Math 104 Final Exam, Section 5 Fall 2008

3 hours. 100 points total. Name ________________________________

1. (10 points) Are the following statements true or false?

   (a) If f_n is a sequence of differentiable functions and $f_n \to f$ uniformly, then $f_n' \to f'$ uniformly.

   (b) If $f : X \longrightarrow Y$ is continuous and $U \subset Y$ is connected, then $f^{-1}(U)$ is connected.

   (c) If $f : X \longrightarrow Y$ is continuous, $E \subset X$ is closed and X is compact, then $f(E) \subset Y$ is closed.

   (d) Let $C(X, \mathbb{R})$ denote the metric space of bounded continuous functions on X. Any closed and bounded subset of $C(X, \mathbb{R})$ is compact.

2. (10 points) Give three equivalent definitions of a continuous map $f : X \longrightarrow Y$ between metric spaces.

3. (5 points) Give an example of a countably infinite subset of $\mathbb{R}$ that is compact.

4. (15 points) Suppose that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the inequality

$$|f(x) - f(y)| \leq |x - y|^2$$

for all x and y in $\mathbb{R}$. Prove that f is constant. (Hint: find the derivative.)

5. (15 points) Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable, and $f'(x)$ is never equal to 1. Prove that there can be at most one fixed point $x \in \mathbb{R}$ so that $f(x) = x$.

6. (10 points) Find a power series $\sum_{n=0}^{\infty} a_n x^n$ which converges pointwise on some subset of $\mathbb{R}$ to $\frac{1}{(1-x)^2}$. What is the radius of convergence of this power series?

7. (20 points) Define the two functions

$$C(x) := \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$$

$$S(x) := \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$$

(a) Show that the power series for $C(x)$ and $S(x)$ converge for all $x \in \mathbb{R}$.

(b) Prove that $S'(x) = C(x)$.

(c) Show that there exists some point x so that $S(x) = 42$.

(d) Why does $S(x)$ achieve a maximum on the set $E := C^{-1}((-\infty, 42])$? (In other words why is there some point $x_0 \in E$ so that $S(x_0) = \sup S(E)$?)

8. (15 points) Prove or find a counter example to the following statement: Given any continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$, there exists a sequence of polynomials $\{p_n(x)\}$ so that $p_n(x) \to f(x)$ pointwise on $\mathbb{R}$.