M. Rieffel | Mathematics 128A Final Examination | December 17, 1998

SHOW YOUR WORK COMPLETELY AND NEATLY. Total points = 140.

18 1. Give an example of an invertible 3×3 matrix A, a vector b, and an approximate solution of $Ax = b$ whose residual error is $\leq 10^{-4}$ but whose error is ≥ 1. Justify your answer. (You may use any norm as long as you specify it.)

4 2. a) Describe briefly the strategy for deriving the Runge-Kutta methods for solving ODE's.

3 b) Define what is meant by the local truncation error, and the localorder, for a single-step method for solving ODE's.

20 c) Derive a specific Runge-Kutta method of local order 3. Include a precise explanation of how you know that your method is of local order 3.

3 3. a) Define precisely what it means for a convergent sequence of numbers to converge linearly.

15 b) View $g(x) = 4x/(3x + 1)$ as an iteration function. Note that 1 is a fixed-point for g. Prove that for any initial guess which is is > 1 the resulting iteration sequence will converge linearly to 1.

(over)

13 4. a) Derive the simple Simpson rule for numerical integration.

4 b) Derive the composite Simpson rule.

3 c) Very briefly discuss precisely the advantages of the composite Simpson rule compared to the composite trapezoid rule.

5. Find the LU decomposition of

17
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 1 & 4 & 1 \end{pmatrix}.$$

Check your answer.

6. Find a positive integer, n, such that if p is the polynomial
20 which interpolates $f(x) = \cos(x)$ at n equispaced points in the interval $[3, 5]$, then $|f(x) - p(x)| < 10^{-4}$ for every x in that interval. Justify your answer.

20 7. Let p be a polynomial of degree $n + 1$ which, for the interval $[2, 6]$ and the weight function $w(x) = 1$, is orthogonal to all polynomials of lower degree. Assume that you have already proved that p then has n distinct roots in the interior of $[2, 6]$. Prove directly from this that the integration rule obtained by interpolating functions at the roots of p and integrating over $[2, 6]$ is exact for polynomials of degree $\leq 2n+1$.