MATH 104 – FINAL EXAM – FALL 2003 D. Geba

① Let $a, b \in \mathbb{R}$ such that $ab > 0$ and consider $f : [a,b] \to [a,b]$ a continuous function. Prove that there exists $c \in (a,b)$ such that $c \cdot f(c) = a \cdot b$.

② Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x \cdot (\sin(x^2))^2$. Prove that f is continuous, but it is not uniformly continuous.

③ Consider the sequence of functions, $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{x}{1+nx^2}$. Prove that $(f_n)_n$ converges pointwise to $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 0$, $\forall x \in \mathbb{R}$. Is this convergence also uniform?

④ Consider the power series $\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$. Find the radius of convergence and the sum of this series.

⑤ Let $f : \mathbb{R} \to \mathbb{R}$ a function differentiable at 0. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(|x|)$. Prove that g is differentiable at 0 if and only if $f'(0) = 0$.

⑥ Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 2\cos x - \sin x, & x \geq 0 \\ ax^2 + bx + c, & x < 0 \end{cases}$

Find a, b, c such that f is two times differentiable.

⑦ Let $f : [0,1] \to \mathbb{R}$ continuous function such that $\int_0^1 f(x)\,dx = \frac{1}{3}$. Prove that $\exists\, c \in (0,1)$ such that $f(c) = c^2$.