# Math 104: Final Exam
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**Name :**

**Student ID Number:**

**Instructions:** This is a closed-book test. Each problem is worth 20 points. Read the questions carefully, and show all your work. All work should be done on the exam paper. Additional white paper is available if needed. Good luck.

| Problem | Score |
|---|---|
| 1 | |
| 2 | |
| 3 | |
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| Total | |

(1) Prove that $\sqrt{2}$ is irrational.

(2) (a) Define the **norm** of $\mathbf{x}$, $|\mathbf{x}|$, for $\mathbf{x} \in \mathbb{R}^k$.
(b) Show $|\mathbf{a_1} + \mathbf{a_2} + \ldots + \mathbf{a_n}| \leq |\mathbf{a_1}| + |\mathbf{a_2}| + \ldots + |\mathbf{a_n}|$.

(3) (a) Define an **open cover** and a **compact** set.
(b) Show explicitly that an open interval $(a, b)$ in $\mathbb{R}$ is not compact.

(4) Prove that every open set in $\mathbb{R}$ is a countable disjoint union of open intervals.

(5) (a) Let $\sum a_n z^n$ be a power series. Define the **radius of convergence.**
(b) Compute the radius of convergence for $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}} z^{2n}$
(c) Do $\sum_{n=1}^{\infty} (-1)^n n^{\frac{-1}{4}}$ and $\sum_{n=1}^{\infty} ((-1)^n n^{\frac{-1}{4}})^2$ converge. Explain your answer.

(6) Let $\{a_n\}$ be a sequence which converges. Show that the limit of $\{a_n\}$ (as $n$ goes to infinity) is unique.

(7) Define the two different types of **discontinuities** and give an example of each of them.

(8) (a) Let $\mathbf{f}$ be a vector valued function which maps $[a, b]$ to $\mathbb{R}^k$. Define the **derivative** of $\mathbf{f}$.
(b) Give an example of a vector valued or complex valued function which does not satisfy the Mean Value Theorem.
(c) What modification of the Mean Value Theorem is satisfied?

(9) Let $f$ be a one-to-one real valued function on an interval $I$. Let $g$ be the inverse function of $f$. Assume that $f$ is continuous at $x \in I$ and that $g$ has a derivative at $y = f(x)$ where $g'(y) \neq 0$. Show that $f'(x)$ exists and that $f'(x) = \frac{1}{g'(y)}$. (Prove directly. Do not state as a consequence of another theorem.)

(10) (a) State Taylor's Theorem and describe in words what the theorem asserts.
(b) Define a **concave** function. When is a function concave?

(11) Define the **Riemann-Stieltjes integral** of a function $f$ with respect to $\alpha$. Define all the terms you use.

(12) Show that a function $f$ bounded on the closed interval $[a, b]$ is Riemann integrable on $[a, b]$ if and only if $f$ has at most finitely many points of discontinuity on $[a, b]$.

(13) State and prove the Fundamental Theorem of Calculus.