Math 128a Midterm Exam 1
Sample Solutions
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1 a: (4 pts) Let $f(x)$ be a continuous function on $[2,3]$. Further, assume that for all $x \in [2,3]$ that $f(x) \in [2,3]$. Prove that $f(x)$ has a fixed point $p \in [2,3]$.

Write $g(x) = f(x) - x$. *Notice that* $g(2) = f(2) - 2 \geq 0$ *as* $f(2) \in [2,3]$. *Further notice that* $g(3) = f(3) - 3 \leq 0$ *as* $f(3) \in [2,3]$. *Thus by the intermediate value theorem,* $g(x)$ *has a root* $p \in [2,3]$. *So* $0 = g(p) = f(p) - p$ *from which it follows that* $f(p) = p$. *Thus* p *is the desired fixed point*

b: (3 pts) Further assume to part **(a)** that $|f'(x)| \leq \frac{1}{11}$ for all $x \in [2,3]$. How many steps of the fixed point method are needed to find the fixed point of $f(x)$ between $[2,3]$ to an accuracy of 10^{-3}.

Let p_0 *be the initial guess somewhere between* 2 *and* 3. *Let* p *be the unique fixed point. Then we know:*

$$|p_n - p| < \left(\frac{1}{11}\right)^n |p_0 - p| < \left(\frac{1}{11}\right)^n$$

So to ensure an accuracy of 10^{-3}, *we would need* 3 *steps of the fixed point method.*

2 a: (3 pts) Let $g(x) = ax + b$ for some fixed, non-zero constants a and b. How many steps of Newton's method are needed to find a root of $g(x)$ to an accuarcy of 10^{-6}. Why?

Notice that the Newton iterate is then

$$\begin{aligned} x - \frac{g(x)}{g'(x)} &= x - \frac{ax+b}{a} \\ &= x - x - \frac{b}{a} \\ &= -\frac{b}{a} \end{aligned}$$

So after one iterate, you get the exact solution of $-\frac{b}{a}$.

Alternately: Newton's method find the intercept of the tanget of $g(x)$ at the starting point. As the tangent to $g(x)$ is $g(x)$, this means that Newton's method finds the intercept of $g(x)$ in one step.

b: (4 pts) Give an example of a, b and c, such that when using three digits rounding,

$$a + (b + c) \neq (a + b) + c$$

Let $a = 1.00$, $b = 0.004$ *and* $c = 0.004$.

Notice that

$$a + (b + c) = 1.00 + fl(0.008) = 1.00 + 0.01 = 1.01.$$

Further notice that

$$(a + b) + c = fl(1.004) + 0.004 = 1.00 + 0.004 = fl(1.004) = 1.00$$

3 a: (3 pts) Define what it means for a sequence $\{p_n\}_{n=0}^{\infty}$ to converge linearly to p.

p_n converges to p linearly if there exists $0 < \lambda < 1$ such that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lambda$$

b: (3 pts) Let $p_n = 10^{-3^n}$. What order of convergence does p_n have?

Notice that $p_n \to 0$. Further notice that

$$\begin{aligned}
\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^3} &= \lim_{n\to\infty} \frac{10^{-3^{n+1}}}{(10^{-3^n})^3} \\
&= \lim_{n\to\infty} \frac{10^{-3^{n+1}}}{10^{3\times(-3^n)}} \\
&= \lim_{n\to\infty} \frac{10^{-3^{n+1}}}{10^{-3^{n+1}}} \\
&= \lim_{n\to\infty} 1 \\
&= 1
\end{aligned}$$

So we see that this converges to 0 with order 3.

c: (4 pts) Give two situations where you would use Bisection method over Newton's method.

Any two of the following

1. *You would use Bisection method if you don't have a sufficiently good approximation of the root to use Newton's method. (For Newton's method, you must start "close" to the root to guarentee convergence).*
2. *You would use Bisection method if the function was not differentiable, or if you did not know the derivative of the function.*
3. *If $f'(p) = 0$.*

4 a: (3 pts) Let $p_n \to p$. Give two conditions that are necessary to use accelerated convergence to compute $\hat{p}_n$.

1. *p_n must converge to p linearly.*
2. *$p_n > p$ for all n, or $p_n < p$ for all n*
3. *Alternately,* $\lim \frac{p_{n+1}-p}{p_n-p}$ *must exist*

b: (2 pts) Given $p_1 = 9, p_2 = 7, p_3 = 4$, compute $\hat{p}_1$.

$$\begin{aligned} \hat{p}_1 &= p_1 - \frac{(p_2 - p_1)^2}{p_3 - 2p_2 + p_1} \\ &= 9 - \frac{(7-9)^2}{4 - 2 \times 7 + 9} \\ &= 9 - \frac{4}{4 - 14 + 9} \\ &= 9 - \frac{4}{-1} \\ &= 13 \end{aligned}$$

5 a: (3 pts) Explain how to derive Secant Method from Newton's method.

For Newton's method, we start use the fixed point iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

For Secant method, we approximation the derivative $f'(x_n)$ by

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Thus we get the Secant method of

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

b: (2 pts) Let $f(x) = x^4 - 17$. Compute the first two steps of Bisection method on the interval $[0, 4]$.

Notice $f(0) = -17, f(4) = 4^4 - 17 > 0$. *So there is a root somewhere in* $[0, 4]$.
Notice $f(2) = 2^4 - 17 = 16 - 17 = -1 < 0$. *So there is a root somewhere in* $[2, 4]$
Notice $f(3) = 3^4 - 17 = 81 - 17 > 0$ *So there is a root somewhere in* $[2, 3]$.

6 a: (4 pts) Use Horner's method to evaluate $p(x) = x^3 + x^2 - x - 1$ at $x = 2$.

Using Horner's method gives:

Coeffient of x^3	Coeffient of x^2	Coeffient of x	Constant term
$a_3 = 1$	$a_2 = 1$	$a_1 = -1$	$a_0 = -1$
$b_3 = 1$	$b_2 = 3$	$b_1 = 5$	$b_0 = 9$

Thus $p(2) = b_0 = 9$

b: (2 pts) Using the results of part (**a**), find $q(x)$ where

$$p(x) = (x - 2)q(x) + c$$

where c is your answer in part (**a**).

We see that $q(x) = b_3x^2 + b_2x + b_1 = x^2 + 3x + 5$ *So*

$$x^3 + x^2 - x - 1 = (x - 2)(x^2 + 3x + 5) + 9.$$

Simple expansion of the right hand side ensures that we did our calculation correctly.