MATH 110, SECTION 4, FALL 2008, FINAL EXAM

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Please enter the following information in capital letters:

Last Name: ____________________ First Name: ____________________

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The Final Exam consists of 7 problems. The maximal number of points is 100.

You may use Page 2 for notes or auxiliary calculations which will not be graded and you cannot refer to the results on Page 2 in your solutions. All pages must be submitted (do not unstaple).

Please enter your solution to Problem 1 on Page 3. Please begin your solutions to Problems 2–7 on the even page just below the statement of the problem and – if necessary – continue your solution on the subsequent odd page.

If you need extra space for a solution to one of the problems, you may use Page 16. In this case you must provide the problem number in the top line of Page 16.

Please use a pen with black or blue ink. Any other devices or documents, such as calculators, lecture notes, etc. must not be used.

Problem	1	2	3	4	5	6	7
Points							

Total

Additional space for your notes (<u>not</u> graded):

End (of additional space for notes)

Problem 1 (18 points: 2 each). Answer TRUE or FALSE (a justification is not required).

a) If W is any subspace of a finite-dimensional inner product space V, then $V = W \oplus W^{\perp}$.

b) The set $\{A \in M_{n\times n}(\mathbb{C}) : \det(A) = 0\}$ is a subspace of the vector space $M_{n\times n}(\mathbb{C})$.

c) If $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable, then the characteristic polynomial of A splits.

d) For all matrices $A \in M_{n\times n}(\mathbb{C})$ with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ the eigenspace E_{λ_k} is equal to the generalized eigenspace K_{λ_k} for all $1 \leq k \leq n$.

e) For all $A, B \in M_{n\times n}(\mathbb{R})$ with $\det(A) = 0$ it holds $\text{rank}(AB) < \text{rank}(B)$.

f) Let V be a finite-dimensional real inner product space. Every normal operator on V is self-adjoint.

g) The largest eigenvalue of a symmetric matrix $A \in M_{n\times n}(\mathbb{R})$ is given by $\max\{x^t Ax : \|x\| = 1\}$.

h) The function $H : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $H\left(\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}\right) = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is a bilinear form.

i) If T is a linear operator on a finite-dimensional inner product space, then $R(T^*)^{\perp} = N(T)$.

Solution to Problem 1:

a) TRUE

b) FALSE

c) TRUE

d) TRUE

e) FALSE

f) FALSE

g) TRUE

h) TRUE

i) TRUE

Points (Problem 1)

Problem 2 (14 points: 2+4+8).

a) Let W be a subset of a finite-dimensional inner product space V. Provide the definition of the orthogonal complement $W^\perp$ of W.

b) Consider $V = \mathbb{R}^2$ with the standard inner product and $v = \begin{pmatrix}1\\1\end{pmatrix}$. Compute $\{v\}^\perp$ and the point $w_0 \in \operatorname{span}\{v\}$ which is closest to $x = \begin{pmatrix}1\\3\end{pmatrix}$, i.e. $\|w_0 - x\| \le \|w - x\|$ for all $w \in \operatorname{span}\{v\}$.

c) Let V be a finite-dimensional inner product space and W_1 and W_2 be subspaces of V. Recall from the lecture that their sum is defined as $W_1 + W_2 := \{w_1 + w_2 : w_1 \in W_1,\ w_2 \in W_2\}$. Prove that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$.

Solution to Problem 2:

a) It is $W^\perp := \{v \in V : \langle v, w\rangle = 0 \text{ for all } w \in W\}$.

b) It is

$$\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}^\perp = \operatorname{span}\left\{\begin{pmatrix}-1\\1\end{pmatrix}\right\}$$

and

$$w_0 = \frac{1}{2}\left\langle \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\rangle \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}.$$

c) It is $x \in (W_1 + W_2)^\perp$ if and only if $\langle x, w\rangle = 0$ for all $w \in W_1 + W_2$. Therefore,

$$(W_1 + W_2)^\perp = \{x \in V : \langle x, w_1 + w_2\rangle = 0 \text{ for all } w_1 \in W_1 \text{ and all } w_2 \in W_2\}. \tag{1}$$

On the one hand, if $x \in (W_1 + W_2)^\perp$ it follows from (1) by setting $w_2 = 0$ that $\langle x, w_1\rangle = 0$ for all $w_1 \in W_1$. Similarly, by choosing $w_1 = 0$ it follows $\langle x, w_2\rangle = 0$ for all $w_2 \in W_2$. This shows that $x \in W_1^\perp$ and $x \in W_2^\perp$. In other words $x \in W_1^\perp \cap W_2^\perp$.

On the other hand, if $x \in W_1^\perp \cap W_2^\perp$, then $\langle x, w_1\rangle = 0$ for all $w_1 \in W_1$ and $\langle x, w_2\rangle = 0$ for all $w_2 \in W_2$. Therefore $\langle x, w_1 + w_2\rangle = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$. By (1) it follows that $x \in (W_1 + W_2)^\perp$.

Points (Problem 2)

Problem 3 (14 points: 7+2+2+3). Consider the following real $n \times n$ matrix $A \in M_{n\times n}(\mathbb{R})$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

and the associated linear transformation $L_A : \mathbb{R}^3 \to \mathbb{R}^3$, $L_A v = Av$. As usual, we equip $\mathbb{R}^3$ with the standard (Euclidean) inner product and the corresponding norm.

a) Compute the rank $\operatorname{rank}(A)$, nullity $\dim N(L_A)$, determinant $\det(A)$ and all eigenvalues of A.

b) Compute a basis of the null space of L_A.

c) Is A an orthogonal matrix? Justify your answer.

d) Decide whether or not there exists an ordered orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A. Justify your answer.

Solution to Problem 3:

a) Because the first and the second column of A are linearly independent and the sum of the second column and third column is the zero vector it follows that $\operatorname{rank}(A) = 2$.

   By the Dimension Theorem it follows that $\dim N(L_A) = 3 - 2 = 1$.

   The eigenvalues are the zeros of the characteristic polynomial

$$f(t) = \det \begin{pmatrix} 1-t & 0 & 0 \\ 0 & 1-t & -1 \\ 0 & -1 & 1-t \end{pmatrix} = t(1-t)(t-2),$$

   which means the eigenvalues are $0, 1, 2$.

   In particular, the above computation for $t = 0$ implies $\det(A) = 0$.

b) By the considerations above $v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ satifies $Av = 0$ and, because the dimension of the null space is exactly one it follows that $\{v\}$ is a basis for $N(L_A)$.

c) The matrix A is not orthogonal. If A was orthogonal every eigenvalue λ would satisfy $|\lambda| = 1$ which contradicts the existence of the eigenvalue 2.

d) Since A is symmetric we can apply the Spectral Theorem and it follows that there exists an ordered orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

Points (Problem 3)

Problem 4 (10 points: 2+8).

a) Let $T : V \to V$ be a linear operator on a vector space V over the field F. The scalar $\lambda \in F$ is an *eigenvalue* of T if and only if ... (complete the statement)

b) Is the following matrix $A \in M_{3\times 3}(\mathbb{R})$ diagonalizable (over the field $\mathbb{R}$)?

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

If possible, compute an ordered basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

Solution to Problem 4:

a) ... there exists a nonzero vector $v \in V$ such that $Tv = \lambda v$.

b) First, we compute all eigenvalues:

$$\det \begin{pmatrix} 3-t & -1 & 1 \\ -1 & 3-t & 1 \\ 0 & 0 & 4-t \end{pmatrix} = (4-t)((3-t)^2-1) = (4-t)^2(2-t)$$

Therefore the eigenvalues are $\lambda_1 = 2$ with algebraic multiplicity $m_1 = 1$ and $\lambda_2 = 4$ with algebraic multiplicity $m_2 = 2$.

Second, we compute bases for the eigenspaces. For $\lambda_1 = 2$ we solve

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \text{ and find } E_2 = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}.$$

For $\lambda_2 = 4$ we solve

$$\left(\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \text{ and find } E_4 = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

As a consequence, the matrix A is diagonalizable and

$$\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$$

is an ordered basis for $\mathbb{R}^3$ which consists of eigenvectors of A.

Points (Problem 4)

Problem 5 (12 points: 4+2+6).

a) Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$, $T(f(x)) = f'(x)$. Determine the T-cyclic subspace generated by the polynomial $p(x) = x^2 + x + 1$.

b) Let $A \in M_{n\times n}(\mathbb{C})$ and $f(t)$ be the characteristic polynomial of A. Which of the following statements is a matrix version of the Caley-Hamilton Theorem for A? [Check the box if true]

- [ ] It is $f(\lambda) = 0 \in \mathbb{C}$ for some $\lambda \in \mathbb{C}$ if and only if $\lambda = 0$.
- [X] It holds $f(A) = 0$, where $0 \in M_{n\times n}(\mathbb{C})$ denotes the zero matrix.
- [ ] $f(t)$ is the zero polynomial.

c) Let $A \in M_{2\times 2}(\mathbb{C})$ be a matrix such that $\lambda_1, \lambda_2 \in \mathbb{C}$ are the (not necessarily distinct) eigenvalues.

i) Prove that A is invertible if and only if $\lambda_1\lambda_2 \neq 0$.

ii) Prove that if $\lambda_1 + \lambda_2 = 0$, then $A^2 = \begin{pmatrix} -\lambda_1\lambda_2 & 0 \\ 0 & -\lambda_1\lambda_2 \end{pmatrix}$.

Solution to Problem 5 a) and c):

a) The T-cyclic subspace generated by $p(x)$ is given by

$$\mathrm{span}\{p(x), T(p(x)), T^2(p(x)), \ldots\} = \mathrm{span}\{x^2 + x + 1, 2x + 1, 2\} = P_2(\mathbb{R}).$$

c) Since A has exactly the two eigenvalues λ_1 and λ_2 the characteristic polynomial must be of the form

$$f(t) = \det(A - tI_2) = (\lambda_1 - t)(\lambda_2 - t) = \lambda_1\lambda_2 - (\lambda_1 + \lambda_2)t + t^2.$$

This implies $\det(A) = f(0) = \lambda_1\lambda_2$. Because A is invertible if and only if $\det(A) \neq 0$ the first claim follows. If $\lambda_1 + \lambda_2 = 0$ the second claim follows from the Caley-Hamilton Theorem which asserts that

$$0 = f(A) = \lambda_1\lambda_2 I_2 + A^2.$$

Points (Problem 5)

Problem 6 (20 points: 4+4+2+4+6). Let V be a finite-dimensional inner product space over the field $\mathbb{C}$ and let $T : V \to V$ be a linear transformation.

a) Provide the defining property for the *adjoint* $T^* : V \to V$ and calculate the adjoint of $T : \mathbb{C}^2 \to \mathbb{C}^2$, $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$, where $\mathbb{C}^2$ is equipped with the standard inner product.

b) When is T called *self-adjoint*? State the definition and give an example.

c) When is T called *normal*? Give the definition and an example of a normal and non-self-adjoint T.

d) Prove that if $S, T : V \to V$ both are self-adjoint linear transformations such that $ST = TS$, then ST is a self-adjoint linear transformation.

e) Prove that if T is self-adjoint, then $\|T(v) + iv\|^2 = \|T(v)\|^2 + \|v\|^2$ for all $v \in V$.

Solution to Problem 6:

a) The defining property of the adjoint $T^* : V \to V$ is

$$\langle Tx, y \rangle = \langle x, T^*y \rangle \quad \text{for all } x, y \in V.$$

The adjoint for the given T is $T^* : \mathbb{C}^2 \to \mathbb{C}^2$, $T^*\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ -a \end{pmatrix}$.

b) T is self-adjoint if and only $T = T^*$. An example is the identity $I : \mathbb{C}^2 \to \mathbb{C}^2$.

c) T is normal if and only $TT^* = T^*T$. An example of a normal and non-self-adjoint operator is T defined in a).

d) By the properties of the adjoint we compute

$$(ST)^* = T^*S^* = TS = TS,$$

were we used the self-adjointness of S and T in the second step and the assumption that S and T commute in the last step.

e) For every $v \in V$ we calculate, using the properties of the inner product,

$$\langle T(v) + iv, T(v) + iv \rangle = \langle T(v), T(v) \rangle + i\langle v, T(v) \rangle - i\langle T(v), v \rangle - i^2\langle v, v \rangle.$$

By the definition of the norm and $i^2 = -1$ it remains to show that

$$i\langle v, T(v) \rangle - i\langle T(v), v \rangle = 0,$$

which is equivalent to the fact that

$$\langle v, T(v) \rangle = \langle T(v), v \rangle.$$

This is true because T is self-adjoint.

Points (Problem 5)

Points (Problem 6)

Problem 7 (12 points: 2+4+6).

a) Which of the following formulae defines the product AB of two matrices $A, B \in M_{n\times n}(\mathbb{C})$?
[Check the box if true]

☐ $$(AB)_{ij} = \sum_{k=1}^{n} A_{ki}B_{jk} \quad (1 \le i, j \le n)$$

☐ $$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{jk} \quad (1 \le i, j \le n)$$

[X] $$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} \quad (1 \le i, j \le n)$$

b) We define the *trace* $\mathrm{tr}(A)$ of a matrix $A \in M_{n\times n}(\mathbb{C})$ to be the sum of the diagonal entries, that is $\mathrm{tr}(A) := \sum_{i=1}^{n} A_{ii}$. Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all $A, B \in M_{n\times n}(\mathbb{C})$.

c) Let $A \in M_{n\times n}(\mathbb{C})$ be a matrix with the distinct eigenvalues $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and corresponding algebraic multiplicities $m_1, \ldots, m_k$. Prove that $\mathrm{tr}(A) = \sum_{i=1}^{k} m_i\lambda_i$.
[Hint: Find a matrix which is similar to A and for which this formula is obvious.]

Solution to Problem 7 b) and c):

b) We compute

$$\mathrm{tr}(AB) = \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik}B_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} B_{ki}A_{ik} = \sum_{k=1}^{n}(AB)_{kk} = \mathrm{tr}(BA).$$

c) For the matrix A we know from the lecture that there exists a Jordan canonical form J such that $A = Q^{-1}JQ$ for some invertible matrix Q. The diagonal entries of J are exactly the eigenvalues of A and each eigenvalue λ_i occurs exactly m_i times, since m_i is the dimension of the generalized eigenspace corresponding to λ_i. This implies $\mathrm{tr}(J) = \sum_{i=1}^{k} m_i\lambda_i$. By Part b) it follows

$$\mathrm{tr}(A) = \mathrm{tr}(Q^{-1}JQ) = \mathrm{tr}(JQ^{-1}Q) = \mathrm{tr}(J) = \sum_{i=1}^{k} m_i\lambda_i$$

which proves the claim.

Points (Problem 7)

Extra space to complement your solution to Problem _____:

End of extra space