2+bx+c=0$ for all x."

3. Suppose that F, G, and Q are polynomials and that

$$\frac{F(x)}{Q(x)} = \frac{G(x)}{Q(x)}$$

for all x except when $Q(x)=0$. Prove that $F(x)=G(x)$ for all x.
[Hint: Use continuity and the function $h(x)=F(x)-G(x)$.]

5. Rationalizing Substitutions and Integration Strategies

Questions

1. (a) For each of the following integrals. write down:
   what seems to be the best substitution for x,
   what dx would be if you used this substitution.

   i. $\displaystyle\int \frac{dx}{\sqrt[3]{x}+\sqrt[4]{x}}$

   ii. $\displaystyle\int \frac{dx}{\sqrt{1+e^x}}$

   iii. $\displaystyle\int \frac{x^3}{\sqrt[3]{x^2+1}}\,dx$

   (b) Evaluate the three integrals using your "best" substitution.

2. (a) Let $t = \tan(\frac{x}{2})$. Write $\cos(\frac{x}{2})$ and $\sin(\frac{x}{2})$ in terms of t.

   (b) Use this information to find $\cos x$ and $\sin x$ in terms of t. Which trig identities did you use?

   (c) If you are using the substitution $t = \tan(\frac{x}{2})$ to evaluate the integral

$$\int \frac{1}{\sin x - \cos x}\,dx$$

   what would you replace dx by?

Problems

1. For each of the following integrals, decide which techniques of integration would work. If you've listed more than one integration technique, indicate which seems to be the best choice.

   - If you choose a substitution as the best technique, write down the substitution explicitly, including whatever would replace dx.
   - If the choice is partial fractions, write down the structure of the partial fractions, but leave the coefficients undetermined.
   - If it's integration by parts, write down your choices for u and dv.

   Do not evaluate the integrals completely! That's problem 2.

(a) $\displaystyle\int \frac{dx}{x^2 + x^4}$

(b) $\int \cos(\ln x)\, dx$

(c) $\displaystyle\int \frac{dx}{x + x(\ln x)^2}$

(d) $\int \cos^3 2x \sin 2x\, dx$

(e) $\int \sqrt{1 + x^2}\, dx$

(f) $\displaystyle\int \frac{x^2}{(x-3)(x+2)^2}\, dx$

(g) $\displaystyle\int \frac{2\, dx}{\sqrt{e^x}}$

(h) $\displaystyle\int \frac{dx}{(1 - 9x^2)^{3/2}}$

(i) $\int x^2 \tan^{-1} x\, dx$

(j) $\displaystyle\int \frac{x^2 - 5x - 8}{x^3 + 4x^2 + 8x}\, dx$

(k) $\displaystyle\int \frac{\sec^2(\sin x)\, dx}{\sec x}$

(l) $\int \tan^3 x \sec^7 x\, dx$

(m) $\displaystyle\int \frac{x^3 + 1}{(x+1)^3}\, dx$

(n) $\int \sin^5 x\, dx$

(o) $\displaystyle\int \frac{dx}{(x-1)\sqrt{x^2 - 2x - 3}}$

2. Each person in your group should choose two different integrals from the previous list, and evaluate them. Did the substitution(s) you discussed as a group turn out to be useful?

3. Assume that $a \neq 0$, and evaluate

$$\int \frac{dx}{a \sin^2 x + b \sin x \cos x + c \cos^2 x}$$

[Hint: Substitute $u = \tan x$, and consider different cases depending on the sign of $b^2 - 4ac$.]

Additional Problems

1. (a) Use the Weierstrass substitution $t = \tan(x/2)$ to prove the formula

$$\int \sec x \, dx = \ln \left| \frac{1 + \tan(x/2)}{1 - \tan(x/2)} \right| + C$$

   (b) Show that

$$\int \sec x \, dx = \ln \left| \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right| + C$$

   by using the formula for $\tan(x + y)$ and part (a).

   (c) Show that the formula from part (a) agrees with the formula

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

6. Approximate Integration

Questions

1. (a) Explain what each of the following terms means for a function on an interval $[a, b]$, and then arrange them in increasing order: upper bound, lower bound, absolute maximum, absolute minimum.

   (b) Draw the graph of a function f that has no upper bound on $[0, 2]$.

   (c) If $g(x)$ is an increasing function, what are upper and lower bounds for $g(x)$ on the interval $[a, b]$?

   (d) Let $f(x) = -e^{-x^2}$. Find a K so that $|f''(x)| \leq K$ for all x in the interval $[0, 2]$.

2. Why is it useful to be able to approximate integrals?

3. (a) What is meant by the error of an approximation?

   (b) What is the difference between "error bound" and "error"?

   (c) Write down the formulas for the Midpoint and Trapezoidal error bounds:

      i. $|E_M| \leq$

      ii. $|E_T| \leq$

   (d) Where do the a and b come from? What does the n refer to? How is K defined?

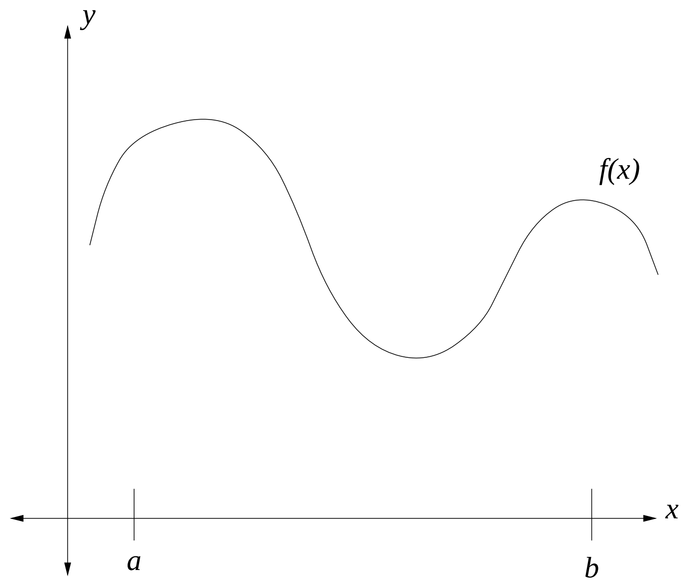
   (e) How big must n be so that $\frac{5 \cdot 7^3}{12n^2} < 10^{-2}$?
   How about $\frac{7 \cdot 5^3}{24n^2} < 10^{-6}$?

4. (a) Write down Simpson's Rule. What is the pattern of the coefficients of the $f(x_i)$?

   (b) What is the error bound formula for Simpson's Rule? How is the K different than the K in the Midpoint or Trapezoidal error estimates?

Problems

1. Based on this graph of $f(x)$ on the interval $[a, b]$, answer the following questions.



(a) In terms of the picture, what does $\int_a^b f(x)\,dx$ represent?

(b) Sketch L_n, R_n, M_n, and T_n for $n = 4$. If you have enough people, sketch a different approximation on each worksheet.

(c) In terms of the picture(s), what does Δx represent in each case?

(d) For each approximation, what determines the height of the approximating polygons? Which approximation is not like the other ones?

(e) Which approximations seem to be overestimates? Which ones underestimates?

(f) Which appears to be the best approximation?

2. (a) Sketch the graphs of $y = \sin x$ and $y = |\sin x|$.

   (b) Find an upper and lower bound for $\sin x$ on the interval $[-\pi/2, \pi/2]$.

   (c) Find an upper bound for $|\sin x|$ on $[-\pi/2, \pi/2]$.

   (d) Is every lower bound for $\sin x$ also a lower bound for $|\sin x|$? How about vice versa? Why or why not?

   (e) Find another upper bound and lower bound for $\sin x$ and $|\sin x|$. How many possible bounds are there?

   (f) How would the bounds you chose on $\sin x$ and $|\sin x|$ have changed if the interval was $[-\pi/2, -\pi/4]$ instead?

3. Let $I = \int_0^1 \frac{dx}{1+x^2}$.

   (a) Compute I by integrating.

   (b) Now approximate I by the Trapezoidal Rule with $n = 4$.

      i. What is the error bound for this approximation?

      ii. How does the error estimate compare with the actual error?

iii. What is the least number of subdivisions for which you are guaranteed an approximation to within 0.001 accuracy?

(c) Repeat the second part for Simpson's Rule.

(d) Starting from the Simpson's Rule estimate for the integral I, find an approximate value for π.

Additional Problems

1. (a) Show that if $p(x)$ is a quadratic polynomial, then

$$\int_a^b p(x)\,dx = \frac{b-a}{6}\big[p(a) + 4\,p((a+b)/2) + p(b)\big].$$

This is the basis for Simpson's Rule. Can you see why?

(b) Explain why n, the number of subdivisions, must be even in Simpson's Rule.

2. Show that if $q(x)$ is a polynomial of degree 3 or smaller, then Simpson's Rule gives the exact value of $\int_a^b q(x)\,dx$.
(**Hint:** First, show that the statement is true when $q(x)$ is a quadratic polynomial. Then prove that it is true when $f(x) = x^3$. After this, you should be able to extend it to any cubic polynomial.)

7. Improper Integrals

Questions

1. Give examples and draw pictures of:
   (a) A convergent type 1 integral.
   (b) A divergent type 1 integral.
   (c) A convergent type 2 integral.
   (d) A divergent type 2 integral.

2. Which of the following integrals are improper? Express the improper integrals in terms of limits of one or more proper integrals.
   (a) $\displaystyle\int_2^3 \frac{dx}{x^2-1}$
   (b) $\int_0^{\pi/2} \sec x\, dx$
   (c) $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt{|2x+1|}}$
   (d) $\displaystyle\int_0^2 \frac{dt}{t^2+1}$

Problems

1. (a) What is the definition of $\int_{-\infty}^{\infty} f(x)\,dx$?
   (b) What does it mean for $\int_{-\infty}^{\infty} f(x)\,dx$ to be convergent?
   (c) If $\int_{-\infty}^{\infty} f(x)\,dx$ is convergent, and $a < b$ are real numbers, show that

$$\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx$$

   [Hint: Sketch a graph of f, a, and b.]
   (d) Draw a picture of $\int_{-\infty}^{\infty} x\,dx$. Evaluate this integral.
   (e) Draw a picture of $\int_{-t}^{t} x\,dx$ and show that $\lim_{t\to\infty} \int_{-t}^{t} x\,dx = 0$. This example illustrates why for some f:

$$\int_{-\infty}^{\infty} f(x)\,dx \neq \lim_{t\to\infty} \int_{-t}^{t} f(x)\,dx.$$

2. (a) What does the Comparison Test say?
   (b) Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ on the interval $[1, \infty)$. What is the relationship between f and g on this interval? (Which is always larger?)
   (c) Graph f and g. Indicate the areas representing $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$.
   (d) Which integral is convergent? Which is divergent? How do you know that?
   (e) Relate the convergence or divergence of an improper integral to the area under the corresponding curve.
   (f) Does the Comparison Test apply to these improper integrals? Why or why not?
   (g) Graph $h(x) = \frac{1}{x^3}$ and $i(x) = \frac{1}{x^{1/2}}$ along with f and g. List the four functions f, g, h, and i in order of increasing value on the interval $x \geq 1$.
   (h) To which pairs of the functions f, g, h, and i does the Comparison Theorem apply? What does it tell you about their convergence or divergence?
   (i) By making the connection between the area under a graph and the convergence of an improper integral, explain why you can draw the same conclusions as you would from the Comparison Theorem simply by looking at the graph of the four functions, f, g, h, and i.

3. (a) Find a t such that $\int_1^t \frac{dx}{x} > 100$. For which s is $\int_1^s \frac{dx}{\sqrt{x}} > 100$?
   (b) For what values of p is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ convergent? What is the value of the integral in those cases?
   (c) How about $\int_0^1 \frac{1}{x^p}\,dx$? For what values of p is the integral convergent, and what is the value in those cases?
   (d) What about $\int_0^\infty \frac{1}{x^p}\,dx$?
   (e) Explain in words what you have discovered about the relationship between the values of p for which the integral $\int \frac{1}{x^p}$ converges and the interval over which you are integrating.
   (f) Find a function $f(x)$ such that $\lim_{x \to 0^+} f(x) = \infty$ and $\int_0^\infty f(x)\,dx$ converges.

4. A non-negative function $f(x)$ such that $\int_{-\infty}^\infty f(x)dx$ converges and equals 1 is called a **probability density function**.

   (a) Let c be a positive constant and define

$$f(x) = \begin{cases} ce^{-cx}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

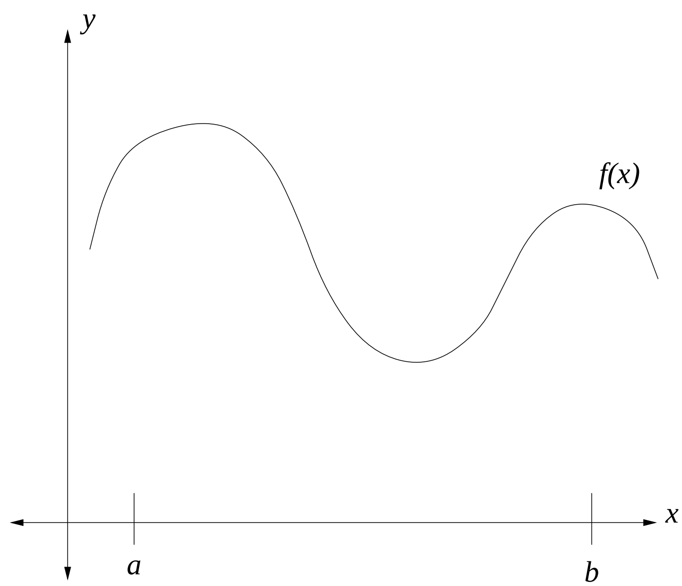
   Show that $f(x)$ is a probability density function.
   (b) Graph $f(x)$ for two choices of c.
   (c) Compute the mean $\mu = \int_{-\infty}^\infty x f(x)\,dx$.
   (d) Compute the variance $\sigma^2 = \int_{-\infty}^\infty (x - \mu)^2 f(x)\,dx$.

Additional Problems

1. Show that $\displaystyle\int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0$. [Hint: Consider $\int_0^1 \frac{\ln x}{1+x^2}dx$ and make the substitution $u = 1/x$.]

8. Arc Length

Questions

1. (a) What is the mathematical definition of a smooth function?
   (b) Sketch the graph of a continuous function whose derivative is not continuous, e.g., $f(x) = |x|$. Does the mathematical definition of "smooth" correspond to the usual notion of "smooth" in this case?
2. Explain (in words) the difference between $\frac{d^2x}{dy^2}$ and $\frac{dx}{dy}^2$.
3. Given the line segment connecting $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$, what is the formula for computing the arc length of this curve? Draw a diagram illustrating this formula.
4. Use the arc length formula to compute the length of the curve $y = mx + b$ from $x = x_0$ to $x = x_1$. Does your answer agree with what you already know about the distance between two points? Why or why not?
5. 

   (a) Using four line segments, sketch an approximation of the length of f between a and b on the diagram.
   (b) Write a formula for your approximation. (You'll need to label the coordinates of the ends of your line segments.)
   (c) Write a formula for an approximation that uses a partition $x_0, x_1, \ldots, x_n$ of $[a, b]$.
   (d) In order to take the limit of your approximation as $||P|| \to 0$ it is helpful to factor $x_i - x_{i-1}$ out of each radical. How can you do this?

(e) In order to simplify further it is helpful to change $\frac{f(x_i)-f(x_{i+1})}{x_i-x_{i+1})}$ to something simpler. How can you do this?

(f) What does your approximation become as $n \to \infty$?

Problems

1. In Question 5, you found a formula for s (the arc length of a curve). Now, think of $s(x)$ as a function—the distance along a curve from the initial point $(a, f(a))$ to the point $(x, f(x))$.

   (a) Explain how the arc length function $s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$ is different from the arc length formula $L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$.

   (b) What very important calculus theorem relates integrals and derivatives? Use this theorem to find $\frac{ds}{dx}$.

   (c) Can you find a lower bound on $\frac{ds}{dx}$? Thinking geometrically, does this make sense? Why or why not?
   (**Hint**: What is the longest side of a right triangle?)

2. (a) Sketch the graph of $y = x^2$ for $0 \le x \le 1$. Using only a geometric argument, show that $\sqrt{2} <$ (arc length of $y = x^2$ from 0 to 1) < 2.

   (b) Now, set up a definite integral equaling that arc length.

   (c) Use the Midpoint Rule to approximate the arc length to within 0.01.

   (d) Evaluate the integral exactly. (**Hint**: This amounts to integrating $\int \sec^3 x\, dx$.)

   (e) How close was your approximation in (c)?

Additional Problems

1. (a) Sketch the graph of $y = x \sin(\frac{1}{x})$ for $0 < x \le 1$.

   (b) Is the arc length of this graph finite or infinite?

2. The arc length formula was based on the premise that one can approximate the length of a smooth curve by using little line segments, and then adding up the lengths of these segments. What if one used another method to approximate the length. Would the approximation remain valid? The following example shows what can happen in this case.

   (a) What's the length of the line segment L from $(0, 0)$ to $(1, 1)$? Sketch this line, along with the coordinate axes.

   (b) We wish to construct successive approximations to L, and compute their lengths. Let the first approximation P_0 be the path from $(0, 0)$ right to $(1, 0)$ and then up to $(1, 1)$. Draw P_0 on your graph. How long is P_0?

(c) Draw in the next approximation, P_1: start at $(0,0)$, go right to $(\frac{1}{2},0)$, go up to $(\frac{1}{2},\frac{1}{2})$, go right to $(1,\frac{1}{2})$, and then up to $(1,1)$. How long is the path P_1?

(d) For each n, let P_n be the path which starts at $(0,0)$, goes right for a distance of $\frac{1}{2^n}$, then up for the same distance, then right again, then up again, then ... (always for the same distance), until reaching $(1,1)$. Sketch P_n for $n = 2$, 3, and 4.

(e) How many line segments make up the total path P_n? What is the length of P_n? Why does this make sense?

(f) Convince yourself that the path P_n approaches the line segment L as $n \to \infty$.

(g) Does the length of P_n approach the length of L as $n \to \infty$? Can you explain what is happening here?

3. Recall that a wire or chain hanging from two points of the same height takes the shape of a catenary $y = a\cosh(x/a)$ where the midpoint of the wire occurs at $x = 0$.

(a) If you hang a wire between two poles at $x = -b$ and $x = b$ such that it doesn't touch the ground between them, how long is the wire? (**Hint**: your answer will be in terms of a and b.)

(b) If you are given 1000 feet of wire to string between two poles that are 900 feet apart, and it hangs according to the above equation, how far above the ground is the lowest point of the wire? The highest point?

(c) If you are given 1000 feet of wire to string between two poles, how close can the poles be before the wire touches the ground?

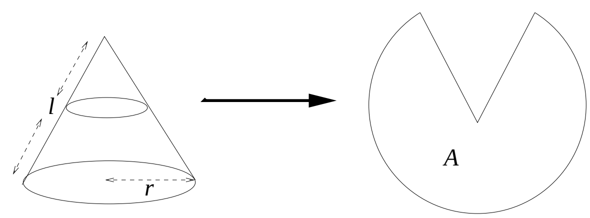
(d) How does the above equation fail to accurately model reality? (**Hint**: don't think too hard, it's not too fancy.)

9. Area of a Surface of Revolution

Questions

1. Sketch a cylinder and label the relevant dimensions. What is the formula for its surface area? Explain how your drawing supports this formula; think of cutting open the cylinder and flattening it out.

2. (a) A cone of base radius r and slant height l was cut open and flattened into a sector of a circle:



   Label each boundary segment of the sector with an appropriate value arising from the cone's dimensions.

   (b) We would like to determine the area A of the sector of the circle, since it's the same as the surface area of the cone. Explain why the area of the sector is the same as the surface area of the cone.

   (c) What is the length of the curved edge of the sector in terms of dimensions of the cone? Using these same dimensions, r and l, figure out what the circumference and the area of the complete circle would be.
   [Hint: It might help to draw in the complete circle around the sector.]

   (d) Using ratios of the area and circumference of the sector and the complete circle, come up with a formula for the surface area A of the cone.

3. True or false? If $f(x) \geq g(x)$ for all x in $[a, b]$, then the length of the graph of f between a and b is greater than or equal to the length of the graph of g between a and b.

Problems

1. Set up an integral to find the area of the surface obtained by rotating the curve about the x-axis. Evaluate it.

   (a) $y = \sqrt{x}$ from 2 to 4.

   (b) $y = x^3$ from 1 to 7.

   (c) $y = \cos x$ from 0 to $\pi/4$.

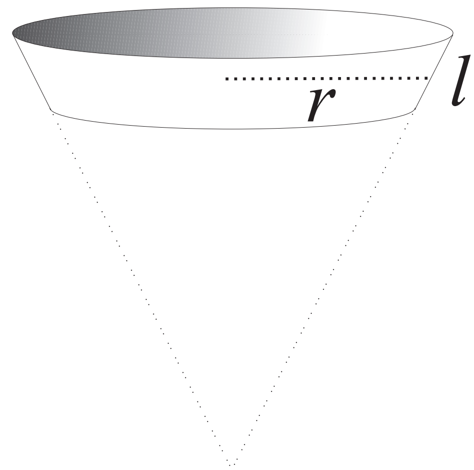
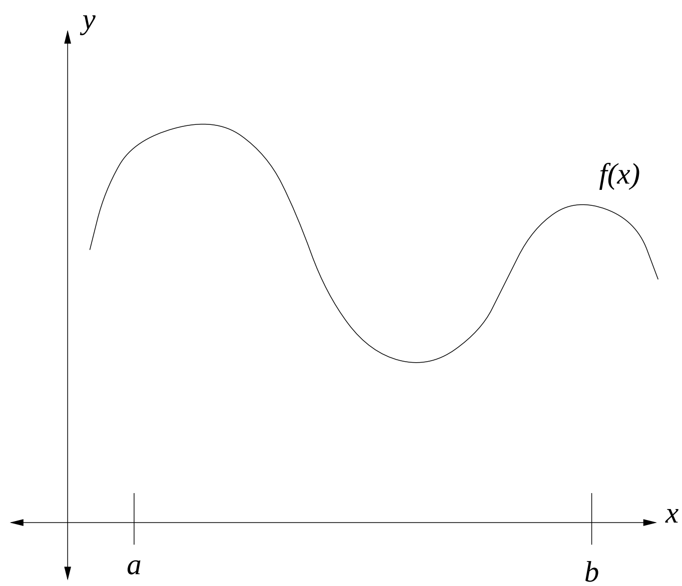


Figure 1: A **band** of slant height l and radius r.

2.



(a) Using four line segments, sketch on the diagram an approximation of the surface area obtained by taking the graph between a and b and revolving it around the x-axis.

(b) Write a formula for your approximation. (You'll need to label the coordinates of the ends of your line segments.)

(c) Write a formula for an approximation that uses a partition $x_0, x_1, \ldots, x_n$ of $[a, b]$.

(d) In order to take the limit of your approximation as $||P|| \to 0$ it is helpful to factor $x_i - x_{i-1}$ out of each radical. How can you do this?

(e) In order to simplify further it is helpful to change $\frac{f(x_i)-f(x_{i+1})}{x_i-x_{i+1}}$ to something simpler. How can you do this?

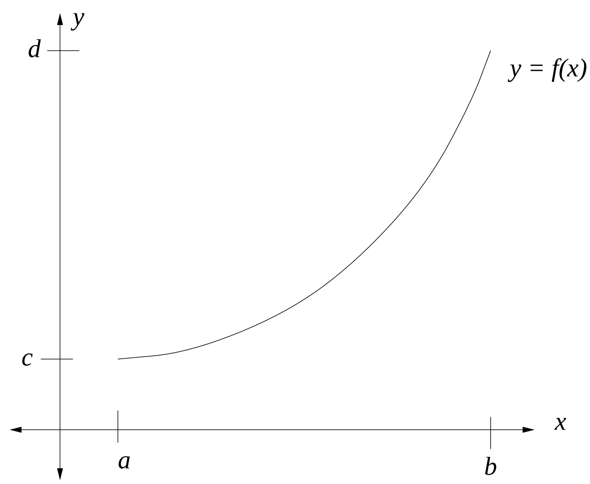
(f) What does your approximation become as $n \to \infty$?

(g) If A is the surface area of $\mathcal{T}$, then A is approximated by a Riemann sum in terms of ΔA's. Write this sum in terms of Δs's. What is the corresponding integral?

(h) Since the curve is written as a function of x, we choose x as our variable of integration. Write A as an integral in terms of x, complete with limits of integration. Do you remember how to write ds in terms of dx?

(i) If, instead, the formula for the curve were given to you as $x = g(y)$, how would the integral formula for surface area change?

3. (a) Suppose instead that we revolve the graph about the y-axis.



Starting from Δs, come up with a formula for the surface area.

(b) How would this formula for the surface area change if the equation for the graph were $x = g(y)$?

4. (a) Sketch the graph of $x = \ln y$ for $1 \leq y \leq e$. Set up (but don't evaluate) the integral for the area of the surface generated when this graph is revolved around the x-axis. What is the integral if the graph is revolved around the y-axis instead?

(b) Now express the same graph as $y = f(x)$. Over what interval will you be integrating? What are the two surface integrals now?

(c) Evaluate the integrals that you can. If you can't evaluate an integral, explain why not and give a rough estimate of its value.

5. (a) Sketch the graph of $y = \frac{1}{x}$ for $x \geq 1$. Show that the region under the graph (and above the x-axis) has infinite area.

(b) If the graph is rotated around the x-axis, show that the area of the resulting surface is infinite.

(c) Is the volume inside the surface finite or infinite? How did you tell?

Additional Problems

1. Suppose that you are given a function $f(x)$ on the interval $a \leq x \leq b$. Show that when $f(x)$ is not necessarily positive ($f(x) < 0$ for at least some x), the formula for surface area (around the y-axis) becomes

$$S = \int_a^b 2\pi|f(x)|\sqrt{1 + (f'(x))^2}\, dx$$

(**Hint**: Start by sketching a function $f(x)$ with positive and negative values, and the corresponding surface of rotation.)

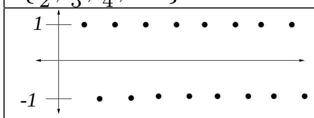
2. (a) If the curve $y = f(x)$, $a \leq x \leq b$, is rotated around the horizontal line $y = c$, where $f(x) \leq c$, find a formula for the area of the resulting surface.

   (b) Find the area of the surface obtained by rotating the circle $x^2 + y^2 = r^2$ about the line $y = r$.

10. Sequences

Questions

1. For each of the following sequences assume that the pattern given by the first few terms continues. Fill in the table.

	bounded?	monotonic?	convergent?	limit is?
$\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$				
				
$\{1, -2, 3, -4, 5, -6, \ldots\}$				
$a_n = (1 + \frac{1}{n})^n$				
$\{1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{8}, 1, \frac{1}{16}, \ldots\}$				
$a_n = 3^n$				
$\{2, 2, 2, 2, 2, \ldots\}$				
$a_n = \cos(\pi n)$				

2. Give examples of sequences which have the following properties. When it's not possible for a sequence to have the three properties, explain why not.

   (a) Bounded, monotonic, and convergent.

   (b) Bounded, not monotonic, and convergent.

   (c) Not bounded, monotonic, and convergent.

   (d) Not bounded, not monotonic, and convergent.

   (e) Bounded, monotonic, and not convergent.

   (f) Bounded, not monotonic, and not convergent.

   (g) Not bounded, monotonic, and not convergent.

   (h) Not bounded, not monotonic, and not convergent.

3. (a) Can a sequence have more than one limit? Why?

   (b) Explain the difference between "a_n can be made as close as we like to L for some (large) value of n" and "a_n, *and all following terms*, can be made as close as we like to L for n sufficiently large." Which phrase better captures the meaning of $\lim_{n\to\infty} a_n = L$

4. (a) *Without* looking at the textbook, try to write down the (ε, N) definition of $\lim_{n\to\infty} a_n = L$. Compare your answers with the definition in the book.
   (b) Draw a picture illustrating this (ε, N) definition for $\lim_{n\to\infty} \frac{(-1)^{n+1}}{n}$. How big must n be when $\varepsilon = \frac{1}{100}$?
   (c) Again, *without* looking at the textbook write the (M, N) definition of $\lim_{n\to\infty} a_n = \infty$. Compare your answers with the definition in the book.
   (d) Draw a picture illustrating this (M, N) definition for $\lim_{n\to\infty} \ln(\ln n)$. How big must n be when $M = 5$? Is the sequence $\ln(\ln n)$ convergent?

Problems

1. (a) Sketch a picture showing the sequence $\{r^n\}$ for the different cases: $r = -2$, -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, 2. For what values of r does the sequence $\{r^n\}$ converge?
   (b) Sketch a picture of the sequence $\{nr^n\}$ showing the different possible cases. For what values of r does the sequence $\{nr^n\}$ converge?

2. Suppose that $P(x) = c_0x^l + c_1x^{l-1} + \cdots + c_l$ and $Q(x) = d_0x^m + d_1x^{m-1} + \cdots + d_m$. Define a sequence $a_n = \frac{P(n)}{Q(n)}$ and use the limit laws to decide whether $\lim_{n\to\infty} a_n$ exists in the following cases. When the limit exists, what is it?

   (a) $\deg P < \deg Q$
   (b) $\deg P = \deg Q$
   (c) $\deg P > \deg Q$

3. (a) Draw a picture which illustrates the theorem "if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$".
   (b) Prove this theorem.
   [Hint: Use the Squeeze Theorem for sequences.]
   (c) True or False: If $a_n > 0$ for all n and $\lim_{n\to\infty} (-1)^n a_n = L$, then $L = 0$. If you answer "true," explain why. If you answer "false," come up with a counterexample.

Additional Problems

1. (a) If $\{a_n\}$ is convergent, show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$.
   (b) A sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = 1/(1 + a_n)$ for $n \geq 1$. Assuming that $\{a_n\}$ is convergent, find its limit.
   [Hint: Let $f(x) = \frac{1}{1+x}$. Suppose that L is the limit. What should $f(L)$ be?]

2. (a) Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.

(b) If $a_1 = 1$, and $a_{n+1} = 1 + \frac{1}{1+a_n}$, find the first eight terms of the sequence $\{a_n\}$. Then use part (a) to show that $\lim_{n\to\infty} a_n = \sqrt{2}$. This gives the **continued fraction expansion**

$$\sqrt{2} = 1 + \cfrac{1}{2 + \frac{1}{2+\dots}}$$

[Hint: First, show that the limits of $\{a_{2n}\}$ and $\{a_{2n+1}\}$ exist. Are these sequences monotonic and bounded? Once you know the limits exist, then find $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+1}$ using Additional Problem 1.]

3. The Fibonacci sequence $\{F_n\}$ is defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$.

(a) Write down the first ten terms in the Fibonacci sequence. Does it have a limit?

(b) Define a new sequence by $a_n = F_{n+1}/F_n$. Assuming that $\lim_{n\to\infty} a_n = L$, show that L is a root of $x^2 = x + 1$. Which root is it?
[Hint: It might help to find a recurrence relation for $\{a_n\}$ and use Additional Problem 1.]

(c) If x is either of the roots of $x^2 = x + 1$, use induction to prove that for $n \geq 1$,

$$x^n = xF_n + F_{n-1}$$

(d) Let y and z denote the two roots of $x^2 = x+1$; say, $y = \frac{1}{2}(1+\sqrt{5})$ and $z = \frac{1}{2}(1-\sqrt{5})$. Then, from part (c), we know that $y^n = yF_n + F_{n-1}$ and $z^n = zF_n + F_{n-1}$. Subtract these equations to show that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

(e) Use part (c) to show that $\{F_{n+1}/F_n\}$ does indeed converge.

11. Series

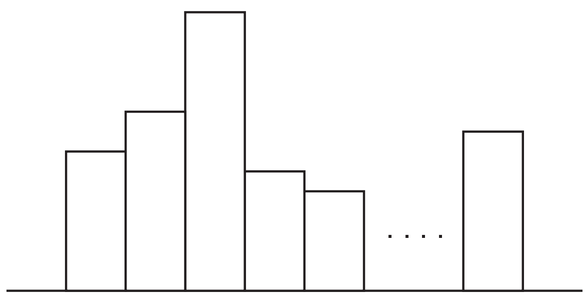
Questions

1. (a) Explain (in your own words) the difference between the *sequence* $\{a_n\}_{n=1}^{\infty}$ and the *series* $\sum_{n=1}^{\infty} a_n$.

   (b) Write out the first three terms of the series $\sum_{n=1}^{\infty} a_n$ and of the series $\sum_{i=1}^{\infty} a_i$. Is there any real difference between the two series, $\sum_{n=1}^{\infty} a_n$ and $\sum_{i=1}^{\infty} a_i$?

   (c) What is meant by the sequence of *partial sums*? For the series $\sum_{i=1}^{\infty} a_i$, write down the first, second, third, and nth partial sums s_1, s_2, s_3, and s_n in terms of the a_i.

   (d) On the graph below, if the rectangles have width 1 and heights a_1, a_2, a_3, a_4, a_5, . . . , a_n, what corresponds to the first, second, third, and nth partial sums of the series $\sum_{i=1}^{\infty} a_i$?



   (e) Draw a picture like the one above for a convergent series $\sum_{n=1}^{\infty} a_n$. What should be happening to the a_n and how does that show up in your picture?

   (f) What does it mean for a series $\sum_{n=1}^{\infty} a_n$ to be convergent? Make sure that your answer includes ideas like partial sums.

2. True or false? When your answer is "true," give an explanation. When your answer is "false," give a counterexample.

   (a) If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

   (b) If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

   (c) If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

3. (a) If $\sum a_n$ converges and $\sum b_n$ diverges, show that $\sum(a_n + b_n)$ diverges.

   (b) If $\sum a_n$ *and* $\sum b_n$ diverge, does $\sum(a_n + b_n)$ necessarily diverge? If "yes," then prove it. If "no," then give a counterexample.

Problems

1. (a) What is the name of the series $\sum_{n=1}^{\infty} \frac{1}{n}$?
   (b) Write out the first ten terms of the sequence $\{\frac{1}{n}\}$ and sketch a graph of these points. Does the sequence converge? If so, what is the limit?
   (c) Write out the first ten terms of the sequence of partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and sketch these points on your previous graph. Does the limit $\lim_{n\to\infty} s_n$ exist? [Hint: In general, $s_{2^n} > 1 + \frac{n}{2}$.]
   (d) Does the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converge? Why or why not?

2. Consider the telescoping series below. Do they converge? What do they converge to? Prove your assertions.
   (a) $\displaystyle\sum_{n=1}^{\infty} \arctan(n+1) - \arctan(n)$
   (b) $\displaystyle\sum_{n=1}^{\infty} \ln\Big(\frac{n+1}{n}\Big)$
   (c) $\displaystyle\sum_{n=1}^{\infty} \cos(2\pi(n+1)^2) - \cos(2\pi n^2)$
   (d) Do all possible telescoping series converge? If not, how can you tell whether or not a telescoping series converges?

3. In the formulas below, each value of p determines a series. Which values of p determine a convergent series? Investigate values of p between 0 and 3.
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$.
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$.

4. Assume that $a \neq 0$.
   (a) What's the difference between $\sum_{n=0}^{N} ar^n$ and $\sum_{n=1}^{N} ar^{n-1}$? Is there a difference between $\sum_{n=0}^{\infty} ar^n$ and $\sum_{n=1}^{\infty} ar^{n-1}$? Explain why or why not.
   (b) Show that $\sum_{n=0}^{\infty} ar^n$ diverges if $|r| \geq 1$.
   (c) Suppose now that $0 < r < 1$. Show that
   $$\sum_{n=0}^{N} ar^n = \frac{a(1-r^{N+1})}{1-r}.$$
   (d) Using the definition of convergence of an infinite series, show that $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

Additional Problems

1. (a) Draw a picture illustrating Zeno's paradox, assuming that the total distance from you to the wall is 1.

   (b) Use your picture to express this paradox as an infinite series in sigma notation. What is the sum of that series?

   (c) Even though the sum of the series is obvious, explain why Zeno was bothered by this.

2. Why is $.999\ldots$ equal to 1?

3. (a) If a and b are digits, show that

   i. $0.aaa\ldots = \frac{a}{9}$

   ii. $0.ababab\ldots = \frac{10a+b}{99}$

   (b) Show that every infinite series representation of a decimal number converges.

4. Don't forget that a series is *not* a finite sum, and that we must always be careful of the sort of algebraic manipulations we subject a series to. For example, we know that moving parentheses around does not affect the value of a finite sum, so $1-1+1=(1-1)+1=1+(-1+1)=1$. Explain what is wrong with the following calculation:

$$\begin{aligned} 0 &= 0+0+0+\ldots \\ &= (1-1)+(1-1)+(1-1)+(1-1)\ldots \\ &= 1-1+1-1+1-1+1-1\ldots \\ &= 1+(-1+1)+(-1+1)+(-1+1)+\ldots \\ &= 1+0+0+0+\ldots = 1 \end{aligned}$$

12. The Integral Test and Estimates of Sums

Questions

1. Which of the following is a geometric series? Which is a p-series, or can be treated like a p-series? How can you tell? Decide which of these are convergent.

(a) $\displaystyle\sum_{n=0}^{\infty} 3^{-n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

(c) $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \ldots$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{3\sqrt{n}}{n^3}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n}$

(f) $\frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 5^2} + \frac{1}{4 \cdot 5^3} + \ldots$

(g) $32 + 16 + 8 + 4 + \ldots$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

(i) $\displaystyle\sum_{n=5}^{\infty} \frac{1}{\sqrt[3]{n-3}}$

(j) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

(k) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n^2}$

(l) $\displaystyle\sum_{n=3}^{\infty} \frac{3}{1.01^{2n}}$

Problems

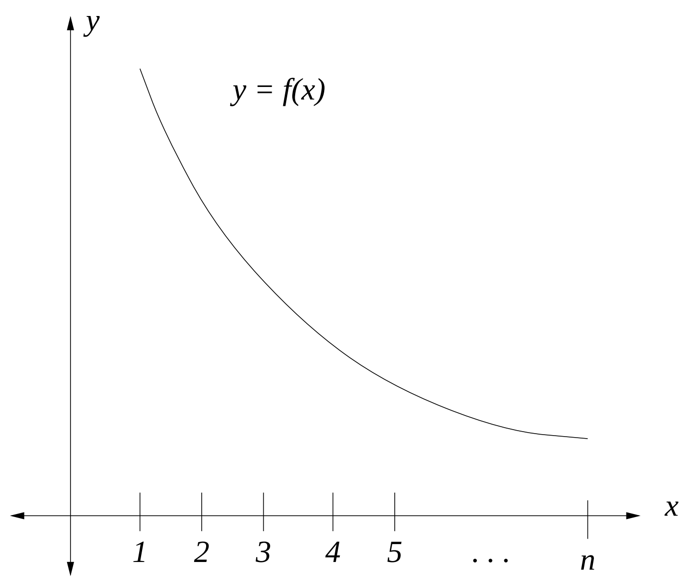
1. (a) For which x does $\sum_n (\ln x)^n$ converge?

(b) For which x does $\sum_n (\ln n)^x$ converge?

2. (a) For what values of p is $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent? [Hint: Use the divergence and integral tests.]

   (b) Now try to decide for which values of p the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is convergent.

3. Suppose that f is a continuous, decreasing and positive function on $[1, \infty)$ and that $a_i = f(i)$ for all positive integers i.



   (a) Sketch $\sum_{i=1}^{n-1} a_i$ and $\sum_{i=2}^{n} a_i$ on the graph above. Which sum is larger than $\int_1^n f(x)\,dx$? Which sum is smaller?

   (b) Using your picture, explain why the series $\sum_{i=1}^{\infty} a_i$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. What properties of f did you need to use?

   (c) We wish to approximate the sum of a convergent series using the nth partial sum:

$$s = \sum_{i=1}^{\infty} a_i = \sum_{i=1}^{n} a_i + \sum_{i=n+1}^{\infty} a_i = s_n + R_n.$$

   Looking at your graph, come up with upper and lower bounds for the remainder R_n in terms of integrals of $f(x)$.

   (d) Now, still using the picture, show that

$$\sum_{i=n+1}^{\infty} a_i \leq \int_n^{\infty} f(x)\,dx \leq \sum_{i=n}^{\infty} a_i.$$

(e) Starting from the previous inequality, show that

$$-a_n \leq \sum_{i=1}^{\infty} a_i - \left[\sum_{i=1}^{n} a_i + \int_n^{\infty} f(x)\,dx \right] \leq 0.$$

(f) For a particular value of n, which more likely to be the better approximation of $s = \sum_{i=1}^{\infty} a_i$, the sum $\sum_{i=1}^{n} a_i$ or $s_n + \int_n^{\infty} f(x)\,dx$?

4. Approximate $\sum_{i=0}^{\infty} \frac{1}{i^2}$ to within 0.001 error using s_n and $s_n + \int_n^{\infty} f(x)\,dx$. Which was the faster approximation?

Additional Problems

1. Explain why it is necessary that the function f be positive and decreasing in order to apply the Integral Test. Does f have to be *always* decreasing or is it enough if f is *eventually* decreasing? Was it really necessary for f to be continuous? Can you think of a weaker condition than continuity?

2. What is the comparison test for integrals? Draw a graph showing how it might be applied to determine the convergence or divergence of each of the following series. Then find out if they converge or diverge.

   (a) $\sum \frac{2^n}{n^2 + 2^n}$

   (b) $\sum \frac{\ln n}{n^2}$

   (c) $\sum \left(\frac{n^2 - 1}{n^3 + 1} \right)^2$

13. The Comparison Tests

Questions

1. (a) What is the formula for a general geometric series? When does such a series converge? Give an example of a geometric series.

   (b) What is the formula for a general p-series? When does such a series converge? Give an example of a p-series.

2. When testing the convergence or divergence of a series, it helps to make a good "guess" at the right answer — and then show that your guess was correct! As a group, decide quickly (without proof) whether each of the following series converges or diverges.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+11}}$

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n+4}{n^{\pi}}$

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{1+e^n}$

   (e) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\ln n}$

   (f) $\displaystyle\sum_{n=4}^{\infty} n \sin\left(\frac{1}{n}\right)$

   (g) $\displaystyle\sum_{n=7}^{\infty} \frac{1}{25n+16}$

   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-8}$

   (i) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2-1}{n^3+1}$

(j) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

(k) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

3. It's not enough to guess about the convergence of a series; you have to play by the rules and use a test to actually *show* convergence. Go back and try to prove that your answers for the previous part were correct. Be sure to state which test you use and why it applies. Occasionally your guess will have been wrong! How will you know when this is the case, and what should you do about it?

Problems

1. Think of the Comparison Test for integrals, and what you already know about the relationship between series with positive terms and the convergence of improper integrals. After working through the following questions, you will have a better understanding of the Comparison Test for series.

   (a) State the Comparison Test. Make sure to include all hypotheses, conclusions, and cases.

   (b) Beginning with the coordinate axes, sketch a graphical representation of the four series: $\sum \frac{1}{n}$, $\sum \frac{1}{n^2}$, $\sum \frac{1}{n^3}$, and $\sum \frac{1}{n^{1/2}}$.
   [Hint: For example, when sketching $\sum \frac{1}{n}$, start by thinking of the graph of $f(x) = \frac{1}{x}$ and the area under the curve.]

   (c) Does $\sum \frac{1}{n}$ converge? How about $\sum \frac{1}{n^2}$? How does the convergence or divergence of these series relate to the areas you have sketched?

   (d) Does the Comparison Test apply to the series $\sum a_n$ and $\sum b_n$ where $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$? Why or why not?

   (e) Look at the four series on your graph: $\sum \frac{1}{n}$, $\sum \frac{1}{n^2}$, $\sum \frac{1}{n^3}$, and $\sum \frac{1}{n^{1/2}}$. Decide which pairs of the fours series the Comparison Test applies to. What does it tell you about their convergence or divergence?

2. Suppose that $\sum a_n$ is a convergent series. Show that

   (a) if each a_n is nonzero, then $\sum \frac{1}{a_n}$ diverges.

   (b) if each a_n is nonnegative, then $\sum a_n^2$ converges.

   (c) if each a_n is nonnegative, then $\sum \sqrt{a_n}$ is sometimes convergent and sometimes divergent. (In this case, you should give at least one example of each possibility, but *don't* try to prove it.)

3. Let p_n denote the nth prime number. For example:

$$p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7, \ldots$$

It is a very difficult theorem from number theory (the *Prime Number Theorem*) that $\frac{p_n}{n \ln n} \to 1$. Use this fact to show that $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \ldots$ diverges to ∞.

4. Give an example of a pair of series $\sum a_n$ and $\sum b_n$ with $a_n, b_n > 0$ such that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ diverges, but $\sum a_n$ converges. Explain why this does *not* contradict the Limit Comparison Test.

Additional Problems

1. (a) If $\sum a_n$ is a convergent series with positive terms, is it true that $\sum \sin(a_n)$ is also convergent? Justify your answer.

   [Hint: can you find a good approximation for $\sin\theta$ when θ is only slightly larger than 0?]

   (b) If $\sum a_n$ and $\sum b_n$ are both convergent series with positive terms, is it true that $\sum a_n b_n$ is also convergent? Again, justify your answer.

2. Suppose a_n and b_n are sequences of positive numbers. Using the definition of convergence of a series, prove that:

   (a) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ also converges.

   (b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

14. Alternating Series

Questions

1. We usually divide series into two groups: series with only positive terms and series with mixed positive and negative terms. What about series with only negative terms, $\sum a_n$ where $a_n \leq 0$? Explain how you would determine the convergence of this type of series and why series with only negative terms don't need their own category.

2. (a) Give an example of an alternating series which begins with a positive term, and one which begins with a negative term.

   (b) Is $1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + -\frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \ldots$ an alternating series? Why or why not?

3. True or false: When "false," try to come up with a counterexample. In either case, justify your answer.

   (a) If $a_n > 0$ for all n, but $\{a_n\}$ is not eventually decreasing, then $\sum a_n$ diverges.

   (b) If $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is a convergent series and the b_n are positive, then eventually $b_n \geq b_{n+1}$ for all n (i.e., there is a N such that for all $n \geq N$, $b_n \geq b_{n+1}$).

   (c) If $a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.

   (d) If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Problems

1. Determine convergence or divergence of each of the following series.

   (a) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right)\cos(n^2)$

   (b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{\pi}{n}\right)$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\sqrt{n}}{n+1}$

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n}{3^n}$

   (e) $\displaystyle\sum_{n=1}^{\infty}(-1)^n(1+n!)^{1/n}$

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n^3 3^n}{n!}$

2. Which of the above series are alternating? Which converge conditionally? Which converge absolutely?

3. (a) What three conditions must a series satisfy in order for the Alternating Series Test to guarantee its convergence? Is the series $\sum_{n=1}^{\infty} \frac{-1^{(n-1)}}{n}$ convergent?
   (b) Come up with an example of an alternating series which does not converge. Why does your series diverge?
   (c) Using the series $\sum_{n=1}^{\infty} \frac{-1^{(n-1)}}{n}$, draw a number line and sketch the partial sums s_n. Explain why this picture supports the Alternating Series Test.
   (d) If $\sum(-1)^{n-1} b_n = s$ is a convergent alternating series with partial sums s_n, use your previous illustration of the Alternating Series Test to explain why $|R_n| = |s - s_n| \leq b_{n+1}$. That is, explain why if the partial sums are used to approximate the alternating series, the remainder is bounded by the first neglected term.

4. (a) Show that, although $\lim_{n\to\infty} b_n = 0$, the series $\sum(-1)^{n-1} b_n$, where

$$b_n = \begin{cases} 1/n & \text{if } n \text{ is odd} \\ 1/n^2 & \text{if } n \text{ is even} \end{cases}$$

   is divergent. Why doesn't the Alternating Series Test apply?
   (b) Show that the alternating series test fails for

$$\frac{1}{3} - \frac{1}{3} + \frac{1}{2} - \frac{1}{2} + \frac{1}{5} - \frac{1}{5} + \frac{1}{4} - \frac{1}{4} + \frac{1}{7} - \frac{1}{7} + \frac{1}{6} - \frac{1}{6} + \ldots$$

   Does the series converge?
   (c) Does $\frac{2}{3} - \frac{3}{5} + \frac{4}{7} - \frac{5}{9} + \cdots$ converge?

5. Is the 50th partial sum s_{50} of the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ an overestimate or an underestimate of the total sum? Explain your answer.

Additional Problems

1. (a) For $|x| < 1$, express $\frac{1}{1+x}$ as a series.
   (b) Integrating both $\frac{1}{1+x}$ and its series expression gives us the equality

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

   for $-1 < x < 1$. We'd like to show that this equation holds when $x = 1$ as well. First, for $|x| < 1$, show that

$$\left[\ln(1+x) - \sum_{n=1}^{N} \frac{(-1)^{n+1} x^n}{n}\right] < \frac{1}{N+1}.$$

(c) Next, by first taking the limit as $x \to 1^-$, and then taking the limit as $N \to \infty$, show that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots .$$

2. This problem is an example of the kinds of *severe* difficulties you can encounter by treating an infinite series with both positive and negative terms like a finite sum.

   (a) Using the fact that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, show that

   $$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 + \cdots = \frac{1}{2}\ln 2.$$

   (b) Add the series (term by term) for $\frac{1}{2}\ln 2$ to the series for $\ln 2$ to show that

   $$\frac{3}{2}\ln 2 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \cdots .$$

   (c) Look very closely at the series for $\frac{3}{2}\ln 2$. Have we shown that $\ln 2 = \frac{3}{2}\ln 2$?!? Try to explain what is happening in this problem.

3. If s is any real number, show that the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ can be rearranged (write the same terms, but in a different order) in such a way that the resulting series converges to s.
   [Hint: Take just enough positive terms to get above s, then just enough negative terms to get below s, then...]

15. Absolute Convergence and the Ratio and Root Tests

Questions

1. (a) Explain (in words) what absolute convergence, convergence, and conditional convergence of a series $\sum a_n$ mean. Be certain that the differences among the three are clear.

   (b) Decide whether the following statements are true or false:

   i. Every convergent alternating series is conditionally convergent.
   ii. Every absolutely convergent series is convergent.
   iii. Every convergent series is absolutely convergent.
   iv. Every alternating series converges.
   v. If $\sum a_n$ is conditionally convergent, then $\sum |a_n|$ diverges.
   vi. If $\sum |a_n|$ diverges, then $\sum a_n$ is conditionally convergent.

2. (a) Suppose that $\sum a_n$ is such that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$. What are the three possible outcomes of the Ratio Test and how do they depend on L? Give an example of a particular series for each case.

   (b) What if, instead, $\lim_{n\to\infty} \sqrt[n]{a_n} = L$. What are the three possible outcomes of the Root Test and how do they depend on L? Given an example of a particular series for each case.

Problems

1. Using the ratio and root tests, determine convergence or divergence of each of the following series.

   (a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{e}{n}\right)^n$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{3n} \cdot \frac{1}{3^n}$

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

2. (a) What does it mean to say that the Ratio Test fails for a particular series?

   (b) For which of the following does the ratio test fail? Decide if the series converges or diverges, even if the ratio test fails.

      i. $\displaystyle\sum_{n=1}^{\infty} n^n$

      ii. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

      iii. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 - n + 5}{4n^7 - 1}$

      iv. $1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \cdots$

   (c) Show that the Ratio Test fails for $\sum a_n$ whenever a_n is a rational function of n.

3. A series $\sum a_n$ is defined by the equations

$$a_1 = 1, \qquad a_{n+1} = a_n \cdot \left(\frac{2 + \cos n}{\sqrt{n}}\right)$$

   Determine whether $\sum a_n$ converges or diverges.

4. (a) Does $\sum \sin(\frac{1}{n})$ converge?

   (b) How about $\sum \frac{1}{n} \sin(\frac{1}{n})$?

5. True or false: When "false," try to come up with a counterexample. In either case, explain why you chose the answer you did.

   (a) If $\sum a_n$ converges and the sequence $\{b_n/a_n\}$ converges, then $\sum b_n$ converges. (Assume $a_n > 0$ and $b_n > 0$.)

   (b) If $a_n > 0$ for all n and $a_n \to 0$, then $\sum (-1)^n a_n$ converges.

   (c) If $a_n > 0$ for all n and $\sum a_n$ converges, then $\sum (-1)^n a_n$ converges.

   (d) If $\sum a_n$ converges, then $\sum \frac{(-1)^n a_n}{n}$ converges.

   (e) If $\sum a_n$ converges, then $\sum a_n^2$ converges.

   (f) If $a_n > 0$ for all n and $\sum a_n$ converges, then $\sum a_n^2$ converges.

(g) If $\sum a_n$ converges and the sequence $\{a_n/b_n\}$ converges, then $\sum b_n$ converges. (Assume $a_n > 0$ and $b_n > 0$.)

(h) If $\lim_{n\to\infty} (a_{n+1}/a_n) = L$ and $L < 1$, then $\sum a_n$ converges.

6. For which x does $\sum_{n=0}^{\infty} nx^n$ converge?

7. Show that if $\sum a_n$ converges and $|x| < 1$, then $\sum a_n x^n$ converges absolutely.

8. Stirling's formula says that

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1.$$

(a) Use Stirling's formula to show that $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$.

(b) Apply the root test to the following series:

i. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

ii. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$

iii. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$

Additional Problems

1. Let $\sum a_n$ be a series of positive terms with the property that there exists a number $r < 1$ and a positive integer n_0 such that $a_{n+1}/a_n \le r$ for all $n \ge n_0$. Show that $\sum a_n$ converges even though $\lim_{n\to\infty} (a_{n+1}/a_n)$ may not exist.

16. Power Series

Questions

1. The series

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is called a power series in $(x-a)$.

(a) For what value(s) of x does this power series always converge? What is the sum of the series in these cases?

(b) Let $a < b < c < d$. Graph on the real line the intervals $[a,b)$, $[b,c)$, and $[c,d)$. Is it possible that $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges on the intervals $[a,b)$ and $[c,d)$ but *not* on the interval $[b,c)$? Why or why not?

2. (a) What is meant by the **radius of convergence**? How is it related to the **interval of convergence** of a power series?

(b) What is the midpoint of the interval of convergence of the power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n?$$

(c) If $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence $R = 0$, then for what values of x is the series convergent?

(d) If instead $R = \infty$, then where is the power series convergent?

(e) If the Ratio Test tells you that $0 < R < \infty$, then what do you know about the convergence of the power series at the points $x = a \pm R$? List some possible intervals of convergence in this case.

3. If $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence $R > 0$, then what is the radius of convergence of the series $\sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$? How about $\left(\sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1}\right) + C$? Do you know automatically what the intervals of convergence are? Why or why not?

4. What is wrong with the following computation (where $x > 0$)?

$$1 = \frac{1-x}{1-x} = \frac{1}{1-x} + \frac{-x}{1-x} = \frac{1}{1-x} + \frac{1}{1-\frac{1}{x}}$$

$$= (1 + x + x^2 + \ldots) + (1 + \frac{1}{x} + \frac{1}{x^2} + \ldots) > 2$$

Problems

1. Calculate the radius of convergence for each power series.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} x^n$

   (b) $\displaystyle\sum_{n=1}^{\infty} x^n \ln(1 + \frac{1}{n})$

   (c) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2n-1}{3n+1}\right) \frac{x^n}{\pi^n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n} x^n$

2. (a) Show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent for all x.

   (b) What does this tell you about $\lim_{n\to\infty} \frac{x^n}{n!}$?

3. (a) Find the power series for each of the following functions by modifying the geometric series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots = \frac{1}{1-x}$.

      i. $\displaystyle\frac{1}{1+x}$

      ii. $\displaystyle\frac{1}{(1-x)^2}$. [Hint: derivative of...]

      iii. $\tan^{-1} x$. [Hint: integral of...]

      iv. $\displaystyle\frac{\ln(1+x)}{x}$

   (b) Now, again using the geometric series, identify the function which corresponds to the following power series.

      i. $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n}$ [Hint: Let $y = x^2$]

      ii. $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$

      iii. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

      iv. $\displaystyle\sum_{n=1}^{\infty} n x^{2n-1}$

   (c) What is the radius of convergence for each of the preceding power series?

4. Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ and find the intervals of convergence for f, f', and f''. What appears to be happening to the intervals of convergence as you take derivatives?

5. (a) Show that if $\lim_{n\to\infty} \sqrt[n]{|c_n|} = c$, then the radius of convergence of the power series $\sum c_n x^n$ is $R = 1/c$.
   (b) Suppose that the radius of convergence of the power series $\sum c_n x^n$ is R. What is the radius of convergence of the power series $\sum c_n x^{2n}$?

6. Construct a power series whose interval of convergence is $[-17, 17)$. Construct similar series whose intervals are $[-17, 17]$, $(-17, 17)$, and $(-17, 17]$.

Additional Problems

1. Term-by-term differentiation and integration works only for some very special kinds of series, like power series. For instance, let $f_n(x) = (\sin nx)/n^2$. Show that the series $\sum f_n(x)$ converges for all values of x but that the series of derivatives $\sum f_n'(x)$ diverges when $x = 2n\pi$, n an integer. For what values of x does the series $\sum f_n''(x)$ converge?

2. If infinitely many coefficients of a power series with radius of convergence R are nonzero integers, show that $R \leq 1$.

17. Taylor and Maclaurin Series

Questions

1. (a) If f has a power series expansion at a point a,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{for} \qquad |x-a| < R,$$

   what is the formula for the coefficients c_n?

   (b) What is meant by the "Taylor series of a function f at a point a"?

   (c) What is meant by a Maclaurin series? How is this different than a Taylor series expansion at 0?

2. Start with the polynomial $P(x) = 2 - 3x^2 + 5x^3$.

   (a) Compute $P^{(n)}(0)$ for $n = 1, 2, 3, 4, 5$.

   (b) Writing $P(x) = c_0 + c_1 x + c_2 x^2 + \cdots$ what is the connection between $P^{(n)}(0)$ and c_n?

   (c) Given any polynomial $Q(x)$, what is its Taylor series at $x = a$?

3. Estimating the remainder of an approximation can be difficult! We always want to find an upper bound M on the absolute value of some function f over an interval $[a, b]$, $|f(x)| \leq M$ for all x where $a \leq x \leq b$. Practice finding upper bounds on the following functions.

   It often helps to sketch the graph of the function.

   (a) e^x on $(-\infty, 0)$

   (b) $\sin x$ on $(-\pi/2, \pi/4]$

   (c) $\cos x - 1$ on $[0, 2\pi]$

   (d) $\dfrac{1}{x}$ on $(1, 10)$

   (e) $3e^x - 5e^{-x}$ on $[0, 1]$.

4. Why doesn't $x^{1/2}$ have a Taylor series representation about $x = 0$? [Hint: give geometric reasons.]

Problems

1. Determine the Maclaurin series for each the following functions, and find their radii of convergence. Before you start computing derivatives, see if you can manipulate known series to get what you want.

   (a) $\ln\left(\dfrac{1+x}{1-x}\right)$

   (b) $\sin^2(x)$

   (c) $\displaystyle\int \frac{\sin x}{x}dx$

   (d) $\ln\sqrt{1-x^2}$

   (e) $\sqrt{1+x}$

   (f) e^{-x^2}

2. (a) How would you show that a function $f(x)$ equals its Taylor series?

   (b) Find the Taylor series of $f(x) = \cos x$ at $a = \pi/3$, and prove that $f(x)$ is equal to this Taylor series for all x.

3. (a) What is the coefficient of x^{100} in the power series for e^{2x} about $x = 0$?

   (b) Evaluate $f^{(100)}(0)$ for the function f, where

$$f(x) = \begin{cases} \frac{1-\cos x}{x^2} & \text{for } x \neq 0 \\ \frac{1}{2} & \text{for } x = 0. \end{cases}$$

4. It is very important to know when a function f equals the sum of its Taylor series. In other words, we would like to know exactly when it is true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

   if f has derivatives of all orders.

   (a) Why is it necessary for f to have derivatives of all orders if we want to talk about the Taylor series of f?

   (b) Thinking in terms of convergent series, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ if $f(x)$ is the limit of the sequence of partial sums. Write down the formula for the partial sums $T_n(x)$ of a Taylor series. What is $T_n(x)$ called?

   (c) What is Taylor's formula for the remainder $R_n(x)$? Be sure to explain where all of the letters in the formula (like a, n, z, and x) come from.

   (d) $f(x)$ equals the sum of its Taylor series if, and only if, $f(x) = \lim_{n\to\infty} T_n(x)$. Show that if $\lim_{n\to\infty} R_n(x) = 0$, then $f(x) = \lim_{n\to\infty} T_n(x)$.

(e) If $\lim_{n\to\infty} R_n(x) = 0$ for all x such that $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$. When this happens, we say that f is **analytic** at a.

5. (a) What is Taylor's formula for the remainder term R_n?
   (b) Use Taylor's remainder formula to derive the following handy inequalities:
      i. $|\ln(1+x) - x| \le \frac{1}{2}x^2$ if $x \ge 0$
      ii. $|\sin x - x| \le \frac{1}{6}|x|^3$ for all x

6. (a) Find the Maclaurin series and intervals of convergence for each of the following functions.
      i. $\dfrac{1}{1-x}$
      ii. e^x
      iii. $\sin x$
      iv. $\cos x$
      v. $\tan^{-1} x$
   (b) Modify series from part (a) to derive Taylor series about $x = 0$ for the following functions.
      i. e^{-x^2}
      ii. $\cos\sqrt{x+1}$
      iii. $\sin^2 x$.
   (c) For each of the functions in part (b) find $f^{(73)}(0)$.

7. Find the functions whose Maclaurin expansion is as given.
   (a) $1 - x^2 - x^3 + x^5 + x^6 - x^8 - x^9 + x^{11} + x^{12} - x^{14} - x^{15} + \cdots$
   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$
   (c) $\displaystyle\sum_{n=1}^{\infty} nx^n$
   (d) $\displaystyle\sum_{n=1}^{\infty} nx^{2n-1}$
   (e) $4 + 5x + 6x^2 + 7x^3 + 8x^4 + 9x^5 + \cdots$ (and then compute $4 + \frac{5}{2} + \frac{6}{4} + \frac{7}{8} + \frac{8}{16} + \cdots$)
   (f) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{nx^{2n}}{(2n+1)!}$
   (g) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{3n+2}$ [Hint: What is the Maclaurin series for $\dfrac{x}{1-x^3}$?]

8. Find the Taylor polynomial $T_4(x)$ at $x = 0$ for the function $\ln(1 + e^x)$.

9. (a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = ?$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2^n} = ?$ (How can you do this? Try expanding $f(x) = \dfrac{x}{(1-x)^2}$ as a power series first.)

   (c) Compare your answers to (a) and (b). Are you surprised? Are you shocked? Are you correct?

   (d) Here is another way to compute $\sum_{n=0}^{\infty} \frac{n}{2^n}$.

      i. Show that the series converges absolutely. Conclude that rearranging terms will yield a series with the same sum.

      ii. Make a triangular grid of numbers with the n^{th} row containing n numbers (like Pascal's triangle), but with each entry in the n^{th} row being $\frac{1}{2^n}$.

      iii. By adding up the rows, show that the sum of all the entries in the triangle is $\sum_{n=0}^{\infty} \frac{n}{2^n}$.

      iv. By adding up the "columns", compute the value of the sum in (iii).

Additional Problems

1. (a) What is the Maclaurin series for $\sinh x$?

   (b) What is $\cosh x + \sinh x$?

   (c) Find the Maclaurin series for $\cosh x$ without calculating any derivatives.

2. Find

$$\lim_{x \to 0} \frac{(\sin x - x)^3}{x(1 - \cos x)^4}$$

   by approximating $\sin x$ and $\cos x$ with suitable Taylor polynomials, not by using l'Hospital's Rule.

3. An analytic function $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ has derivatives of all orders at the point a. However, simply because f has derivatives of all orders at a point a, does *not* mean that f is analytic at a.

   (a) Let $f(x) = \begin{cases} e^{-1/x^2} & \text{when } x \neq 0, \\ 0 & \text{when } x = 0 \end{cases}$

   Sketch the graph of $y = f(x)$.

   (b) Show that $f'(0) = 0$ by evaluating $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$.

   (c) It can be shown that $f^{(n)}(0) = 0$ for all $n \geq 0$, but don't worry about proving this. Instead, use this information to find the Taylor series for f about $x = 0$. What is its radius of convergence?

(d) For what values of x does $f(x)$ equal the sum of this series?

4. Show that within a certain interval of convergence,

$$\frac{\ln(1-x)}{x-1} = \sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) x^n.$$

What is the interval (not radius!) of convergence?

18. The Binomial Series

Questions

1. (a) Simplify: $\binom{7}{5}$, $7!/5!$, $\binom{12345}{1}$, $\binom{12345}{0}$, $\binom{12345}{12345}$, and $\binom{12345}{12344}$.
   (b) Simplify $\binom{\frac{1}{2}}{3}$.
   (c) Simplify $\binom{k}{1}$, $\binom{k}{k-1}$, $\binom{k}{k}$, and $\binom{k}{0}$.
   (d) What is 0! defined to be? Explain why this definition is reasonable.
   [Hint: Here's one reason. What do you need to multiply by to get from one 2! to 3!? How about from 1! to 2!?]
   (e) Suppose k and n are positive integers and $1 \leq n \leq k$. Express $\binom{k}{n}$ as a quotient of factorials.
   Does this formula work when $n = 0$?
   Does this formula work if k isn't an integer?
   (f) What does your formula from part (e) tell you about $\binom{k}{n}$ and $\binom{k}{k-n}$, if k and n are positive integers and $1 \leq n \leq k$?

2. (a) Find the fourth-degree Taylor polynomial at $x = 0$ for $f(x) = 5x^3 - 3x^2 + 4x - 7$.
   (b) For a polynomial of degree n, what can you say in general about its mth degree Taylor polynomial at $x = 0$?

Problems

1. The coefficients in the Binomial Theorem form a nice pattern called Pascal's Triangle. Shown here are the first five rows:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

   (a) Looking only at the rows of the triangle, figure out how one row depends on the row immediately above it. What is the pattern? Using this pattern, write down the next five rows of the triangle.
   (b) Multiply out $(1+x)^0$, $(1+x)^1$, $(1+x)^2$, $(1+x)^3$, and $(1+x)^4$. The Binomial Theorem tells us that in general $(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n$. Using the polynomials you've already calculated, show that $(1+x)^k$ has the same coefficients as the $(k+1)$'st row of Pascal's Triangle when $k = 0$, 1, 2, 3, and 4.

(c) Now use Pascal's Triangle to write out $(1+x)^5$, $(1+x)^6$, $(1+x)^7$, $(1+x)^8$, and $(1+x)^9$. Was that easier than using the Binomial Theorem?

2. Thinking of Pascal's Triangle, show algebraically that $\binom{k}{n} + \binom{k}{n+1} = \binom{k+1}{n+1}$ when k and n are positive integers, and $n + 1 \leq k$.

3. (a) If $P(x)$ is a polynomial of degree n and a is any number, show that

$$P(x) = P(a) + \frac{P'(a)}{1!}(x-a) + \cdots + \frac{P^{(n)}(a)}{n!}(x-a)^n.$$

   (b) Find the Taylor series expansion of the following functions, and then check your expansions using algebra.

      i. $3x^2 - 5x + 7$ in powers of $x - 1$

      ii. x^3 in powers of $x + 2$

4. Suppose that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for all x.

   (a) If f is an odd function, show that

$$c_0 = c_2 = c_4 = \cdots = 0$$

   [Hint: if $f(x)$ is odd what is $f'(x)$? How about when $f(x)$ is even?]

   (b) If f is an even function, show that

$$c_1 = c_3 = c_5 = \cdots = 0$$

5. If $f(x) = e^{x^2}$, show that $f^{(2n)}(0) = \frac{(2n)!}{n!}$.

Additional Problems

1. Use the following steps to prove that if k is any real number and $|x| < 1$, then

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

   (a) Let $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n$. Differentiate this series to show that

$$g'(x) = \frac{kg(x)}{1+x} \qquad -1 < x < 1$$

   (b) Let $h(x) = (1+x)^{-k} g(x)$ and show that $h'(x) = 0$.

   (c) Deduce that $g(x) = (1+x)^k$

2. Here you prove why $\binom{k}{n}$ is read as "k choose n". For example, if you have k friends, and you want to choose one of them to be your best friend, then you have $\binom{k}{1} = \frac{k}{1!} = k$ different possible choices for your best friend. Show that this interpretation makes sense for all n when $0 \leq n \leq k$. Start by assuming that $\binom{k}{n}$ is the number of was to choose n from k when $k < n$ to show inductively that $\binom{k}{n+1}$ is the number of ways to choose $n+1$ from k. Interpret the equality $\binom{k}{n} + \binom{k}{n+1} = \binom{k+1}{n+1}$ in terms of choosing $n+1$ of your closest $k+1$ friends to attend a party.
   [Hint: Consider the two cases: in the first case, you have already invited your best friend. In the second case, your best friend is unfortunately out of town and not able to come to the party.]

19. Applications of Taylor Polynomials

Questions

1. The error in approximating f by its first-degree Taylor polynomial at $x = a$, is $f(x) - [f(a) + f'(a)(x - a)]$. This error is approximately $\frac{1}{2}f''(a)(x - a)^2$ for x near a. Suppose this error is positive. Is the tangent line to the graph of f at $x = a$ then above or below the graph near $x = a$?

2. Why is it correct to say that $\sin x \approx x$ is a *quadratic* approximation of $\sin x$ near $x = 0$?

3. Sometimes it is not necessary to use the remainder formula when estimating with Taylor series. Using the Maclaurin series for $\sin x$, express $\int_0^1 \frac{\sin x}{x}\,dx$ as an alternating series. Approximate the integral with error at most $5 \cdot 10^{-7}$.

Problems

1. (a) Draw $1 + x$, $1 + x + \frac{1}{2}x^2$ and e^x on the same graph.
   (b) Prove that $e^x \geq 1 + x$ for all x. Is $e^x \geq 1 + x + \frac{1}{2}x^2$ for all x?
   (c) Use Taylor's remainder formula to compute an upper bound for the error in approximating e^x by $1 + x + \frac{1}{2}x^2$ (its second-degree Taylor polynomial at 0) on the interval $[-1, 1]$.
   (d) Now estimate this same error graphically. Try to do so *without* thinking about your previous answer.
   (e) How do you explain the discrepancy between your answers for parts (c) and (d)?
   (f) Approximate e with an error of at most 0.0001 by using a Taylor polynomial for e^x at $x = 0$.

2. Find an interval centered on $x = 0$ within which $\sin x$ can be approximated by $x - \frac{x^3}{3!}$ with four-decimal places of accuracy. Find a corresponding interval for approximating $\sin x$ by $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ with the same degree of accuracy.

3. (a) Consider the integral $\int_0^\infty \frac{xe^{-x}}{1-e^{-x}}\,dx$. It is a hard-to-prove fact that the antiderivative for the function $\frac{xe^{-x}}{1-e^{-x}}$ cannot be expressed in terms of familiar functions. Explain why this improper integral converges.
   [Hint: This question has two parts: Investigate what happens as $x \to 0$ and what happens as $x \to \infty$.]
   (b) Demonstrate that you haven't forgotten your skill in integral substitutions by making the substitution $u = 1 - e^{-x}$. What will you substitute for x?

(c) Continue working with the integral in u that you created in part (b). Use your knowledge of power series to write the integrand as a series. Integrate this power series to get a series of numbers which adds up to this integral.

(d) Approximately how many terms of this series would you need to add up to get a sum that is within 0.0001 of the true value of the integral?

Additional Problems

1. Use the following outline to prove that e is an irrational number.

   (a) If e were rational, then it would be of the form $e = p/q$, where p and q are positive integers and $q > 2$. Use Taylor's formula to write

   $$\frac{p}{q} = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{q!} + \frac{e^z}{(q+1)!} \text{ where } 0 < z < 1.$$

   (b) For simplicity, let $s_q = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{q!}$. Show that $q!(e - s_q)$ is an integer.

   (c) Show that $0 < q!(e - s_q) < 1$.

   (d) Use parts (b) and (c) to deduce that e is irrational.

20. Differential Equations

Questions

1. (a) In each of the following cases, give an example of a *non-zero* function which:
   i. Equals its derivative.
   ii. Equals its second derivative.
   iii. Equals its second derivative and is *not* a scalar multiple of the previous function. (A function f is a scalar multiple of a function g if there is a number c such that $f = cg$. For example, $(1/2)\sin x$ is a scalar multiple of $\sin x$.)
   iv. Equals the negative of its second derivative.
   v. Equals the negative of its second derivative and is *not* a multiple of the previous function.

   (b) For each function that you wrote down in part a, find a differential equation that the function satisfies.

   (c) For each function that you wrote down in part a, find numbers a, b, and c such that your function satisfies the differential equation $ay'' + by' + cy = 0$.

2. (a) Explain how a particular solution of a differential equation differs from a general solution.

   (b) How is solving an initial-value problem different than solving just a differential equation? What extra information do you have in an initial-value problem? How does that change the sort of answer you get?

Problems

1. For each differential equation, determine if the given function is a solution:

   (a) $y' = e^x + y; \quad y = xe^x$
   (b) $\frac{dy}{dx} = 1 + y^2; \quad y = \tan x$
   (c) $(y')^2 = 4 + y^2; \quad y = e^x - e^{-x}$
   (d) $y' = \frac{1}{e^y}; \quad y = \ln(x + c)$

2. (a) Find the general solution of $y\frac{dy}{dx} = x$.

   (b) In each of the following cases, find a particular solution which satisfies the given initial condition: $y(2) = 1$, $y(-1) = 2$, $y(-2) = 2$.

   (c) On the same axes, sketch the three particular solutions from part (b).

3. Consider the differential equation $\frac{dy}{dx} = \sqrt{1 - y^2}$.

   (a) Find the general solution of this differential equation.

   (b) Sketch a graph showing several representative curves from the family of solutions. Be sure to consider the *sign* of $\frac{dy}{dx}$ in plotting the solution curves. For solutions of this particular equation, should the curves be always increasing, always decreasing, or some of each?

   (c) Should solution curves from a first-order differential equation cross each other? If yours do, reconsider your answers to part (b).

4. Using separation of variables, solve $\frac{dy}{dx} = ky$ and sketch some of the solution curves for $k = 1$ and $k = -1$. Remember that, when solving a differential equation like $\frac{dy}{dx} = ky$, you have to treat the case $y = 0$ separately.

Additional Problems

1. Show that the functions $y = e^x$ and $y = \cos x$ can not be solutions to the same first-order equation $y' = F(x, y)$ on any interval containing the origin.

21. Differential Equations Related to the Aerodynamics of Baseballs

Introduction

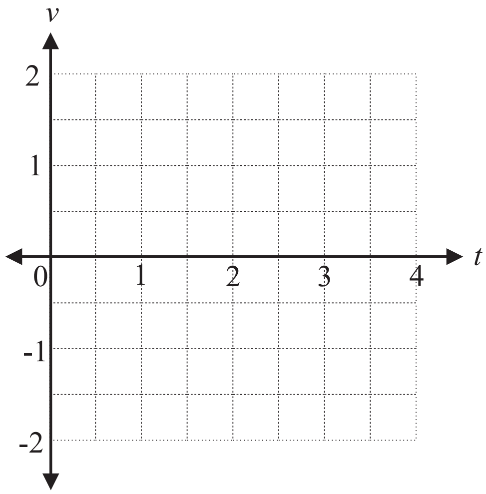
A great controversy developed during the 1996 Major League Baseball season. Batters were hitting more home runs than in previous seasons. Different explanations were given for this Power Surge of 1996. Were the batters suddenly stronger? Were the pitchers suddenly worse? Or, was it something to do with the baseball itself? If the baseball actually did fly through the air with greater ease, could this be enough to account for the Power Surge?

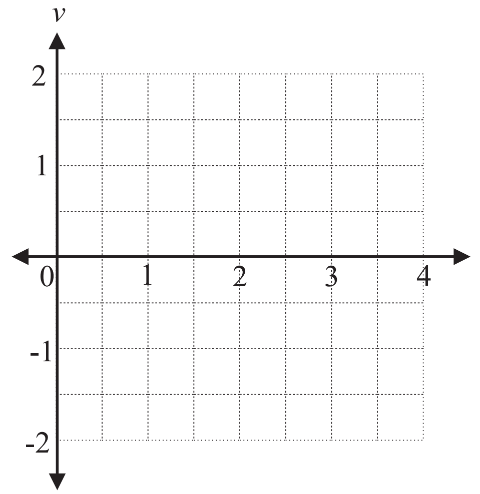
A reporter for the *Santa Rosa Press-Democrat* came to campus to discuss the question with Mechanical Engineering Professor George Johnson. To investigate the effect of aerodynamic drag on baseballs, Professor Johnson and two of his graduate students decided that dropping a 1996 baseball and a 1994 baseball from the Campanile would provide a reasonable way to compare them. The study of motion under the influence of gravitational and frictional forces, like dropping a baseball from the Campanile, is governed by differential equations. This worksheet explores some of the the mathematical aspects of this experiment.

Questions

1. (a) Suppose a dropped baseball of mass m is subject to a gravitational force mg, where g is the acceleration due to gravity. If this is the only force present as the baseball falls, its velocity is governed by the differential equation $dv/dt = g$. Use Newton's law of motion, $F = ma$, to explain why the velocity obeys this differential equation. What is the general solution of this differential equation? What is the particular solution for the initial condition $v = 0$ when $t = 0$?

   (b) Draw a direction field on the first grid below, and use it to give a qualitative description of the solutions of the differential equation $dv/dt = 1$. Using this direction field for v, what happens to the velocity of the ball if it falls for a really long time? Does the speed of the ball keep getting larger and larger? Is there some maximum speed that it does not exceed?





$$\frac{dv}{dt} = 1 \qquad\qquad \frac{dv}{dt} = 1 - v^2$$

In an effort to provide a more accurate model of the behavior of a falling baseball, we assume that the frictional force on a moving baseball is proportional to the square of its speed. (This is a debatable but reasonable assumption.) This assumption and Newton's law imply that v, the velocity of the baseball, satisfies the following equation:

$$\frac{dv}{dt} = g - bv^2$$

where g is the acceleration due to gravity and b is the coefficient of friction.

2. You can use direction fields to explore the solutions of a simpler related differential equation. Draw a direction field on the second grid above, and use it to give a qualitative description of the solutions of the differential equation $dv/dt = 1 - v^2$. How do these solutions behave as $t \to \infty$?

Problems

The acceleration due to gravity, g is known to be approximately 9.8 meters/sec ^{2}. The friction coefficient b was measured by experiment (it is roughly 0.005 meter^{-1}).

1. (a) Find the general solution to

$$\frac{dv}{dt} = g - bv^2.$$

(**Hint**: It is separable.)

(b) Then find the particular solution which satisfies the initial condition of $v = 0$ when $t = 0$.

Answer: $v = \sqrt{g/b}\left(\dfrac{e^{(2t\sqrt{bg})} - 1}{e^{(2t\sqrt{bg})} + 1}\right)$.

2. Find the limit of the velocity as $t \to \infty$. Is this consistent with what you found out using the direction field in Problem 2?

3. Using the result of Problem 1b, find the distance x (in meters) that the baseball falls after time t.

   Answer: $x = (1/b)\ln(e^{2t\sqrt{bg}} + 1) - t\sqrt{g/b}$.

4. Write down and solve the differential equation which models a pitched baseball. For simplicity, assume that the motion is horizontal, with friction being the only force. Determine how much a pitched ball slows down by the time it reaches the batter who is 18 meters ($\sim$ 60 feet) away. Do this for a very fast pitcher who releases the ball with an initial velocity of 45m/s ($\sim$ 100 mph). Compare this to a much slower one 30m/s ($\sim$ 65 mph)?

   (**Hint**: Make a substitution of the form $u = e^x$. By taking appropriate initial conditions you should be able to get rid of the constant of integration.)

Reference

Adair, Robert K. *The Physics of Baseball.* Harper Perennial. 1994.

22. Separable and Homogeneous Equations

Questions

1. (a) Write a general second-order differential equation with the solution $y = \sin x$.
   (b) Write a general second-order differential equation with the solution $y = \cos x$.
   (c) Are both or either of your differential equations satisfied by $y = 2\sin x + 5\cos x$? Explain why or why not. If not, find a differential equation that is satisfied by $y = \sin x$, $y = \cos x$, and $y = 2\sin x + 5\cos x$.

2. Some first-order differential equations have the form $\frac{dy}{dx} = F(x, y)$. If $F(x, y) = \frac{g(x)}{h(y)}$, what is the differential equation called and how can it be solved?

3. (a) Using the example

$$\frac{dy}{dx} = \frac{x^2 - xy + y^2}{x^2 - y^2} + \ln x - \ln y + \frac{x + y}{x + 2y}$$

   explain what it means for a first-order differential equation to be **homogeneous**.
   (b) Explain why $\dfrac{dy}{dx} = \dfrac{x^2 - xy + y^2}{x - y^2}$ is *not* homogeneous.
   WARNING: You will soon encounter another meaning of the word "homogeneous", and it is *very* important not to mix up the two meanings! You should be able to tell from the context which meaning is the right one for each situation. `:-(`

Problems

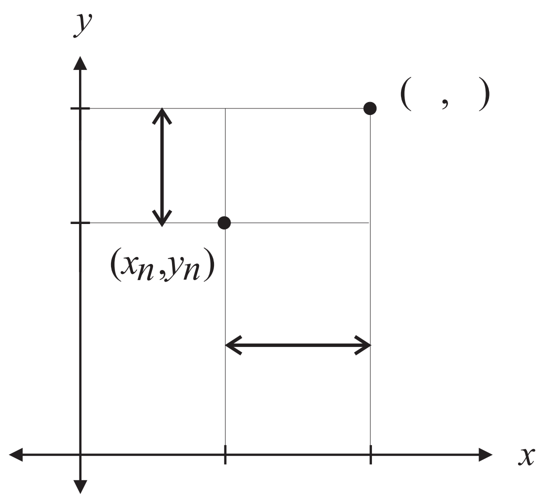
1. Solve each of the following differential equations by separating the variables.
   (a) $\dfrac{dy}{dx} = e^{x+y}(e^x + 1)^{-1}$
   (b) $\dfrac{dy}{dx} = \dfrac{y}{x^2 - 1}$
   (c) $\dfrac{dy}{dx} = (x + 1)^{-\frac{1}{3}}$
   (d) $\dfrac{dy}{dx} + xy = 0$

2. Find the orthogonal trajectories of each family of curves. Sketch some of the graphs.
   (a) $kx^2 + y^2 = 1$

(b) $y = \dfrac{k}{1+x^2}$

3. Euler's method can be used to approximate solutions of differential equations when finding an explicit solution is too difficult or impossible. This method is based on making a series of "corrections" to the tangent line approximation.

   (a) The general formula for Euler's method is $y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$. Where do x_0 and y_0 come from? What is h? What is F? How do you find x_n, given x_{n-1}? Label the missing coordinates and lengths on the following graph.



   (b) Use Euler's method with a step size of 0.25 to find approximate values for $y(1)$ when $y' = 5x - 3\sqrt{y}$, $y(0) = 2$. Be sure to make a table of your values.

Additional Problems

1. A curve in the plane has the property that a normal line to the curve at the point $P = (x, y)$ always passes through the point $(2, 0)$. Find and graph the equation of the curve if the curve passes through the point $(1, 1)$. [Hint: It helps greatly to choose the right form for the equations of the normal lines.]

2. Solve $\dfrac{dy}{dx} = ky(M - y)$, where k and M are some constants, and $y(0) = y_0$.

3. A droplet of liquid evaporates at a rate proportional to its surface area.

   (a) Explain how this yields the differential equation:

$$\frac{dV}{dt} = -KV^{-\frac{2}{3}}.$$

   (b) The value of K depends on the particular liquid and the atmospheric conditions, and can be determined experimentally using this equation. If you have a drop with initial volume $v_0 = 64$ of a chemical such that $K = 3$. How long will it take the drop to evaporate?

23. Competitive Cooling

Introduction

It's a hot day, and you just bought a can of iced tea at room temperature. Your favorite TV show is on in an hour, and you want the tea to be as cold as possible by then. Should you cool it in the freezer or in an ice bucket?

Comparing the two cooling methods involves a competition between two factors–the lower temperature of the air in the freezer against the more efficient heat conduction of the ice water in the bucket. In this worksheet, we will find and solve a differential equation to see which method wins.

To find the differential equation, we will equate the heat leaving the can to the heat entering the surrounding fluid (air or water). The rate of change of the amount of heat in the can is proportional to the rate of change of temperature, dT/dt, and to the mass of the can (including its contents). The mass is in turn equal to the product of the volume V and the average mass density ρ. The constant of proportionality, c, is called the *heat capacity* of the can of tea.

On the other hand, the rate at which heat enters the surrounding fluid is proportional to the surface area A of the can and to the temperature difference $T - T_\infty$, where T_∞ is the temperature of the fluid. (We assume that T_∞ is kept constant by the freezer mechanism, or by renewal of the ice in the water.) The constant of proportionality, h, is called the *convection coefficient* of the fluid.

Since all the heat leaving the can must enter the surrounding fluid, we get the differential equation

$$-\rho c V \frac{dT}{dt} = hA(T - T_\infty). \tag{1}$$

We will measure time in seconds, length in meters, mass in kilograms, and temperature in degrees Celsius. Heat is measured in units of energy called Joules. In these units, the values of some of our parameters are: $\rho = 1000\,\text{kg/m}^3$; $c = 4200\,\text{J/kg}^\circ\text{C}$; $h = 40\,\text{J/m}^2\text{s}^\circ\text{C}$ for freezer air; $h = 160\,\text{J/m}^2\text{s}^\circ\text{C}$ for ice water; $T_\infty = -20\,^\circ\text{C}$ for freezer air, $T_\infty = 0\,^\circ\text{C}$ for ice water.

The (12 ounce) can is cylindrical, 0.13 m tall and 0.065 m in diameter, and is originally at 30 °C.

Questions

1. Why is there a minus sign in the differential equation (1)?
2. Why do you think we used the notation T_∞ for the temperature of the fluid?
3. Which cooling method do you think will lead to a colder can of tea after an hour? Why?

4. On which of the parameters above do you think that the answer to Question 3 will depend?

5. Solve the differential equation $dx/dt = -x$, where x is a function of t with the initial value x_0 when $t = 0$. Sketch a graph of this solution.

Problems

1. What is the cooling rate when $t = 0$ for each environment (freezer air or ice water)? What is the cooling rate for each when the tea is at 10°C? What is it at 1°C ?

2. Solve Equation (1) for temperature as a function of time, with the initial temperature denoted by T_0. (Don't substitute any numbers for the parameters at this point.) [Hint: Solve the equation by separation of variables, or make a change of variables to reduce to the form in Question 5. Recall that $\frac{d}{dt}(z + \text{constant}) = \frac{d}{dt}z$.]

3. How cold will the tea be after an hour (3600 seconds) in the freezer? in the ice water?

4. If you put a can in the ice water and a can in the freezer at the same time, is there a time when the two cans have equal temperature again? Use the intermediate value theorem and a graph of both solutions.

Additional Problems

1. How long does it take to cool the iced tea to 10°C in the freezer? In the ice water? How about to cooling it to 1°C ?

2. How do the answers to the Problems above depend on the values of the parameters? For instance, if you had a glass bottle of juice instead, with a volume to surface area ratio of 0.30 m and a density of 1200 kg/m^3, would you have had a cold drink sooner? What if you had two hours instead of just one? Would these changes affect your choice of cooling method?

3. An important concept in business and banking is the time required for an investment of money to double. A common rule of thumb, called the "rule of 72," is that the doubling time in years is approximately 72 divided by the yearly interest rate r. When interest is compounded annually, money will grow according to the formula $A(t) = P(1 + \frac{r}{100})^t$, where P is the principal on deposit, r is the annual interest rate as a percentage, and t is time in years.

   (a) Use the formula for A given above to calculate the doubling time as a function of r.

   (b) Compare the exact time required and the value $72/r$ for the following values of r: 2%, 4%, 8%, 12%, and 18%.

(c) When interest is compounded continuously, then $A(t) = Pe^{\frac{r}{100}t}$. Find the exact doubling time in this case. Explain why a "rule of 69" is better than a "rule of 72" for doubling time in this case.

(d) Make a table showing doubling time in years when $r = 2\%$, 4%, 8%, 12%, and 18% when interest is compounded continuously. Compare the answer you get when using a "rule of 72," "rule of 69," and the exact doubling time. Why would business people use the "rule of 72" instead of the "rule of 69"?

24. First-Order Linear Differential Equations

Questions

1. (a) Show that $xy' = y$ is a separable, homogeneous, and linear differential equation by putting the differential equation in the appropriate forms.

   (b) Solve $xy' = y$ in each of the three ways. Did you get the same answer each time? Do they satisfy the differential equation?

2. An important aspect of solving differential equations is deciding which technique to use; so far, you have three different techniques to apply to first-order differential equations depending on whether the equation is separable, homogeneous, or linear. For each of the following differential equations, decide which type(s) it is, but *don't* solve the equation!

   (a) $yy' = x\sqrt{1+x^2}\sqrt{1+y^2}$

   (b) $y' + 2xy = 2x^3$

   (c) $1 + 2xy^2 + 2x^2yy' = 0$

   (d) $1 + y^2 - y'\sqrt{1-x^2} = 0$

   (e) $xy' - 2y = x^3$

   (f) $xy' = y + x\cos^2(y/x)$

   (g) $(x^2 + xy)y' = x^2 + y^2$

   (h) $y' = 2 + 2x^2 + y + x^2y$

   (i) $(2y - 3y^2)y' = x\cos x$

Problems

1. Each person in your group should go back and choose two (different) differential equations from the last Question. Make certain that you choose two equations of different types. Solve both equations and explain your methods to the group. Your group should discuss the solution to at least one separable equation, at least one linear equation, and at least one homogeneous equation.

2. This problem investigates the source of the mysterious "integrating factor."

   (a) Solve the differential equation $(\sin x)y' + (\cos x)y = 2x$ by recognizing the left-hand side as the derivative of some function.

(b) The equation $y' + p(x)y = q(x)$ is equivalent to the equation

$$\boxed{\texttt{:-)}} \cdot y' + \boxed{\texttt{:-)}} \cdot p(x)y = \boxed{\texttt{:-)}} \cdot q(x)$$

after multiplication by Smiley, the integrating factor. Show that the left-hand side of this equation is recognizable as a derivative *if* $\frac{d}{dx}\boxed{\texttt{:-)}} = \boxed{\texttt{:-)}} \cdot p(x)$.

(c) Show that the function $u(x) = e^{\int p(x)\,dx}$ satisfies the equation $u' = u \cdot p(x)$. Are there any other functions which are also solutions to the differential equation $u' = u \cdot p(x)$?

(d) Put these three pieces together. Why does the integrating factor allow you to solve first-order linear differential equations?

3. Solve the following equations, and discuss the behavior of the solutions as $x \to \infty$ for several different initial conditions.

   (a) $y' = x - y$

   (b) $y' = -xy$

   Briefly, sketch the direction fields of these ODE's graphing a few solutions. Compare the behavior of these solutions with the precise solutions you obtained above.

4. Psychologists in learning theory study **learning curves**, the graphs of the "performance function" $P = P(t)$ of someone learning a skill as a function of the training time t. If M represents the maximal level of performance, it is noted that for a certain skill the learning is at first rapid, and then it tapers off (the rate of learning decreases) as $P(t)$ approaches M.

   (a) Explain why solutions to the differential equation $\frac{dP}{dt} = k(M - P)$, k a positive constant, fulfill the above description of P. Sketch a typical learning curve.

   (b) Solve the differential equation under the initial condition $P(0) = P_0$ and sketch a graph of P.

   (c) Suppose that, for a specific learning activity, it is determined that $P_0 = .1M$ and $k = 0.05$ for t measured in hours. How long does it take to reach 90% of M, the maximal level of performance?

5. For an object of mass m freely falling towards the ground, the frictional force from air resistance is proportional to the object's speed. Suppose you drop an object with **coefficient of friction** k off a tall building.

   (a) Draw a free-body diagram for the falling object. There should be two forces acting on it.

   (b) Use Newton's Second Law to show that the height $y = y(t)$ of the object satisfies the differential equation

   $$my'' - ky' = -mg$$

   where g is the acceleration due to gravity.

(c) Rewrite the above differential equation as a first order differential equation for the velocity $v(t)$.

(d) Solve to find velocity as a function of time.

(e) Compute $\lim_{t\to\infty} v(t)$.

(f) Argue that if a building is tall enough, the object will reach the same terminal velocity regardless of whether you release it from rest or throw it downward as hard as you can.

(g) What is the terminal velocity of a 1 kg mass if $k = .2$?

(h) If you drop such a mass from a 60 meter tall building, will it reach terminal velocity before hitting the ground?

6. (a) Show that if $\Phi(x)$ is a solution of $y' + p(x)y = 0$, then so is $c \cdot \Phi(x)$ for any constant c.

   (b) Show that if $\Phi(x)$ and $\Psi(x)$ are a solutions of $y' + p(x)y = 0$, then so is $\Phi(x) + \Psi(x)$.

   (c) Show that $1/x$ satisfies the differential equation $y' + y^2 = 0$, but that c/x doesn't satisfy the differential equation, unless $c = 0$ or $c = 1$.

   (d) Why doesn't part (b) contradict part (a)?

7. Let $y' = 3x(y + x^n)$, where n is any integer.

   (a) For what value(s) of n is this a separable equation?

   (b) For what value(s) of n is this a linear equation?

25. Complex Numbers and Linear Independence

Questions

1. A complex number is often represented in the form $a + bi$, where a and b are real numbers and $i^2 = -1$. A complex number $a + bi$ can also be thought of as a point (a, b) in the complex plane.

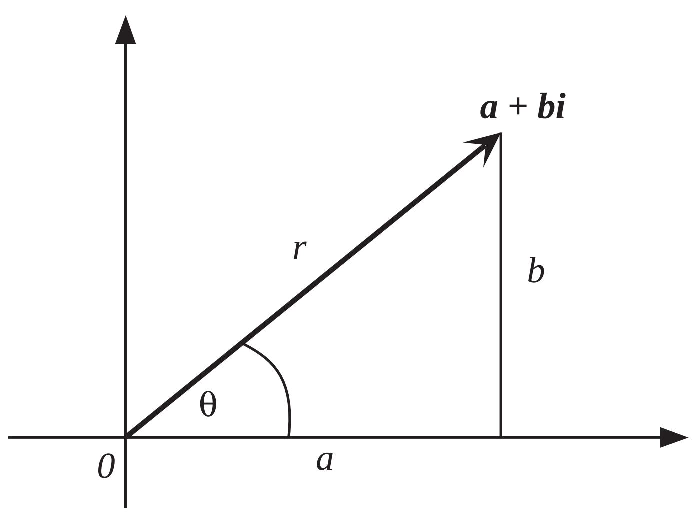
   (a) Sketch a coordinate axis. Which is the real axis? Which is the imaginary axis?

   (b) Graph and label the following points on your complex coordinate axis: 1, -1, i, $-2i$, $1 + i$, $2 - 2i$, $-\sqrt{3} + i$, $\frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

   (c) What is the **complex conjugate** $\bar{z}$ of a complex number $z = a + bi$? Give a geometric and an algebraic representation of $\bar{z}$. Write the algebraic representation and graph the geometric representation.

   (d) Graph the conjugates of the points on your complex plane. What is the distance from $a + bi$ to $\overline{a + bi}$?

2. Looking at the graph below, what is the relationship between a and b in rectangular coordinate and r and θ in polar coordinate?



   (a) Write $z = a + bi$ in terms of r and θ.

   (b) Write 1, -1, i, $-2i$, $1 + i$, $2 - 2i$, $-\sqrt{3} + i$, $\frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ in polar form.

3. (a) What does it mean for two functions $f(x)$ and $g(x)$ to be **linearly independent**?

   (b) If $f(x)$ and $g(x)$ are not linearly independent, then are they necessarily linearly dependent? Why or why not?

(c) Decide by inspection whether each of the following pairs of functions are linearly independent or linearly dependent.

i. e^{-t} and $2e^{-t}$

ii. e^{-t} and e^{-2t}

iii. e^{-t} and e^{t}

iv. e^{-t} and e^{1-t}

[Hint: In order to decide linear independence "by inspection," you might just look at the two functions $f(x)$ and $g(x)$. Or you could pick a point x_0 and show that $f(x_0) \neq c \cdot g(x_0)$ for all constants c. Or you might show that $f(x_0) = 0$ for some point x_0, but that $g(x_0) \neq 0$. How do these methods help you determine linear independence?]

Problems

1. (a) Evaluate:

$$i^2, \qquad i^3, \qquad i^4, \qquad i^5, \qquad i^{273}$$

(b) Write out the power series expansion for e^{ix} and simplify using (a).

(c) Separate the power series into its real and imaginary parts, assuming that x is real.

(d) Show that $e^{ix} = \cos x + i \sin x$, assuming that all the series converge (they do). This is called **Euler's Formula**, and will prove important whenever we work with complex numbers.

2. Consider the equation $x^2 + 6x + 13 = 0$.

(a) Solve it. Call the two solutions r and s.

(b) Compute r^2 and s^2.

(c) On one graph, plot r^2, $6r$, and 13 as vectors on the complex plane. Show geometrically that they add to 0.

(d) Do the same for s^2, $6s$, and 13.

(e) What does it mean to have a solution to $x^2 + 6x + 13 = 0$?

3. (a) Plot the four solutions to $x^4 - 1 = 0$.

(b) Plot the four solutions to $x^4 - 3 = 0$.

(c) Plot the four solutions to $x^4 + 2 = 0$.

(d) Plot the five solutions to $x^5 - 1 = 0$.

4. (a) Use the previous problem to find a geometric interpretation for taking the nth root of a complex number.

(b) Plot the six solutions to $x^6 + 8 = 0$.

(c) How many linear factors should $x^6 + 8$ have?

(d) Factor the polynomial $x^6 + 8$ into linear factors.

5. For most of the functions you will be working with, linear dependence or independence can be determined by inspection (just look at the functions). Sometimes however, we must be more careful. One method is to calculate the **Wronskian** of the two functions f and g.

$$W(f, g) = f \cdot g' - g \cdot f'$$

The functions f and g are linearly independent only if $W(f, g) \neq 0$.

(a) Use the Wronskian to show that e^{at} and e^{bt} are linearly independent if $a \neq b$, where a and b are real numbers.

(b) Show that e^{at} and te^{at} are linearly independent, where a is a real number.

(c) Show that $\cos at$ and $\sin at$ are linearly independent when a is a real number.

6. Test the following pairs of functions for linear dependence:

(a) $(t-1)^2$, t^2

(b) $t \sin t$, $t \cos t$

(c) $e^t \sin t$, $e^t \cos t$

Additional Problems

1. Use the $re^{i\theta}$ form to find a geometric interpretation of multiplication of complex numbers.

2. Prove the following properties of the conjugate of complex numbers:

(a) $\overline{z + w} = \overline{z} + \overline{w}$

(b) $\overline{zw} = \overline{z}\,\overline{w}$

(c) $\overline{(z^n)} = (\overline{z})^n$, where n is a positive integer.

3. Prove that if $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

This is called de Moivre's Theorem. [Hint: Use Euler's Formula.]

26. Second-Order Linear Equations

Questions

1. (a) What does it mean for a second-order linear differential equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

to be **homogeneous**?

(b) Is this the same meaning as the book uses when talking about first-order homogeneous equations? Why or why not?

2. (a) What does it mean for two functions y_1 and y_2 to be **linearly independent**? Give an example of two functions which are linearly independent, and an example of two functions which are linearly dependent.

(b) What does it mean for a function $f(x)$ to be a **general solution** of a differential equation? If $f(x)$ is the general solution to a first-order differential equation, then how many parameters (undetermined coefficients) does it contain? If $f(x)$ were the general solution to a second-order differential equation, how many parameters c_i would it have? In general, if $f(x)$ is the general solution to an nth-order differential equation, how many parameters should it have?

(c) If you have two linearly independent solutions y_1 and y_2 of a second-order linear differential equation $P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$, how can you come up with the general solution? What is it?

3. (a) If $y = e^{rx}$, then what are y' and y''?

(b) Substitute these expressions for y, y', and y'' into the differential equation $ay'' + by' + cy = 0$, where a, b, and c are some constants. What do you need to do to the resulting equation to get the characteristic equation of the differential equation that you started with? Why are you (mathematically) allowed to do this?

(c) If $y = e^{rx}$ is going to be a solution to the differential equation $ay'' + by' + cy = 0$, what *must* be true about r?

4. For second-order differential equations, explain how a boundary-value problem is different from an initial-value problem. In each case, what information are you given besides the differential equation?

5. Consider a violin string of length L that is fastened down at both ends. Imagine that the string vibrates in the (x, y) plane, and that its ends are fastened at $(0, 0)$ and $(L, 0)$. When at rest, the string lies along the x axis. When it is plucked or bowed, its ends

remain fastened, but the string is displaced, and its position, at some unspecified time, is described by a function $y = f(x)$.

When the string is vibrating and producing a pure tone, at any instant of time the second derivative of $f(x)$ with respect to x is proportional to $f(x)$, with a negative constant of proportionality k.

(a) Formulate a differential equation expressing this proportionality.

(b) Formulate additional equations, which express the requirement that the ends of the string are fastened. How many additional equations are needed?

(c) Is this an initial value problem? Explain why, or why not.

(d) One solution of this equation is simply $f(x) = 0$ for all x. Can you find other solutions? Does your ability to do so depend on the constant k?

6. In each part of this problem, $y = f(x)$ is the graph of some function whose second derivative $f''(x)$ is equal to $-f(x)$, for every x. Answer the following questions by thinking about how the graph must look, without using a formula for f.

   (a) Suppose that a graph $y = f(x)$ passes through the point (1,2). Is the graph concave up there, or concave down? Is $f'(x)$ increasing, or decreasing, when x is slightly larger than 1?

   (b) Explain why the graph is concave downwards at any point where $f(x)$ is negative.

   (c) Suppose for the remainder of this problem that $f'(1) = -3$. Sketch the graph, near $(x, y) = (1, 2)$, taking into account the information from parts (i) and (ii).

   (d) Must the graph cross the line $y = 0$ somewhere to the right of $x = 1$? Explain why. (Hint: Base your reasoning on concavity.)

   (e) Sketch a graph $y = g(x)$ that crosses the axis $y = 0$ at some point x_0, is strictly decreasing where $x \geq x_0$, is concave upwards, but approaches $-\infty$ as x approaches $+\infty$.

   (f) Would this be possible if $g'' = -g$? (Hint: If $g \to -\infty$, what happens to g''? How does this affect the graph of g?)

Problems

1. Consider the differential equation $ay''+by'+cy = 0$. For each of the following functions, find a, b, and c so that the differential equation is satisfied by the function(s). Don't just differentiate the functions! Instead, use what you know about the solutions to second-order linear differential equations to decide what values a, b, and c should have. Then differentiate the function(s) and substitute them into your differential equation to check that your values for a, b and c are correct.

   (a) $\cos 3x$

(b) e^{2x} and e^x

(c) $e^{2x}\cos 3x$

(d) xe^x

2. Find the general solution.

   (a) $y'' - y' + y = 0$

   (b) $y'' - y' = 0$

   (c) $y'' + 2y' + y = 0$

3. Solve the given initial or boundary value problems, and sketch each solution.

   (a) $y'' + 4y' + 6y = 0$, $y(0) = 2$, and $y'(0) = 4$

   (b) $y'' + 4y' + 5y = 0$, $y(0) = 1$, and $y'(0) = 0$

   (c) $y'' + 3y' - 10y = 0$, $y(0) = 1$, and $y'(0) = 3$

   (d) $y'' + 2\pi y' + \pi^2 y = 0$, $y(0) = 3$, and $y(1) = 3e^{-\pi} + 1$

4. Show that if y_1 and y_2 are solutions of $P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$, then so is the linear combination $y = c_1y_1 + c_2y_2$, where c_1 and c_2 are any constants.

5. (a) Before you studied second-order linear equations, how would you have solved the differential equation $y'' = 0$? What would the general solution have been?

   (b) Show that you get the same answer as before when you solve $y'' = 0$ using the characteristic equation.

6. (a) Find two different (linearly independent) functions f and g which equal the *negative* of their second derivatives.

   (b) How can you modify f and g to find a general solution to the differential equation $y'' + 4y = 0$?

   (c) Now, using the characteristic equation, find a solution of the form $y = c_1e^{r_1x} + c_2e^{r_2x}$.

   (d) Use Euler's formula to show that your answers from parts (b) and (c) are equivalent.

7. Find α such that f approaches 0 as $x \to \infty$, where f is the solution to the initial-value problem $y'' - y' - 2y = 0$, $y(0) = \alpha$, $y'(0) = 2$.

8. (a) Show that the boundary-value problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$ has only the trivial solution $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.

   (b) For the case $\lambda > 0$, find the values of λ for which this problem has a nontrivial solution and give the corresponding solution.

   (c) Does a second-order initial-value problem always have a solution? How about a boundary-value problem?

9. Consider the differential equation $y'' - 2y' + (1 - \varepsilon^2)y = 0$, for some real number ε.

   (a) Sketch a graph of the characteristic equation. What are its roots?

   (b) Find the specific solution to the differential equation satisfying $y(0) = 0$, $y'(0) = 1$.

   (c) What happens to the differential equation when $\varepsilon \to 0$?

   (d) What happens to the particular solution as $\varepsilon \to 0$? Be very careful when applying l'Hospital's Rule!

   (e) Use this method to find the general solution of

   $$y'' - 2y' + y = 0$$

Additional Problems

1. If r is a repeated root of $ax^2 + bx + c = 0$, show that e^{rx} and xe^{rx} are solutions of $ay'' + by' + cy = 0$.
   [Hint: Start by showing e^{rx} is a solution. Also, express r in terms of a, b, and c. How does that help you in showing that xe^{rx} is a solution?]

2. If the roots of the characteristic polynomial of a second order linear homogeneous differential equation with constant coefficients are $\alpha \pm i\beta$, then the general solution to the differential equation is

   $$C_1e^{\alpha x}(\cos \beta x + i \sin \beta x) + C_2e^{\alpha x}(\cos \beta x - i \sin \beta x).$$

   (a) Show that this solution is real if and only if C_1 and C_2 are complex conjugates of each other.

   (b) Can you find a function, other than a scalar multiple of e^x, that equals its third derivative? [Hint: your knowledge of complex numbers could be helpful.]

27. Nonhomogeneous Linear Equations

Questions

1. (a) What is the complementary equation to the differential equation $ay'' + by' + cy = G(x)$?

   (b) What do the three categories of complementary solutions $y_c = c_1y_1 + c_2y_2$ look like? How do the forms of y_1 and y_2 depend on the roots of the characteristic equation?

2. If f is a solution of $ay'' + by' + cy = G(x)$ and g is a solution of the complementary equation, show that $y = f + g$ is a solution of $ay'' + by' + cy = G(x)$.

3. Decide which of the following functions would work with the method of undetermined coefficients.

$$G(x) = e^{-3x} \quad G(x) = e^{x^2} \quad G(x) = x^2 \quad G(x) = \sqrt{x}$$
$$G(x) = \tan x \quad G(x) = \sin x \quad G(x) = \tfrac{\sin x}{x} \quad G(x) = \cos 3x - \sin 5x$$

$$G(x) = x^5 e^{-(3/2)x}\sin(2.7)x + x^3\cos 4x - 18 + e^{77x}$$

4. The first part of the method of undetermined coefficients is making a first guess about the form of y_p. You will need to make a separate guess for each term in the sum of $G(x)$, and then add those separate guesses together.

If $G(x)$ has a term	then for that term guess
$a_nx^n + \ldots + a_1x + a_0$	$A_nx^n + \ldots + A_1x + A_0$
$(a_nx^n + \ldots + a_1x + a_0)e^{rx}$	$(A_nx^n + \ldots + A_1x + A_0)e^{rx}$
$(a_nx^n + \ldots + a_1x + a_0)\cos\beta x$ $+(b_nx^n + \ldots + b_1x + b_0)\sin\beta x$	$(A_nx^n + \ldots + A_1x + A_0)\cos\beta x$ $+(B_nx^n + \ldots + B_1x + B_0)\sin\beta x$
$(a_nx^n + \ldots + a_1x + a_0)e^{\alpha x}\cos\beta x$ $+(b_nx^n + \ldots + b_1x + b_0)e^{\alpha x}\sin\beta x$	$(A_nx^n + \ldots + A_1x + A_0)e^{\alpha x}\cos\beta x$ $+(B_nx^n + \ldots + B_1x + B_0)e^{\alpha x}\sin\beta x$

In the table above, a_i and b_i denote known numbers (since they are parts of $G(x)$) while A_i and B_i denote unknown numbers which you will eventually need to calculate.

WARNING: if your guess is already a solution to the complementary equation you must multiply your guess by x. If it's still a solution of the complementary equation, you must multiply by x again.

Practice making your first guess on the following functions:

If $G(x) =$	guess $y_p =$
e^x	
x	
x^3	
x^3e^{-2x}	
$\sin 3x$	
$x^2 \sin x$	
$x^3 + xe^x$	
$xe^{-x} \cos 2x$	
$e^x \sin x - 3e^x \cos x$	
$5 - 2\sin x - 3e^{4x} + \cos x$	

Problems

1. (a) To see why you might have to multiply by a factor of x, try to solve the differential equation $y'' - y' - 2y = e^{-x}$ using your first-try guess (according to the table in this worksheet). What happens when you try to determine the coefficients?
   (b) Refine your guess and find the general solution to $y'' - y' - 2y = e^{-x}$.
   (c) Occasionally, multiplying by x will not even be enough! Solve $y'' - 8y' + 16y = e^{4x}$ by making better and better guesses. Why was multiplying your first guess by x not good enough?

2. Each person in your group should choose a different differential equation from the following list and find the general or particular solutions.
   (a) $y'' + y' - 2y = 2x$, $y(0) = 0$, $y'(0) = 1$
   (b) $2y'' - 4y' - 6y = 3e^{2x}$
   (c) $y'' + 4y = x^2 + 3e^x$, $y(0) = 0$, $y'(0) = 2$
   (d) $y'' + 2y' = 3 + 4\sin 2x$
   (e) $y'' + y = 3\sin 2x + x\cos 2x$
   (f) $y'' + 2y' + y = e^x \cos x$

3. (a) Once in a while (especially on exams) you will be faced with a differential equation $ay'' + by' + cy = G(x)$ where $G(x)$ won't work with the method of undetermined coefficients. In this case, you should use the **method of variation of parameters**.

i. Start by *guessing* that $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$. Why is this method called "variation of parameters"?

ii. Differentiate y_p.

iii. We don't want to have to deal with the second derivatives of u_1 and u_2, so we *assume* $u_1'y_1 + u_2'y_2 = 0$. How does this simplify y_p'?

iv. Differentiate the simplified form of y_p' to get y_p''.

v. Substitute y_p, y_p', and y_p'' into $ay'' + by' + cy = G(x)$, and simplify as much as possible. What equation are you left with?

vi. You have two unknown functions u_1' and u_2'. What two equations relating these functions do you have?

vii. By solving the system for u_1' and u_2' and integrating the results, show that:

$$u_1 = \frac{-1}{a}\int \frac{G(x)y_2}{y_1y_2' - y_1'y_2}\,dx, \qquad u_2 = \frac{1}{a}\int \frac{G(x)y_1}{y_1y_2' - y_1'y_2}\,dx.$$

This is how the method of variation of parameters yields a particular solution.

(b) Use the method of variation of parameters to solve the differential equation $y'' - 2y' + y = e^x/(1+x^2)$. Why wouldn't the method of undetermined coefficients work?

4. In this problem we will explore the method of variation of parameters for equations of the form $y' + p(x)y = q(x)$.

(a) Given $p(x)$, let $\Phi(x) = e^{-\int p(x)\,dx}$. Show that the general solution to $y' + p(x)y = 0$ is $f(x) = C\Phi(x)$.

(b) The constant C above is called a **parameter**. The set of particular solutions to the differential equation is **parametrized** by C.

(c) Suppose you don't know about integrating factors, and you wish to solve the equation $y' + p(x)y = q(x)$. Try replacing the parameter C with a function $C(x)$ and show that $C(x)\Phi(x)$ will satisfy the inhomogeneous equation $y' + p(x)y = q(x)$ if $C(x)$ satisfies

$$C'(x) = \frac{q(x)}{\Phi(x)} = q(x)e^{\int p(x)}\,dx.$$

(d) Show that the general solution to $y' + p(x)y = q(x)$ is $f(x) = \Phi(x)\int \frac{q(x)}{\Phi(x)}\,dx$. The idea of replacing the constant parameter C by the variable parameter $C(x)$ is called **variation of parameters**.

(e) Solve the equation $y' - \frac{2x}{x^2+1}y = 1$ using the method of variation of parameters.

(f) How many arbitrary constants should there be in the general solution of a first order differential equation? How many do you see in the solution in (d) (look closely)? Explain the discrepancy.

Additional Problems

1. (a) Show that if a, b, and c are positive constants, then all solutions of $ay''+by'+cy=0$ approach 0 as $x \to \infty$.

   (b) If $a > 0$ and $c > 0$, but $b = 0$, show that the previous result is no longer true, but that all solutions are bounded as $x \to \infty$.

   (c) If $a > 0$ and $b > 0$ but $c = 0$, show that all solutions approach a constant as $x \to \infty$. Determine this constant for the initial-value problem $y(0) = y_0$, $y'(0) = y'_0$.

28. Applications of Second-Order Differential Equations

Questions

1. The differential equation for the simple harmonic motion of a spring is

$$m\frac{d^2x}{dt^2} + kx = 0.$$

   (a) Explain in words how this differential equation is obtained from Newton's Second Law, $F = ma$. Draw and label a picture to illustrate your explanation. What does m refer to? How about k, $\frac{d^2x}{dt^2}$, and x?

   (b) What does $-kx$ represent, where k is a positive constant?

   (c) Solve the differential equation using the notation $\omega = \sqrt{k/m}$.

   (d) Verify that the solution can also be written as $x(t) = A\cos(\omega t + \delta)$, where

$$\begin{aligned} \omega &= \sqrt{\frac{k}{m}} \quad \text{(the frequency)}, \\ A &= \sqrt{c_1^2 + c_2^2} \quad \text{(the amplitude)}, \\ \cos\delta &= \frac{c_1}{A}, \quad \sin\delta = -\frac{c_2}{A} \quad (\delta \text{ is the phase angle}). \end{aligned}$$

2. This model of vibrating springs is a major oversimplification since it ignores all external forces, like damping. We can modify the equation to include the forces as follows:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.$$

   Explain in words what this modified differential equation means, and again draw a picture to illustrate your explanation. What do c and $\frac{dx}{dt}$ refer to? How are m, k, $\frac{d^2x}{dt^2}$, and x different than in the first equation?

Problems

1. Consider the differential equation

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.$$

   (a) In each of the three following cases, find the general solution to the differential equation $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$, sketch and describe the graph of a solution curve, and explain why the case deserves the name it has.

i. Over-damping: $c^2 - 4mk > 0$

ii. Critical damping: $c^2 - 4mk = 0$

iii. Under-damping: $c^2 - 4mk < 0$

(b) How do external forces affect the physical system described by the differential equation? What impact do the external forces have on the algebraic solutions to the differential equation, in comparison to the undamped case?

(c) If the differential equation $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$ is describing the shock absorbers in your car, which kind of damping would you like best? Worst? Explain your answers.

2. If a series circuit has a $C = 0.8 \times 10^{-6}$ Farad capacitor and a $L = 0.2$ Henry inductor, find the resistance R so that the circuit is critically damped.

3. An external force can also be added to the spring system to give it a "kick," yielding the nonhomogeneous equation $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$.

(a) Suppose that the damping constant is so small that the damping force is negligible. If an external force of $F(t) = F_0 \cos \omega_0 t$ is applied, where $\omega_0 \neq \omega$, use the method of undetermined coefficients to show that the motion of the mass is described by

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t.$$

(b) Again, suppose that the damping constant $c = 0$. If an external force $F(t) = F_0 \cos \omega t$ is applied (so that the applied frequency equals the natural frequency), use the method of undetermined coefficients to show that

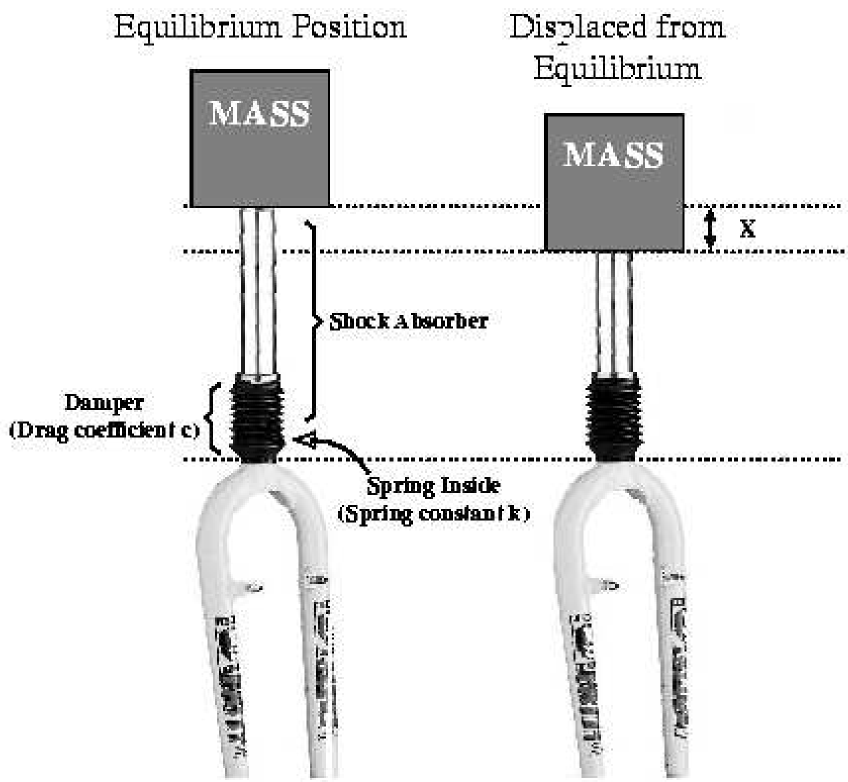
$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{2m\omega} t \sin \omega t$$

29. Oscillations of Shock Absorbers

Introduction

The differential equations for damped oscillations model the shock absorber found on any automobile, or even on a high-end mountain bike.

A shock absorber is essentially a spring and a damper. (See the figure below.) The spring cushions the shock and provides a restoring force $F_{\text{spring}} = -kx$ when it is stretched or squeezed by an amount x from its "neutral" position. (The proportionality coefficient k is called the *spring constant*.) The damper uses the viscosity of oil in a sealed container to produce a drag force which keeps the bike from bouncing up and down too much: $F_{\text{damper}} = -cv$, where c is the *drag coefficient* and $v = dx/dt$ is the (vertical) velocity of the effective mass m; i.e. the portion of the mass of the bike and rider supported by the front wheel.



Newton's law ($F_{\text{total}} = ma$) gives the equation:

$$-kx - cv = m\frac{dv}{dt},$$

which leads to the second order differential equation

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \qquad (2)$$

where m and k are positive and c is non-negative. This differential equation has the following characteristic equation:

$$\omega^2 + \frac{c}{m}\omega + \frac{k}{m} = 0 \tag{3}$$

Questions

1. Make an educated guess of the general form of the solution to Equation (2).

2. Derive Equation (3) from (2).

3. Knowing the general form of $x(t)$, write down the general form of $v(t)$?

Problems

The following problems will familiarize you with the behavior of mass-damper-spring systems.

1. Solve the characteristic equation (3), and find the possible values of ω.

2. Write the general solution of Equation (2) using the roots of the characteristic equation, which you just found. The values of $\frac{k}{m}$ and $\frac{c}{m}$ will determine whether the the characteristic equation has real or complex roots. If you have complex roots, you should use Euler's formula: $e^{p+iq} = e^p(\cos q + i \sin q)$ to rewrite the complex exponentials in terms of sines and cosines.

3. Can the real part of either root of Equation (3) be positive, given that m, c and k are positive? (Consider both cases: One for real roots and another for complex roots.) Can x or v grow without bound as t increases? What is the meaning of this result in terms of a bouncing bicycle?

4. Illustrate the effect of the relationship between the stiffness k/m and the damping c/m by drawing a figure in the first quadrant of the $(c/m, k/m)$ plane which shows the types of solutions you get in different regions (e.g. "x and v oscillate but their magnitudes decrease with time"). Which values of c/m and k/m give solutions which do not oscillate?

Assume now that you and your bike have a mass of 100 kilograms, of which 40% is supported by the front wheel, and that your shock absorber has a rubber spring (k = 4,000 Newton/meter) and uses oil in the damper.

5. Suppose that you forget to fill the damper with oil, so that $c = 0$, and that you hit a bump that gives you a vertical velocity of 0.2 meters/second at a moment when $x = 0$. Find the particular solution of Equation (2) with these initial conditions, draw the phase portrait, and describe the vertical motion.

6. Now you put oil (which has a drag coefficient $c = 500$ Newton-second/meter) in the damper, and you hit the same bump. Find and describe the motion, and draw the phase portrait. How does this differ from the undamped case? Is this amount of damping enough to prevent oscillations?

7. Why don't you want c to be too small or too large?

Additional Problems

1. Now you really want to tweak your bike. You look at some books on shock absorbers, and they say that the best you can do is to 'critically damp' the thing. This means that the damping is just enough to prevent oscillations; mathematically, critical damping occurs at the transition between real and complex characteristic roots–on the boundary between the regions in Problem 4.

   (a) Given the effective mass m and the spring constant k, find the drag coefficient c which you'll need to critically damp your shock. (Hint: look at the discriminant of the quadratic equation you used to find the characteristics roots ω_1 and ω_2.)

   (b) At critical damping there is only one distinct root of Equation (3), i.e. $\omega_1 = \omega_2$, so there is another fundamental solution of the form

   $$ae^{\omega t} + bte^{\omega t}$$

   Use this to find the solution with the initial conditions produced by hitting a bump, as in Problems 5 and 6, draw the phase portrait, and describe the motion.

2. Here is another way to understand the effects of damping. Take the solution $x(t)$ and $v(t)$, which you found in Problems 5 and 6, and express the kinetic energy $K = \frac{1}{2}mv^2$ and potential energy $P = \frac{1}{2}kx^2$ as functions of time. Plot (K, P) as a parametric curve (like the phase portraits you drew before). Look at the difference between the plot for the undamped versus the damped shock absorber. What does the damper do to the total energy. Does the law of conservation of energy apply here?

30. Series Solutions

Questions

1. (a) Is the series $\sum_{n=2}^{10} \frac{1}{n-1}$ the same as the series $\sum_{n=0}^{8} \frac{1}{n+1}$?

   (b) We can also tell that two series are the same by manipulating the indices. Remember that it doesn't matter if we use n or i (or j or k or ...) as our indexing variable. If you don't remember why this is true, decide whether $\sum_{n=1}^{20} \frac{1}{n}$ and $\sum_{i=1}^{20} \frac{1}{i}$ are the same series, as you did in part (b).

   Now, let's show that $\sum_{n=2}^{10} \frac{1}{n-1}$ is the same series as $\sum_{n=0}^{8} \frac{1}{n+1}$ by manipulating indices.

   i. In the first sum $\sum_{n=2}^{10} \frac{1}{n-1}$, we can replace n by $i+2$ to get $\sum_{i+2=2}^{10} \frac{1}{i+2-1}$. Except for substituting $i+2$ for n *everywhere*, did we change anything else about the sum? Why did we choose $i+2$?

   ii. Now, algebraically $\frac{1}{i+2-1} = \frac{1}{i+1}$. Also, if $i+2$ goes from 2 to 10, then i must go from 0 to 8. Convince everyone in your group that this is true. Do you have a better idea now why $i+2$ was chosen?

   iii. So $\sum_{n=2}^{10} \frac{1}{n-1} = \sum_{i+2=2}^{10} \frac{1}{i+2-1} = \sum_{i=0}^{8} \frac{1}{i+1}$ which is just the same as $\sum_{n=0}^{8} \frac{1}{n+1}$

2. Now it's your turn; manipulate the indices of the following series so that the expression inside the summand contains only x^n.

   (a) $\sum_{n=0}^{\infty} \frac{(-1)^{n+2}}{2^{n+1}} x^{n+3}$

   (b) $\sum_{n=1}^{100} n x^{n-1}$

   (c) $\sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2}$

3. Let $y = f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$.

   (a) Find y' and y'', and express these derivatives as power series with only x^n in the expression.

   (b) The crucial step in finding series solutions is called "equating coefficients." In the equation $y'' - y = 0$, does the "0" on the right-hand side represent the number zero or the zero function? Explain the difference between these two kinds of zeros.

   (c) What is the Taylor series expansion about $x = 0$ of the zero function?

   (d) When are two Taylor series equal? If $\sum a_n x^n = 0$, then what can you say about the a_n?

4. Explain why $\sum_{n=1}^{\infty} 2nc_n x^n = \sum_{n=0}^{\infty} 2nc_n x^n$.

Problems

1. Power series may be added term-by-term, as long as you are very careful to match up exponents and indices. Let $y = \sum a_n x^n$. Substitute y and its derivatives into the following expressions, and write the expression as one summand.

   (a) $y' - 6y$

   (b) $xy'' - y$

   (c) $y'' + xy' + y$

2. Power series solutions usually lead to a recursion formula for the coefficients a_n.

   (a) To find the pattern, you will need to keep writing out terms until it becomes clear. Practice by expressing the coefficients defined by each of the following recursion formulas in terms of the given quantities.

      i. $(n+1)a_{n+1} = a_n$. Write a_n in terms of a_0.

      ii. $c_{n+2} = -\frac{c_n}{(n+1)(n+2)}$. Write c_n in terms of c_0 or c_1.

      iii. $n^2 a_n + a_{n-2} = 0$ and $a_1 = 0$. Write a_n in terms of a_0 and 0.

   (b) Explain in words why a formula like $c_{n+2} = -\frac{c_n}{(n+1)(n+2)}$ is called a recursion formula. (Merriam-Webster Dictionary defines "recursion" as the determination of a succession of elements, as numbers or functions, by operation on one or more preceding elements according to a rule or formula involving a finite number of steps.) What is recursive about it?

3. Solve $y' = y$ using power series. Did you get the right answer?

4. Use power series to solve the initial-value problem $y'' + xy' + y = 0$, $y(0) = 0$, $y'(0) = 1$.

5. Verify that the function

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 \cdot 3 \cdot \ldots \cdot (2n+1)} = x + \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} + \ldots$$

   is a solution to the equation $y'' - xy' - y = 0$. Where does the series converge?

Additional Problems

1. The second-order differential equations considered in the text and preceding problems are all linear, which means essentially that the dependent variable y and its derivatives occur only to the first power. The equation

$$y' = 1 + y^2$$

is nonlinear, and it is easy to see directly that $y = \tan x$ is the particular solution for which $y(0) = 0$. Show that

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$$

by assuming a solution for the differential equation in the form of a power series $\sum a_n x^n$ exists, and finding the a_n's in two ways:

(a) by the regular power series solution method. (Note how the nonlinearity of the equation complicates the formulas.)

(b) by differentiating the differential equation repeatedly to obtain

$$y'' = 2yy'; \qquad y''' = 2yy'' + 2(y')^2; \qquad \dots$$

and using the formula $a_n = f^{(n)}(0)/n!$.

Review Problems

Introduction

Below you will find a compilation of some extra problems which you may find useful in reviewing for your quizzes and exams.

There are four sections of problems. Within each section, the problems are organized randomly. Some problems are especially challenging so don't despair if you find yourself stumped by a portion. The complete solutions appear after the four sections of problems.

Integrals and related problems

1. Determine whether $\int_0^2 \frac{dx}{4x-5}$ is improper. If improper either evaluate, or prove that the integral is divergent.

2. True or False: $\int_{-4}^{4} \frac{dx}{5x^{1/3}-4}$ converges by comparison to $\int_{-4}^{4} \frac{dx}{x^{\frac{1}{3}}}$.

3. Find the arc length of $y = \cosh x$ on the interval $0 \le x \le 1$.

4. Evaluate the integral: $\int \tan^4 x\, dx$.

5. Evaluate the integral: $\int \sqrt{2x - x^2}\, dx$.

6. (a) Evaluate the integral $\int \frac{dx}{x^2(x+2)}$.

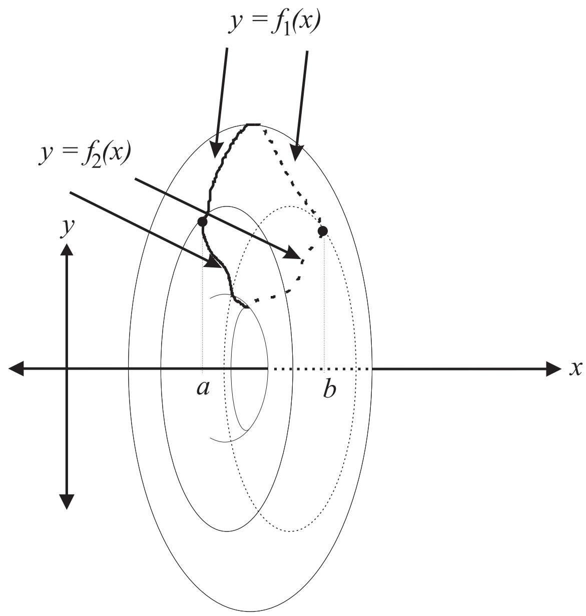
   (b) Evaluate $\int_1^{\infty} \frac{dx}{x^2(x+2)}$ or show that it is divergent.

7. Find the length of the curve defined by:

$$y = \frac{1}{x^2}, \quad 0 < x \le 1.$$

8. Find the partial fractions decomposition (including the values of A, B, etc.) of $\frac{x^3+2x}{x^3+1}$.

9. Below is pictured a surface of revolution generated by rotating the curves $y = f_1(x)$ and $y = f_2(x)$ around the x-axis.



Find a formula for the surface area of this surface, involving $f_1(x), f_2(x)$, and their derivatives.

10. Integrate: $\displaystyle\int \frac{\sin x}{\cos^{101} x}\, dx$

11. Which of the curves below has *both* of the following properties:

- its length is *infinite*.
- the area beneath it and above the x-axis is *finite*.

(a) $y = \dfrac{1}{\sqrt{x} \cdot |\ln x|}, \quad 0 < x \leq e^{-1}.$

(b) $y = \dfrac{1}{|\ln x|}, \quad 1 < x < \infty.$

(c) $y = \dfrac{1}{\sqrt{x}}, \quad e^{-1} < x \leq 1.$

(d) $y = \dfrac{1}{x^2}, \quad 0 < x \leq e^{-1}.$

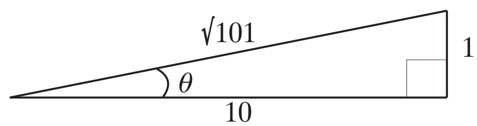
(e) None of the above.

Sequences and Series

1. Does the following series converge or diverge. (Justify.)

$$\sum_{n=0}^{\infty} \frac{(1,000,000)^n}{n!}$$

2. Below you may use the formula $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$. Consider the following figure:



   (a) Using the above, find an expression for θ in terms of an infinite sum.

   (b) Find an approximation for θ with an error less than 0.00001. (You don't need to simplify any fractions that you may have, and needn't express your answer using decimals.)

3. Find a power series representation for each of the following. State the *radius* of convergence in each case.

   (a) $-\ln(1-x)$

   (b) $\ln(1+x)$

   (c) $\ln\left(\frac{1+x}{1-x}\right)$

4. Find the Maclaurin series for $\ln(1-x^3)$. What is the corresponding Taylor polynomial, $T_7(x)$ about $x = 0$?

5. Give an example of each of the following. (No explanation required.)

   (a) An infinite sum whose convergence can be decided by the ratio test.

   (b) An infinite sum whose convergence can be decided by the root test.

   (c) A sequence that is bounded above but diverges.

6. (a) Does $\int_{e^2}^{\infty} \frac{dx}{x(\ln x)^{1.5}}$ converge or diverge?

   (b) Does $\sum_{n=100{,}000}^{\infty} \frac{1}{n(\ln n)^{1.5}}$ converge or diverge?

7. Determine whether or not the given sum converges. Find its value if it does. (Justify)

   (a) $\sum_{n=1}^{\infty} \left[\sin\left(\frac{n+1}{n}\right) - \sin\left(\frac{n+2}{n+1}\right)\right]$

   (b) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

8. For (a) and (b) determine whether the series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, absolutely convergent, or divergent.

   (a) $a_n = \dfrac{n^2 - n + 2}{\sqrt[4]{n^{10} + n^5 + 3}}$

   (b) $a_n = (-1)^n \dfrac{1 + e^{-n}}{n}$

9. Find the Taylor series of the function $f(x) = \dfrac{1}{\sqrt{x}}$ about the point $a = 1$.

10. Determine whether each of the following diverges or converges. (Justify.)

    (a) $.9 - .99 + .999 - .9999 + .99999 - .999999 + \cdots$

    (b) $\dfrac{1}{2^2} + \dfrac{2}{3^2} + \dfrac{3}{4^2} + \cdots$

    (c) $\sum_{n=1}^{\infty} (\dfrac{n^2 + 2n}{n^3 + 1} - \dfrac{1}{2})^n$

11. Find the radius of convergence for:

$$\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} 3^{3n} x^n.$$

12. For (a) and (b) determine if the sequence $\{a_n\}$ converges. If it does, find the limit.

    (a) $a_n = \dfrac{1}{n^{-\ln n}}$

    (b) $a_n = \sqrt{\dfrac{(1+n)n}{\sin n + n^2}}$

13. Obtain the Taylor series of $1 - \sin^2 x$ about $x = 0$.
    [Hint: trigonometric identities.]

Differential Equations

1. Find the general solution of the ODE:

$$y' = \cos^2 y \cdot \ln x$$

2. Solve $y' + \cos x \cdot y = \sin x \cdot \cos x$.

3. Find the general solution of $y' = \dfrac{y}{x} + 2$.

4. Sketch a direction field for $\dfrac{dy}{dt} = \dfrac{y}{t}$. Then for each initial condition below, graph a solution curve for $t \geq 1$ on the direction field which satisfies the condition.

   (a) $y(1) = 0$.

   (b) $y(1) = 1$.

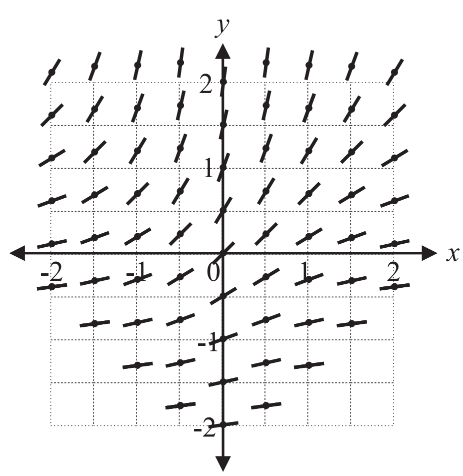
5. True or false: The families of curves $x = ky^2$ and $\frac{1}{2}x^2 + y^2 = c$ (c and k are constants) are orthogonal trajectories. (Justify.)

6. Solve:
$$y'' - 2y' - 3y = 0; \quad y(0) = 3, \quad y'(0) = 1.$$

7. Solve:
$$y'' - 2y' + 5y = 0.$$

8. Solve:
$$y'' - 6y' + 9y = 0; \quad y(0) = 1, y(1) = e^4 + e^3.$$

9. Below is pictured a direction field for a differential equation of the form
$$y' = f(x, y).$$



Which of the following best describes the function $f(x, y)$?

   (a) $\dfrac{y}{x} + e^{\frac{y}{x}}$

   (b) $\dfrac{e^y}{1 + x^2}$

   (c) $\csc x$

   (d) $\sin y$

10. Consider the linear differential equation

$$y' \cdot \cos x = y \cdot \sin x + e^x \cos x$$

   (a) Which of the following is an *integrating factor* for the differential equation:
      i. $I(x) = e^{\int e^x \cos x \; dx}$
      ii. $I(x) = \sin x$
      iii. $I(x) = e^{-\ln|\cos x|}$
      iv. $I(x) = e^{\cos x}$
      v. $I(x) = \cos x$
      vi. $I(x) = e^{-\cos x}$
      vii. None of the above.
   (b) Find the general solution to the above differential equation.

11. Find the solution of $\dfrac{d^2y}{dx^2} = xy; \quad y(0) = 1, \quad \dfrac{dy}{dx}(0) = 0.$

Complex Numbers

1. Let $z = 1 - i\sqrt{3}$.
   (a) Find $|z|$.
   (b) Find $\arg z$.
   (c) Find z^5 . [Hint DeMoivre, or Euler.]
2. (a) Find all solutions to the equation $x^2 - 2x + 5 = 0$.
   (b) For each solution x, write x^2 and $\dfrac{1}{x}$ in the form $a + ib$.
3. (a) Solve $x^6 = -1$.
   (b) Factor $x^6 + 1$ over $\mathbf{C}$.
   (c) Factor $x^6 + 1$ over $\mathbf{R}$.

Solutions to Review Problems

Integrals, etc.

1. Let's integrate using the substitution

$$\left\langle \begin{array}{c} u = 4x - 5 \\ u + 5 = 4x \\ x = \frac{u+5}{4} \end{array} \; ; \; dx = \frac{du}{4} \; ; \; \begin{array}{ccc} x = 2 & \Rightarrow & u = 3 \\ x = 0 & \Rightarrow & u = -5 \end{array} \right\rangle$$

to get the integral: $\displaystyle\int_{-5}^{3} \frac{\frac{du}{4}}{u} = \frac{1}{4}\int_{-5}^{3} \frac{du}{u}$. At this point, we need to be careful because the singularity $u = 0$ occurs *between* the endpoints -5 and 3. A common mistake is to formally take the anti-derivative at this point, and then plug in the end-points to get the answer: $\frac{1}{4}(\ln|3| - \ln|-5|) = \frac{1}{4}\ln\frac{3}{5}$. But this is a **fallacy** because the integral is *improper* due to the point $u = 0$ at which the integrand $\frac{1}{u}$ is undefined!

The correct way to evaluate the integral is by breaking it up into a sum of improper integrals, each with only an *endpoint singularity* as follows:

$$\frac{1}{4}\int_{-5}^{3} \frac{du}{u} = \frac{1}{4}\int_{-5}^{0} \frac{du}{u} + \frac{1}{4}\int_{0}^{3} \frac{du}{u}$$

where the first summand has a right endpoint singularity, and the second summand has a left endpoint singularity. To show that $\frac{1}{4}\int_{-5}^{3} \frac{du}{u}$ *diverges*, it is enough to show that one of the above diverges. E.g., choose the second. By definition of an improper integral of type II we have

$$\int_{0}^{3} \frac{du}{u} = \lim_{t \to 0^+} \left[\int_{t}^{3} \frac{du}{u} \right]$$

$$= \lim_{t \to 0^+} \left[\ln u \Big|_t^3 \right] = \lim_{t \to 0^+} [\ln 3 - \ln t] = \ln 3 - \lim_{t \to 0^+} \ln t = +\infty$$

Therefore, the integral is divergent.

2. FALSE.

Heuristic answer(**WRONG**): $\displaystyle\int_{-4}^{4} \frac{dx}{5x^{1/3} - 4}$ converges because it behaves much like $\displaystyle\int_{-4}^{4} \frac{dx}{x^{\frac{1}{3}}}$ which converges.

Surprising rigorous answer: Use the substitution

$$\left\langle \begin{array}{c} u = 5x^{\frac{1}{3}} - 4 \\ \frac{u+4}{5} = x^{\frac{1}{3}} \\ x = \frac{1}{5^3}(u+4)^3 \end{array} \; ; \; dx = \frac{3}{5^3}(u+4)^2 \, du \; ; \; \begin{array}{ccc} x = 4 & \Rightarrow & u = 5 \cdot 4^{\frac{1}{3}} - 4 \\ x = -4 & \Rightarrow & u = -5 \cdot 4^{\frac{1}{3}} - 4 \end{array} \right\rangle$$

to get the integral

$$\int_{-5\cdot 4^{\frac{1}{3}}-4}^{5\cdot 4^{\frac{1}{3}}-4} \frac{\frac{3}{5^3}(u+4)^2\ du}{u} = \frac{3}{5^3}\left[\int_{-5\cdot 4^{\frac{1}{3}}-4}^{0} \frac{(u+4)^2\ du}{u} + \int_{0}^{5\cdot 4^{\frac{1}{3}}-4} \frac{(u+4)^2\ du}{u}\right]$$

To show that this integral *diverges* we need to show that one of the improper integrals, say $\int_{0}^{5\cdot 4^{\frac{1}{3}}-4} \frac{(u+4)^2\ du}{u}$ diverges, and to show this, it's enough to show that $\int_0^1 \frac{(u+4)^2\ du}{u}$ diverges. This is done by comparing to the divergent integral $\int_0^1 \frac{du}{u}$ by using the fact that $(u+4)^2 > 1$ for $u \geq 0$.

3. The arc length of the hyperbolic cosine:

$$y = \cosh x \Longrightarrow y' = \sinh x$$
$$\Longrightarrow \sqrt{1+(y')^2} = \sqrt{1+\sinh^2 x} = \sqrt{\cosh^2 x} = \cosh x$$

The arc length of the curve given by $y = \cosh x$ is therefore computed via the formula:

$$\mathcal{L} = \int_0^1 \sqrt{1+(y')^2}\ dx = \int_0^1 \cosh x\ dx$$
$$= \sinh x\Big|_0^1 = \sinh(1) - \sinh(0) = \sinh(1)$$

where we've used the fact that $\frac{d}{dx}\sinh x = \cosh x$, and that $\sinh(0) = 0$. Finally, using the formula

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

we get the answer: $\boxed{\sinh(1) = \frac{1}{2}(e - \frac{1}{e})}$.

4. The main point here is to use the trigonometric identity $\tan^2 x = \sec^2 x - 1$:

$$\int \tan^4 x\ dx = \int \tan^2 x \tan^2 x\ dx = \int \tan^2 x(\sec^2 x - 1)\ dx$$
$$= \int \tan^2 x \sec^2 x\ dx - \int \tan^2 x\ dx = \int \tan^2 x \sec^2 x\ dx - \int (\sec^2 x - 1)\ dx$$
$$= \int \tan^2 x \sec^2 x\ dx - \int \sec^2 x\ dx + \int dx = \langle u = \tan x\ ;\ du = \sec^2 x\ dx\rangle - \tan x + x$$
$$= \int u^2\ du - \tan x + x = \frac{u^3}{3} - \tan x + x + C = \boxed{\frac{1}{3}\tan^3 x - \tan x + x + C}$$

5. Here we want to complete the square, so that we have

$$2x - x^2 = -(x^2 - 2x) = -([x^2 - 2x + 1] - 1) = -([x-1]^2 - 1) = 1 - [x-1]^2\ ,$$

and then apply the trigonometric substitution $x-1=\sin\theta$ to integrate using the identity $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$:

$$\begin{aligned}\int \sqrt{2x-x^2}\,dx &= \int \sqrt{1-[x-1]^2}\,dx = \langle\ x-1=\sin\theta\ ;\ dx=\cos\theta\,d\theta\ \rangle\\ &= \int \cos\theta\cdot\cos\theta\,d\theta = \int \cos^2\theta\,d\theta = \int \frac{1+\cos 2\theta}{2}\,d\theta\\ &= \frac{\theta}{2}+\frac{\sin 2\theta}{4}+C = \frac{\theta}{2}+\frac{\sin\theta\cos\theta}{2}+C\end{aligned}$$

where in the last step we used the identity $\sin 2\theta = 2\sin\theta\cos\theta$. Back-substitute using the fact that $x-1=\sin\theta$ so that $\theta=\sin^{-1}(x-1)$ and after setting up an appropriate triangle

$$\cos\theta = \sqrt{1-[x-1]^2} = \sqrt{2x-x^2}$$

so that our integral becomes: $\boxed{\tfrac{1}{2}\sin^{-1}(x-1)+\tfrac{1}{2}(x-1)\sqrt{2x-x^2}+C}$

6. (a) First look for a partial fractions decomposition of the form

$$\frac{1}{x^2(x+2)} = \frac{A}{x^2}+\frac{B}{x}+\frac{C}{x+2}$$

and multiply both side of the equation by the denominator of the left side, then solving for $A, B,$ and C to get $A=\frac{1}{2}, B=-\frac{1}{4}$, and $C=\frac{1}{4}$. Thus we can now integrate the partial fraction:

$$\begin{aligned}\int \frac{1}{x^2(x+2)}\,dx &= \frac{1}{2}\int\frac{dx}{x^2}-\frac{1}{4}\int\frac{dx}{x}+\frac{1}{4}\int\frac{dx}{x+2}\\ &= -\frac{1}{2x}-\frac{1}{4}\ln|x|+\frac{1}{4}\ln|x+2|+C = \boxed{-\frac{1}{2x}+\frac{1}{4}\ln\left|\frac{x+2}{x}\right|+C}\end{aligned}$$

(b) Now take the limit as $x\to\infty$ to find the value of the improper integral:

$$\begin{aligned}\int_1^\infty \frac{1}{x^2(x+2)}\,dx &= -\frac{1}{2x}+\frac{1}{4}\ln\left|\frac{x+2}{x}\right|\Big|_0^{R\to\infty}\\ &= \lim_{R\to\infty}\left[-\frac{1}{2R}+\frac{1}{4}\ln\left|\frac{R+2}{R}\right|\right]-\left[-\frac{1}{2}+\frac{1}{4}\ln\left|\frac{1+2}{1}\right|\right]\\ &= \left[0+\frac{1}{4}\ln\left(\lim_{R\to\infty}\frac{R+2}{R}\right)\right]+\frac{1}{2}-\frac{1}{4}\ln 3 = \frac{1}{4}\ln 1+\frac{1}{2}-\frac{1}{4}\ln 3 = \boxed{\frac{1}{2}-\frac{\ln 3}{4}}\end{aligned}$$

7. This is a trick question. As $y=\dfrac{1}{x^2}$ has a vertical asymptote at $x=0$ conclude that $\boxed{\text{the length of the curve is infinite}}$.

8. First we need to divide the numerator by the denominator. This can be done by polynomial long division or by using the trick:

$$\frac{x^3+2x}{x^3+1} = \frac{(x^3+1)-1+2x}{x^3+1} = 1 + \frac{2x-1}{x^3+1}.$$

The next step is to factor the denominator. This is done by guessing a root and then using synthetic division (or any other method you like for factoring once you know a root). Start by guessing $\pm 1, \pm 2, \ldots$ etc. The guess "-1" works as $(-1)^3+1=0$. Then factor to get $x^3-1=(x+1)(x^2-x+1)$. Now x^2-x+1 is *irreducible* because its discriminant $b^2-4ac=1-4=-3$ is negative, so we cannot factor the quadratic over the real numbers. Thus we look for a partial fractions decomposition of the form

$$\frac{2x-1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}.$$

Multiplying by the denominator of the left side, doing lots of algebra, and solving for A, B, and C we get: $A=-1, B=1$, and $C=0$ so that the answer is:

$$\frac{x^3+2x}{x^3+1} = 1 + \frac{2x-1}{(x+1)(x^2-x+1)} = \boxed{1 - \frac{1}{x+1} + \frac{x}{x^2-x+1}}$$

9. The main point of this problem is to understand how the formula for area of a surface of revolution works. As the surface is the union of the surface generated by rotating $y=f_1(x)$ and the surface generated by $y=f_2(x)$ around the x-axis we get:

$$\boxed{\mathcal{A} = \int_a^b 2\pi f_1(x)\sqrt{1+[f_1'(x)]^2}\ dx + \int_a^b 2\pi f_2(x)\sqrt{1+[f_2'(x)]^2}\ dx}$$

10. There are many ways to do this problem. Here's one way:

$$\begin{aligned}
&\int \frac{\sin x}{\cos^{101} x}\ dx \\
&= \int \frac{1}{\cos^{100} x}\cdot\frac{\sin x}{\cos x}\ dx = \int \sec^{100} x \cdot \tan x\ dx = \int \sec^{99} x \cdot (\sec x \tan x\ dx) \\
&= \langle u = \sec x\ ,\ du = \sec x \tan x\ dx\rangle = \int u^{99}\cdot du = \frac{u^{100}}{100} + C \\
&= \boxed{\frac{\sec^{100} x}{100} + C}
\end{aligned}$$

11. (a)(ANSWER) $y = \dfrac{1}{\sqrt{x}\cdot|\ln x|}, \quad 0 < x \le e^{-1}$.

This answer works because the curve is infinite as it has a vertical asymptote at $x=0$, and has finite area by comparison with $\dfrac{1}{\sqrt{x}}$.

(b) can't work because $\displaystyle\int_1^{\infty} \frac{dx}{\ln x}$ diverges.

(c) can't work because the curve is finite.

(d) doesn't work since $\displaystyle\int_0^{e^{-1}} \frac{dx}{x^2}$ diverges.

Sequences and Series

1. Let's use the ratio test:
$$\frac{|a_{n+1}|}{|a_n|} = \frac{(1,000,000)^{n+1}}{(n+1)!} \cdot \frac{n!}{(1,000,000)^n} = \frac{1,000,000}{n+1} \longrightarrow 0$$
as $n \to \infty$. As $0 < 1$ we conclude that the series $\boxed{\text{converges}}$.

2. (a) $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{1}{10}$. Therefore, $\theta = \tan^{-1}\left(\frac{1}{10}\right)$. Plug this into the power series expansion for $\tan^{-1} x$ to get:
$$\boxed{\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{1}{10}\right)^{2n+1}}$$

(b) The sum above satisfies the hypotheses of the *alternating series test.* Thus we can use the *alternating sum estimation theorem* to deduce that the error we get by summing up the first N terms, is less than the absolute value of the $(N+1)$'st term. So just start writing out the series, and then keep only the part of the series whose terms are bigger than, or equal to (in absolute value) to 0.00001.
So expanding out the series, we get:
$$\theta = \frac{1}{10} - \frac{1}{3\cdot 10^3} + \frac{1}{5\cdot 10^5} + \cdots = \frac{1}{10} - \frac{1}{3,000} + \frac{1}{500,000} - \cdots$$
Notice that $\frac{1}{500,000} = 2 \cdot \frac{1}{1,000,000} = 0.000002$ which is already smaller than 0.00001, so we can stop with just the *first two* terms to get the approximation:
$$\boxed{\theta \approx \frac{1}{10} - \frac{1}{3,000} = \frac{299}{3,000} \text{ radians}}$$

3. (a), (b), and (c) all have the same radius of convergence, $R = 1$. The power series representations can be obtained as follows:

(a) *Integrate* the geometric expansion to get:
$-\ln(1-x) + C = \displaystyle\int \frac{dx}{1-x} = \int \sum_{n=0}^{\infty} x^n\, dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$. Plug in $x = 0$ to get $-\ln(1-0) + C = 0$ so $0 + C = 0$. Therefore:
$$\boxed{-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}}$$

(b) Substitute $y = -x$ in the above to get:
$-\ln(1+x) = -\ln(1-y) = \sum_{n=0}^{\infty} \frac{y^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{n+1}}{n+1}$. Now take the negatives of both sides to get:

$$\ln(1+x) = -\sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{n+1}}{n+1} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n}x^{n+1}}{n+1}}$$

(c)

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n}x^{n+1}}{n+1} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \left[(-1)^n + 1\right] \frac{x^{n+1}}{n+1}$$

Notice that:

$$[(-1)^n + 1] = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

So keep only the *even* part of the series to get the solution:

$$\ln\left(\frac{1+x}{1-x}\right) = \sum_{n \text{ even}} [2] \frac{x^{n+1}}{n+1} = \langle n \to 2k \rangle = \boxed{\sum_{k=0}^{\infty} \frac{2x^{2k+1}}{2k+1}}$$

4. The easiest way to do this problem is to first memorize the expansion $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} x^{n+1}$, or at least to know how to derive this identity by integrating the geometric series for $\frac{1}{1+x}$. Now let $y = -x^3$ and use the expansion for $\ln(1+y)$ to get $\ln(1-x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (-x^3)^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{n+1}}{n+1} x^{3n+3} =$ $\boxed{-\sum_{n=0}^{\infty} \frac{1}{n+1} x^{3n+3}}$.

For T_7 keep only the terms involving powers of x of degree 7 or less to get:
$\boxed{T_7(x) = -x^3 - \frac{1}{2}x^6}$

5. Here are some of the multitudes of possible answers:

(a) $\sum_{n=0}^{\infty} \frac{1}{n!}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^n}$

(c) $\{-n\}_{n=1}^{\infty}$

6. (a) $\displaystyle\int_{e^2}^{\infty} \frac{dx}{x(\ln x)^{1.5}} = \langle u = \ln x \ ; \ du = \frac{dx}{x} \rangle = \int_{2}^{\infty} \frac{du}{u^{1.5}}$
which converges since $p = 1.5 > 1$.

(b) $\displaystyle\sum_{n=100{,}000}^{\infty} \frac{1}{n(\ln n)^{1.5}}$ converges by using the integral test with $\displaystyle\int_{100{,}000}^{\infty} \frac{dx}{x(\ln x)^{1.5}}$ which converges by part (a).

7. (a) $\displaystyle\sum_{n=1}^{\infty} \left[\sin\left(\frac{n+1}{n}\right) - \sin\left(\frac{n+2}{n+1}\right)\right]$ is a *telescoping series* whose partial sums (after crossing out all the middle terms) are given by:

$$s_N = \sin(2) - \sin\left(\frac{N+2}{N+1}\right).$$

Taking the limit of this as $N \to \infty$ we get: $\boxed{\sin(2) - \sin(1)}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges by a simple application of the *ratio test*. To evaluate, notice that powers of $\frac{1}{2}$ appear in each term of the infinite sum. The trick is to let $x = \frac{1}{2}$ in the series, and manipulate it, trying to see what function in x it is equal to:
$\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = x\frac{d}{dx}\sum_{n=0}^{\infty} x^n = x\frac{d}{dx}\frac{1}{1-x} = x\frac{1}{(1-x)^2}$. Now replace x by $\frac{1}{2}$ to get: $\displaystyle\frac{1}{2}\frac{1}{(1-\frac{1}{2})^2} = \boxed{2}$.

8. (a) Divergent by limit comparison with $\displaystyle\sum \frac{1}{\sqrt{n}}$.

(b) Conditionally convergent. It is convergent by the alternating series test, but is *not* absolutely convergent by comparison to $\displaystyle\sum \frac{1}{n}$.

9. The answer is $\boxed{\displaystyle 1 + \sum_{n=1}^{\infty}(-\frac{1}{2})^n \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}(x-1)^n}$. To see this, use the binomial series on:
$f(x) = \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{1+(x-1)}} = (1 + (x-1))^{-\frac{1}{2}}$,
or just use the formula for a Taylor series involving lots of derivatives.

10. (a) $.9 - .99 + .999 - .9999 + .99999 - .999999 + \cdots$
$\displaystyle= \sum_{n=1}^{\infty}(-1)^{n-1}\left[1 - \left(\frac{1}{10}\right)^n\right]$ *diverges* by the test for divergence.

(b) $\frac{1}{2^2}+\frac{2}{3^2}+\frac{3}{4^2}+\cdots=\sum_{n=1}^{\infty}\frac{n}{(n+1)^2}$ *diverges* by limit comparison with $\sum\frac{1}{n}$.

(c) For $\sum_{n=1}^{\infty}(\frac{n^2+2n}{n^3+1}-\frac{1}{2})^n$ use the root test:
$\sqrt[n]{|a_n|}=\left|\frac{n^2+2n}{n^3+1}-\frac{1}{2}\right|=\left|\frac{n^{-1}+2n^{-2}}{1+n^{-3}}-\frac{1}{2}\right|\longrightarrow\frac{1}{2}<1$ as $n\to\infty$. This implies *convergence*.

11. Use the ratio test with $a_n=\frac{(n!)^3}{(3n)!}3^{3n}x^n$. We have:

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{([n+1]!)^3 3^{3[n+1]}|x|^{[n+1]}}{(3[n+1])!}\cdot\frac{(3n)!}{(n!)^3 3^{3n}|x|^n}$$
$$=\frac{(n+1)^3(n!)^3\cdot 3^3 3^{3n}\cdot|x||x|^n}{(3n+3)\cdot(3n+2)\cdot(3n+1)\cdot[(3n)!]}\cdot\frac{(3n)!}{(n!)^3 3^{3n}|x|^n}=\frac{27(n+1)^3|x|}{(3n+3)\cdot(3n+2)\cdot(3n+1)}$$
$$=\frac{27(1+1/n)^3|x|}{(3+3/n)\cdot(3+2/n)\cdot(3+1/n)}\longrightarrow\frac{27}{27}|x|=|x|$$

as $n\to\infty$. Setting this to be less than 1 we get $|x|<1$ and the sum converges (*at least*) on $(-1,1)$. Thus the radius of convergence is $\boxed{R=1}$.

12. (a) $a_n=\frac{1}{n^{-\ln n}}=n^{\ln n}=e^{\ln(n^{\ln n})}=e^{(\ln n)^2}$ *diverges* because $(\ln n)^2\to\infty$ as $n\to\infty$.

(b) $a_n=\sqrt{\frac{(1+n)n}{\sin n+n^2}}=\sqrt{\frac{(\frac{1}{n}+1)\cdot 1}{\frac{\sin n}{n^2}+1}}\longrightarrow\sqrt{\frac{0+1}{0+1}}=\sqrt{1}=\boxed{1}$ as $n\to\infty$ where we use the fact that $\frac{\sin n}{n^2}\to 0$ as $n\to\infty$ (you can use the *squeeze theorem* to prove this).

13. Use the identity

$$\sin^2 x=\frac{1}{2}(1-\cos 2x)$$

and plug in $y=2x$ into the known Taylor series for $\cos y$, and simplify algebraically in order to get $\boxed{1-\sin^2 x=1+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n-1}}{(2n)!}x^{2n}}$.

Differential Equations

1. The ODE $y' = \cos^2 y \cdot \ln x$ is separable:

$$\frac{dy}{dx} = \cos^2 y \cdot \ln x \Rightarrow \frac{dy}{\cos^2 y} = \ln x\ dx \Rightarrow \int \sec^2 y\ dy = \int \ln x\ dx$$

$$\Rightarrow \tan y = \left\langle \begin{matrix} u & = & \ln x \\ du & = & \frac{dx}{x} \end{matrix} ; \begin{matrix} dv & = & dx \\ v & = & x \end{matrix} \right\rangle = \ln x \cdot x - \int x \cdot \frac{dx}{x}$$

$$= x \ln x - \int dx = x \ln x - x + C.$$

So we have $\tan y = x \ln x - x + C$ which implies: $\boxed{y = \tan^{-1}(x \ln x - x + C)}$.

2. $y' + \cos x \cdot y = \sin x \cdot \cos x$ is a linear first order ODE. The integrating factor is $I(x) = e^{\int \cos x\ dx} = e^{\sin x}$. Multiplying the differential equation on both sides of by the integrating factor, and using the "reverse" of the product rule one gets:

$$(e^{\sin x} y)' = e^{\sin x} \sin x \cdot \cos x \tag{1}$$

We want to integrate both sides, and to do this the hard part is integrating the right hand side.

$$\int e^{\sin x} \sin x \cdot \cos x\ dx = \langle t = \sin x\ ;\ dt = \cos x\ dx \rangle = \int e^t t\ dt$$

$$= \left\langle \begin{matrix} u & = & t \\ du & = & dt \end{matrix} ; \begin{matrix} dv & = & e^t\ dt \\ v & = & e^t \end{matrix} \right\rangle = te^t - \int e^t\ dt = te^t - e^t + C$$

$$= e^t(t-1) + C = e^{\sin x}(\sin x - 1) + C$$

Now integrating both sides of equation (1) and dividing by $e^{\sin x}$ we get:
$\boxed{y = \sin x - 1 + Ce^{-\sin x}}$.

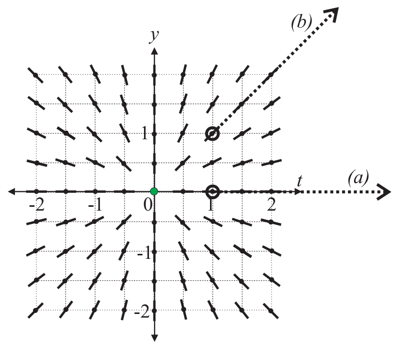
3. $y' = \frac{y}{x} + 2$ is a homogeneous first order ODE. So use the substitution $\langle v = \frac{y}{x}\ ;\ y' = v + xv' \rangle$ to get:

$$v + xv' = v + 2 \Rightarrow x\frac{dv}{dx} = 2 \Rightarrow dv = 2\frac{dx}{x}$$

$$\Rightarrow \int dv = 2 \int \frac{dx}{x} \Rightarrow v = 2 \ln |x| + C \Rightarrow \frac{y}{x} = 2 \ln |x| + C$$

$$\Rightarrow \boxed{y = 2x \ln |x| + Cx}$$

4. Notice that at any point, the direction field has as its slope its vertical coordinate divided by its horizontal coordinate. So at any point, the slope is the slope of the line between the point and the origin. Thus our direction field points radially away from the origin and we get the following picture:



5. **False**: Let's find the differential equation of each family and then check if the slopes are negative reciprocals of each other.

   - $x = ky^2$: Implicit differentiation with respect to x gives:
     $1 = k \cdot 2yy' \Rightarrow y' = \dfrac{1}{2ky}$. To find the differential equation, we need to replace the constant k by a function in x and y gotten from the original family of curves. In our case, $x = ky^2$ implies that $k = \dfrac{x}{y^2}$ so this is how we should replace k. Plugging this into $y' = \dfrac{1}{2ky}$ we get the family's differential equation:

     $$y' = \frac{1}{2y\frac{x}{y^2}} = \boxed{\frac{y}{2x}}.$$

   - $\frac{1}{2}x^2 + y^2 = c$: Implicit differentiation gives:
     $x + 2yy' = 0$ so this time we got rid of the constant c and we don't have to worry about replacing it, as in the first family. Solving for y' we get the differential equation:

     $$y' = \boxed{-\frac{x}{2y}}$$

   Apparently $\dfrac{y}{2x}$ is *not* the negative reciprocal of $-\dfrac{x}{2y}$ so that the two families are not orthogonal trajectories of each other.

   (Question: how could you change the families a bit to turn them into orthogonal trajectories?)

6. The characteristic equation of $y'' - 2y' - 3y = 0$ is $r^2 - 2r - 3 = (r-3)(r+1) = 0$ so $r = 3, -1$. Thus the general solution is $y = C_1e^{3x} + C_2e^{-x}$ whose derivative is $y' = 3C_1e^{3x} - C_2e^{-x}$. Plugging in the initial conditions we get: $3 = y(0) = C_1 + C_2$ and $1 = y'(0) = 3C_1 - C_2$. We want to solve for C_1 and C_2. For example, add the equations together to get $4C_1 = 4$ so $C_1 = 1$. Plug this into the first equation to get $C_2 = 2$. Thus the solution is: $\boxed{y = e^{3x} + 2e^{-x}}$.

7. The characteristic equation of $y''-2y'+5y=0$ is $r^2-2r+5=0$. Use the quadratic formula to get $r=1\pm 2i$. The complex solution gives a solution involving exponentials and sines/cosines. The real part of the root is 1 in our case implying that we'll have $e^{1\cdot x}=e^x$ as part of our solution. The imaginary part of the root is $\pm 2i$ meaning that we'll have sines/cosines of the form $\sin 2x, \cos 2x$. The solution is: $\boxed{y=e^x(C_1\cos 2x+C_2\sin 2x)}$

8. The characteristic equation of $y''-6y'+9y=0$ is $r^2-6r+9=(r-3)^2=0$ meaning that we have the *repeated* root $r=3$. e^{3x} will be a particular solution. Another *linearly independent* solution is gotten by multiplying the first solution by x to get xe^{3x}, giving us the general solution $y=e^{3x}(C_1+C_2x)$. Let's plug in the left boundary condition: $1=y(0)=(C_1+0)$ so $C_1=1$ and we can replace C_1 to get $y=e^{3x}(1+C_2x)$. Plug in the right hand boundary condition: $e^4+e^3=y(1)=e^3(1+C_2)$ so dividing by e^3 we get $e+1=1+C_2$ so that $C_2=e$ and our solution is: $\boxed{y=e^{3x}(1+ex)}$.

9. The best formula for the function $f(x,y)$ is given by choice (b) $\dfrac{e^y}{1+x^2}$. There are many ways to do this problem. A recommended method would be to use a process of elimination, and this can be done many ways. For example, notice that for our direction field, the slopes depend on the height, so the variable y *must* appear in the formula for $f(x,y)$. This eliminates (c) $\csc x$. Also, for our direction field, all slopes are positive, this eliminates (d) $\sin y$, as $\sin y$ is negative, for example, at $y=-0.5$. Finally, (a) cannot be the answer as the function $\dfrac{y}{x}+e^{\frac{y}{x}}$ is *not even defined* on the y-axis, since $x=0$ is being divided. This leaves (b) $\dfrac{e^y}{1+x^2}$ as the only possibility.

10. (a) Rewrite $y'\cdot\cos x=y\cdot\sin x+e^x\cos x$ as $\cos x\cdot y'-\sin x\cdot y=e^x\cos x$ so that we can use the *reverse* of the product rule to get:

$$(\cos x\cdot y)'=e^x\cos x$$

The *integrating factor* is just the function appearing next to the "y" after applying the reverse of the product rule. Therefore $I(x)=\cos x$ is the integrating factor and the answer is (v).

(b) The only difficult bit is integrating $e^x\cos x$. This is done by integrating by parts twice until the formula appears again and solving for the integral... One gets $\int e^x\cos x\,dx=\frac{e^x}{2}(\sin x+\cos x)+C$ So after integrating both sides of $(\cos x\cdot y)'=e^x\cos x$ and dividing both sides by $\cos x$ we get:

$$\boxed{y=\frac{e^x}{2}(\tan x+1)+C\sec x}$$

11. Use the method of *series solutions*:

$$y=\sum_{n=0}^{\infty}c_nx^n,\quad y'=\sum_{n=0}^{\infty}nc_nx^{n-1},\quad y''=\sum_{n=0}^{\infty}n(n-1)c_nx^{n-2}.$$

(Remember, it is possible *when necessary* to drop the first term of the sum for y' and the first two terms of y'', and re-index...) Continue:

$\frac{d^2y}{dx^2} = xy \Longrightarrow \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = x\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+1}$. After dropping the first two terms of y'' and re-indexing it via $\langle n \mapsto n+3 \rangle$ we get:

$\sum_{n=-1}^{\infty} (n+3)(n+2)c_{n+3}x^{n+1} = \sum_{n=0}^{\infty} c_n x^{n+1}$

$\Longrightarrow 2c_2 + \sum_{n=0}^{\infty} [(n+3)(n+2)c_{n+3} - c_n]x^{n+1} = 0$. For this last equation to be true, each coefficient of x^n should be zero. That is, $c_2 = 0$ and the following recursion must also hold:

$$(n+3)(n+2)c_{n+3} = c_n, \quad n \geq 0.$$

Re-indexing by $\langle n \mapsto n-3 \rangle$ and solving for c_n we get:

$$c_n = \frac{1}{n(n-1)} c_{n-3}; \quad n \geq 3. \tag{2}$$

Use the initial conditions $y(0) = 1, \frac{dy}{dx}(0) = 0$ to conclude that $c_0 = 1$, and $c_1 = 0$. As the recursion skips by three, and $c_1 = c_2 = 0$, we infer that for n *not* a multiple of $3, c_n = 0$. This leaves us only needing to calculate c_{3k}. After applying equation 2 repeatedly, or finding $c_3, c_6, c_9 \ldots$ and looking for a pattern, one gets:

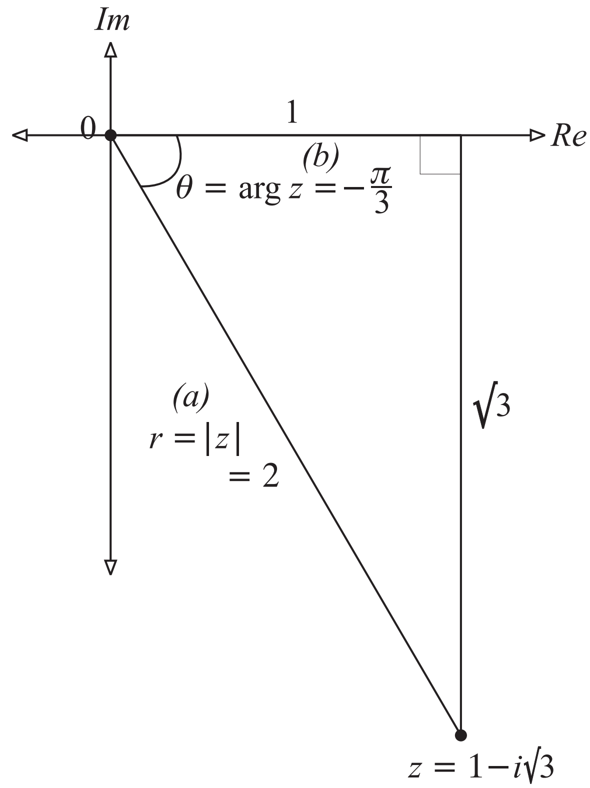
$$c_{3k} = \begin{cases} \frac{1}{(3k)(3k-1)\cdot(3k-3)(3k-4)\cdots 3\cdot 2}, & k > 0 \\ 1, & k = 0. \end{cases}$$

Thus we get the solution:

$$\boxed{y = 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(3k)(3k-1)\cdot(3k-3)(3k-4)\cdots 3\cdot 2}}.$$

Complex Numbers

1. It is very useful to construct a triangle in the complex plane to do this problem:



(a) If $z = a + ib$ then the *modulus* of z is defined by the Pythagorean-like formula: $|z| = \sqrt{a^2 + b^2}$ so that we get $|1 - i\sqrt{3}| = \sqrt{1+3} = \boxed{2}$.

(b) If $z = a + ib$ then the *argument* of z is defined by: $\tan(\arg z) = \dfrac{b}{a}$ so that we have $\tan(\arg(1 - i\sqrt{3})) = \dfrac{-\sqrt{3}}{1} = \dfrac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$. So we look for an angle in the fourth quadrant (see the triangle above) whose tangent is $\dfrac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$, e.g. $\arg z = \boxed{-\dfrac{\pi}{3}}$.

(c) Convert to polar form: $z = re^{i\theta} = 2e^{-i\frac{\pi}{3}}$. Then it is easy to raise this expression to the fifth power:

$$z^5 = (2e^{-i\frac{\pi}{3}})^5 = 2^5(e^{-i\frac{5\pi}{3}}).$$

The angle $-\dfrac{5\pi}{3}$ is the same as the angle $\dfrac{\pi}{3}$ so in terms of polar coordinates our expression becomes $32(e^{i\frac{\pi}{3}})$. Now use Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$ to convert back to rectangular coordinates:

$$z^5 = 32(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 32(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = \boxed{16 + i16\sqrt{3}}.$$

2. (a) Use the quadratic formula to get: $\boxed{x = 1 \pm 2i}$

(b) For $x = 1 + 2i$:

$$x^2 = (1+2i)^2 = 1 + 2(2i) + (2i)^2 = 1 + 4i - 4 = \boxed{-3 + 4i}.$$

To find $\frac{1}{x}$ use the conjugate trick:

$$\frac{1}{x} = \frac{1}{1+2i} = \frac{1}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{1-2i}{1+4} = \boxed{\frac{1}{5} - \frac{2}{5}i}$$

Similarly, for $x = 1-2i$: $x^2 = \boxed{-3-4i}$ and $\frac{1}{x} = \boxed{\frac{1}{5} + \frac{2}{5}i}$.

3. (a) Solve $x^6 = -1$ as follows. Look for a solution of the polar form $x = re^{i\theta}$, and express -1 in polar coordinates but allow for the fact that every time you go 2π around a circle you get back where you started. That is, instead of saying $\arg(-1) = \pi$ we write $\arg(-1) = \pi + 2\pi k$ with k an integer. As $x^6 = (re^{i\theta})^6 = r^6 e^{6i\theta}$, our original equation $x^6 = -1$ becomes in polar notation:

$$r^6 \cdot e^{i6\theta} = 1 \cdot e^{i(\pi + 2\pi k)}.$$

Setting the moduli, and arguments equal separately we get:

$$r^6 = 1, \quad 6\theta = \pi + 2\pi k.$$

The first equation must be solved by a *non-negative* real number since absolute values are positive. This forces $r = 1$. The solution of the second equation is $\theta = \dfrac{\pi + 2\pi k}{6}$ for which we keep only the first six solutions (after that, we get back to where we started on the circle). So we get: $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}$, and $\frac{11\pi}{6}$. Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ to convert to rectangular coordinates we get: $\boxed{x = \frac{\sqrt{3}}{2} + i\frac{1}{2}, i, -\frac{\sqrt{3}}{2} + i\frac{1}{2}, -\frac{\sqrt{3}}{2} - i\frac{1}{2}, -i, \text{ and } -\frac{\sqrt{3}}{2} + i\frac{1}{2}}$.

(b) $x^6 + 1 = (x - x_1)(x - x_2) \cdots (x - x_6)$
$= \boxed{(x - \frac{\sqrt{3}}{2} - i\frac{1}{2})(x - i)(x + \frac{\sqrt{3}}{2} - i\frac{1}{2})(x + \frac{\sqrt{3}}{2} + i\frac{1}{2})(x + i)(x - \frac{\sqrt{3}}{2} + i\frac{1}{2})}$.

(c) To factor $x^6 + 1$ over $\mathbf{R}$, we need to pair up the above expression with complex-conjugates next to each other. Then use the fact that for any complex number a, with complex-conjugate $\overline{a}$ we have:

$$(x + a)(x + \overline{a}) = x^2 + (a + \overline{a})x + a\overline{a}$$

where $a + \overline{a}$ and $a\overline{a}$ are *real* numbers. Thus we get:
$x^6 + 1 = (x + i)(x - i) \cdot (x + \frac{\sqrt{3}}{2} + i\frac{1}{2})(x + \frac{\sqrt{3}}{2} - i\frac{1}{2}) \cdot (x - \frac{\sqrt{3}}{2} + i\frac{1}{2})(x - \frac{\sqrt{3}}{2} - i\frac{1}{2})$
$= \boxed{(x^2 + 1) \cdot (x^2 + x\sqrt{3} + 1) \cdot (x - x\sqrt{3} + 1)}$.