Math 128a Midterm Exam 2
Sample Solutions
Oct 26, 2003

K. Hare

1 a: (4 pts) Consider

$$A = \begin{bmatrix} 1 & 4.25 & 1.25 \\ 4 & 1 & 1 \\ 1 & 1.25 & 4.5 \end{bmatrix}$$

Use Gaussian elimination, with partial pivoting to compute the determinate of A.

$$\begin{aligned}
\det\left(\begin{bmatrix} 1 & 4.25 & 1.25 \\ 4 & 1 & 1 \\ 1 & 1.25 & 4.5 \end{bmatrix}\right) &= -\det\left(\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4.25 & 1.25 \\ 1 & 1.25 & 4.5 \end{bmatrix}\right) \\
&= -\det\left(\begin{bmatrix} 4 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 1 & 4.25 \end{bmatrix}\right) \\
&= -\det\left(\begin{bmatrix} 4 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}\right) \\
&= -4 \times 4 \times 4 \\
&= -64
\end{aligned}$$

b: (3 pts) If it takes 10 seconds to compute the determinate of a random 1000×1000 matrix, how long would it take to compute the determinate of a random 5000×5000 matrix?

Computing a determinate take $\mathcal{O}(n^3)$ time. So if for $n = 1000$, it takes 10 seconds, for $n = 5000$ it would take 5^3 times as long, so 1250 seconds. (About 20 minutes).

2 a: (4 pts) A matrix A is positive definite if $x^t Ax > 0$ for all $x \neq 0$. Prove that the diagonal entries $a_{i,i} > 0$

Consider $x_i = [0, 0, \cdots, 1, \cdots 0]^t$ *which contains a* 1 *in the* i^{th} *position, and* 0 *everywhere else. We see that* $x_i^t Ax = a_{i,i}$. *By the definition of positive definite this is greater than 0. Thus this proves the result.*

b: (3 pts) Find $\alpha > 0$ such that the following system is strictly diagonally dominate.

$$\begin{bmatrix} \alpha & 2 & 3 \\ 10 & 20 & 7 \\ \alpha & 3 & 10 \end{bmatrix}$$

We see from the first row that $\alpha > 5$. *We see from the last row that* $\alpha < 7$. *Combining these together we see that* $5 < \alpha < 7$.

3 a: (4 pts) Let A be a $n \times n$, non-singular lower triangular matrix. How many step of the Jacobi Iterative method are needed to solve $Ax = b$? (Justify your answer.)

We see on the first step that x_1 will be solved exactly. Thus we see on the second step that both x_1 and x_2 will be solved exactly. In general on the j^{th} step, $x_1, x_2, \cdots x_j$ will be solved exactly.

Thus, this will take n steps to solve the matrix exactly.

b: (4 pts) Compute the first two steps of the Jacobi Iterative method, with starting point (0,0), to the system

$$\begin{bmatrix} 10 & 3 \\ 2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

We see that we have

$$T = \begin{bmatrix} 0 & -3/10 \\ -2/10 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 7/10 \\ 3/10 \end{bmatrix}$$

$$x_{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x_{(1)} = \begin{bmatrix} 7/10 \\ 3/10 \end{bmatrix} \qquad x_{(2)} = \begin{bmatrix} 61/100 \\ 16/10 \end{bmatrix}$$

4 a: (4 pts) Given $f(-1) = -4$, $f(0) = -3$ and $f(1) = 0$, use Neville's method to approximate $f(2)$.

x_i	$f(x_i)$		
-1	-4		
0	-3	-1	
1	0	3	5

So we estimate that $f(2) \approx 5$.

b: (5 pts) Use a variation of the Newton Divide Difference method for Hermite polynomials to find the unique polynomial, of degree at most three, such that

$$P(-1) = -4, P(0) = -1, P'(0) = 2, P(1) = 2$$

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, \cdots, x_{i+3}]$
-1	-4			
0	-1	3		
0	-1	2	-1	
1	2	3	1	1

So we get that $P(x) = -4 + 3(x+1) - 1(x+1)x + 1(x+1)x^2 = -1 + 2x + x^3$ *A quick check does in fact show that* $P(-1) = -4, P(0) = -1, P'(0) = 2$ *and* $P(1) = 2$.

5: (5 pts) A natural cubic spline S on $[0, 2]$ is defined by

$$S(x) = \begin{cases} x^3 & \text{if } 0 \leq x \leq 1 \\ a + b(x-1) + c(x-1)^2 + d(x-1)^3 & \text{if } 1 \leq x \leq 2 \end{cases}$$

Find a, b, c and d.

We know that $S_0(1) = S_1(1), S_0'(1) = S_1'(1), S_0''(1) = S_1''(1)$ *Further we know that* $S_1''(2) = 0$. *These give us the four equations.*

$$\begin{aligned} a &= 1 \\ b &= 3 \\ 2c &= 6 \\ 2c + 6d &= 0 \end{aligned}$$

Thus we have that $a = 1$, $b = 3$, $c = 3$, *and* $d = -1$.

6 a: (4 pts) Complete the factorization below

$$\begin{bmatrix} 2 & 0 & -1 \\ 4 & -3 & -5 \\ -2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \underline{2} & \underline{-3} & 0 \\ \underline{-1} & \underline{0} & 1 \end{bmatrix} \begin{bmatrix} \underline{2} & \underline{0} & \underline{-1} \\ 0 & 1 & \underline{1} \\ 0 & 0 & \underline{2} \end{bmatrix}$$

b: (4 pts) Prove that there do not exist lower and upper triangular matrices L and U satisfying

$$\begin{bmatrix} 0 & -2 & 0 \\ 2 & 1 & 0 \\ 6 & 2 & -1 \end{bmatrix}$$

Assume that there does exist a factorization. We see from a_{12} *that* $u_{12}l_{11} = -2$ *and hence* $l_{11} \neq 0$. *We see from* a_{21} *that* $u_{11}l_{21} = 2$ *and hence* $u_{11} \neq 0$. *We see from* a_{11} *that* $u_{11}l_{11} = 0$. *This is a contradiction.*