Aldi

MATH 110-S2 FALL 2008
FINAL EXAM

Please write your name on each blue-book, and the number of blue-books used, if you use more than one. You have until 3:30pm. Write all proofs in *full sentences* and show your work whenever possible. There are four problems, skip ahead if you get stuck. Good luck!

(1) (24 pts)

(a) Define the notion of linear transformation. Define the notion of isomorphism. State the classification of finite-dimensional vector spaces up to isomorphism.

(b) Define the notion of characteristic polynomial. Define the notion of minimal polynomial. State the Primary Decomposition Theorem.

(c) Define the notion of dual space. Define the notion of symmetric bilinear form. State Sylvester's Law of Inertia.

(2) (24 pts) Label each of the following statements as True or False. Justify your answers.

(a) If $A \in M_{n\times n}(\mathbb{R})$ is such that $A^2 = 2A$, then A is diagonalizable.

(b) If $A \in M_{n\times n}(\mathbb{R}) \subseteq M_{n\times n}(\mathbb{C})$, then $L_A \in \mathcal{L}(\mathbb{R}^n)$ and $L_A \in \mathcal{L}(\mathbb{C}^n)$ have the same eigenvalues.

(c) If V is an infinite dimensional vector space, then there exists a subspace $W \subseteq V$, not equal to V, such that W is also infinite dimensional.

(d) The sum of two elementary matrices is never an elementary matrix.

(3) (26 pts)

Let $\mathbb{F}$ be a field and for each $a \in \mathbb{F}$, consider the function

$$T_a : M_{2\times 2}(\mathbb{F}) \longrightarrow M_{2\times 2}(\mathbb{F})$$

such that

$$T_a(A) := A + aA^t .$$

(a) Show that T_a is a linear operator for all $a \in \mathbb{F}$.

(b) For each $a \in \mathbb{F}$, write down the matrix representative of T_a with respect to the standard basis of $M_{2\times 2}(\mathbb{F})$.

(c) Assuming that $\text{char}(\mathbb{F}) \neq 2$, compute the characteristic polynomial and the minimal polynomial of T_a, for all $a \in \mathbb{F}$.

(d) Assuming that $\text{char}(\mathbb{F}) \neq 2$, find all values of $a \in \mathbb{F}$ for which T_a is diagonalizable and, for those values, write down a basis of eigenvectors for T_a.

(EXTRA CREDIT, 5pts) Answer questions c) and d) if $\mathbb{F}$ is a field of characteristic equal to two

(4) (26 pts)
Let $\mathbb{F}$ be the field with three elements. Recall that a congruence class is an equivalence class of matrices with respect to the equivalence relation given by congruence.

(a) Compute the number of elements in the set $M_{2\times 2}(\mathbb{F})$.

(b) Compute the number of invertible matrices in $M_{2\times 2}(\mathbb{F})$.

(c) Compute the number of distinct characteristic polynomials of matrices in $M_{2\times 2}(\mathbb{F})$.

(d) Compute the number of congruence classes of symmetric matrices in $M_{2\times 2}(\mathbb{F})$.

(EXTRA CREDIT, 5pts) Answer questions a), b) and d), replacing $M_{2\times 2}(\mathbb{F})$ with $M_{n\times n}(\mathbb{F})$, for $n > 2$.