# Final Exam for **MATH 125A**

## Fall 2008, December 19, UC Berkeley

**Problem 1 [20P]**

(a) Consider the 3-place truth function $f : \{T, F\}^3 \to \{0, 1\}$ given by

| $A_0$ | $A_1$ | $A_2$ | $f(A_0, A_1, A_2)$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ |
| $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $T$ |

Find a formula $\varphi = \varphi(A_0, A_1, A_2)$ such that $f_\varphi = f$, where $f_\varphi$ is the truth function derived from $\varphi$.
You can use the connectives $\wedge$ (AND), $\vee$ (OR), $\to$, and $\neg$.

(b) Does there exist a propositional formula $\varphi$ such that $\varphi$ is a contradiction and there exists a propositional variable $A_i$ for which the following holds: If we replace *every* occurence of $A_i$ in $\varphi$ by $(\neg A_i)$, we obtain a new formula $\varphi'$ that is a tautology?

Give an example or show that such a formula does not exist.

**Problem 2 [25P]**

(a) Suppose $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure. State what it means that a subset $Y \subseteq M$ is *definable from parameters in* $X \subseteq M$.

(b) Let $\mathcal{A} = \{+, -, \cdot, \underline{0}, \underline{1}\}$, where $+, -, \cdot$ are binary function symbols, and $\underline{0}, \underline{1}$ are constant symbols. Consider the $\mathcal{L}_{\mathcal{A}}$-structure $\mathbb{R} = (\mathbb{R}, +, -, \cdot, 0, 1)$.

- Show that the set $\{x \in \mathbb{R} : x > 0\}$ is definable in $\mathbb{R}$ (without parameters). Infer that the relation $\{(x, y) \in \mathbb{R}^2 : x < y\}$ is definable in $\mathbb{R}$, too.
- Show that for every rational number $q$, the set $\{q\}$ is definable without parameters in $\mathbb{R}$.
- Show that for any two $a, b \in \mathbb{R}$, $a \neq b$, the 1-types of $a$ and $b$ in $\mathbb{R}$ are different.

(c – Extra Credit) Give an example of a finite language $\mathcal{L}_{\mathcal{A}}$ and an infinite $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M}$ with exactly four definable (without parameters) subsets of $M$. Justify your answer.

**Problem 3 [15P]**

Using the axiom system for first order logic introduced in class (axioms are given below), show that for every term $\tau$ in $\mathcal{L}$,

$$\vdash (\tau \hat{=} \tau),$$

i.e. there exists a proof of $(\tau \hat{=} \tau)$ that uses only the axioms and consequences thereof. Justify your steps.

**Problem 4 [40P]**

(a) State the *Compactness Theorem* for first order logic.

(b) Suppose $T_1 \subsetneq T_2 \subsetneq T_3 \subsetneq \ldots$ is a strictly increasing sequence of $\mathcal{L}_{\mathcal{A}}$-theories. Suppose further that each $T_i$ is *closed under logical consequences*, i.e. for each $i$, if $T_i \models \sigma$ for some sentence $\sigma$, then $\sigma \in T_i$.

- Show that $\bigcup_{i \in \mathbb{N}} T_i$ is consistent.
- Show that $\bigcup_{i \in \mathbb{N}} T_i$ is not finitely axiomatizable, i.e. there does not exist a finite set of sentences $\Gamma$ such that $\mathcal{M} \models \Gamma$ if and only if $\mathcal{M} \models \bigcup_{i \in \mathbb{N}} T_i$.

(c) State the definition of a *complete theory*.

(d) Give an example of a language $\mathcal{L}_{\mathcal{A}}$ and an $\mathcal{L}_{\mathcal{A}}$-theory that is complete.

(e – Extra Credit) Is the theory $\bigcup_{i \in \mathbb{N}} T_i$ from part (b) necessarily complete? Prove or give a counterexample.

**Axiom System for First Order Logic**

The set of logical axioms, denoted $\Delta$, is the smallest set of $\mathcal{L}$ formulas which satisfies the following closure properties.

1. (Instances of Propositional Tautologies) Suppose that $\phi_1$, $\phi_2$ and $\phi_3$ are $\mathcal{L}$ formulas. Then each of the following $\mathcal{L}$ formulas is a logical axiom:
   (a) $((\phi_1 \to (\phi_2 \to \phi_3)) \to ((\phi_1 \to \phi_2) \to (\phi_1 \to \phi_3)))$
   (b) $(\phi_1 \to \phi_1)$
   (c) $(\phi_1 \to (\phi_2 \to \phi_1))$
   (d) $(\phi_1 \to ((\neg\phi_1) \to \phi_2))$
   (e) $(((\neg\phi_1) \to \phi_1) \to \phi_1)$
   (f) $((\neg\phi_1) \to (\phi_1 \to \phi_2))$
   (g) $(\phi_1 \to ((\neg\phi_2) \to (\neg(\phi_1 \to \phi_2))))$
2. Suppose that $\phi$ is an $\mathcal{L}$ formula, $\tau$ is a term, and that $\tau$ is substitutable for $x_i$ in $\phi$. Then
$$((\forall x_i \phi) \to \phi(x_i; \tau)) \in \Delta.$$
3. Suppose that $\phi_1$ and $\phi_2$ are $\mathcal{L}$ formulas. Then
$$((\forall x_i(\phi_1 \to \phi_2)) \to ((\forall x_i \phi_1) \to (\forall x_i \phi_2))) \in \Delta.$$
4. Suppose that $\phi$ is an $\mathcal{L}$ formula and that $x_i$ is not a free variable of $\phi$. Then
$$(\phi \to (\forall x_i \phi)) \in \Delta.$$
5. For every variable $x_i$, $(x_i \doteq x_i) \in \Delta$.
6. Suppose that $\phi_1$ and $\phi_2$ are $\mathcal{L}$ formulas and that $x_j$ is substitutable for $x_i$ in $\phi_1$ and in $\phi_2$.
$$\text{If } \phi_2(x_i; x_j) = \phi_1(x_i; x_j),$$
$$\text{then } ((x_i \doteq x_j) \to (\phi_1 \to \phi_2)) \in \Delta.$$
7. Suppose that $\phi \in \Delta$. Then $(\forall x_i \phi) \in \Delta$.