185 final exam

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Name: L. Bartholdi

There are two parts to this test, on the front and back page. The back page asks you to apply the results seen in class to the theory of elliptic functions.

Do not, under any circumstances, give answers on a separate sheet. They will not be considered for your grade.

(1) Mark the following as true / false. Wrong answers count negatively.

- ____: There exists a linear fractional transformation ϕ with $\phi(0) = 0$, $\phi(1) = 1$, $\phi(-1) = -1$, and $\phi(i) = -i$.
- ____: There exists a branch of $\sqrt{z}$ on the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.
- ____: There exists a branch of $\sqrt{z}$ on the annulus $\{z \in \mathbb{C} : 2 < |z| < \infty\}$.
- ____: If $f(z)$ is holomorphic, then $f(z^2)$ is holomorphic.
- ____: If $f(z)$ is holomorphic, then $f(z)^2$ is holomorphic.
- ____: The function $\sin(z)$ is one-to-one (= univalent) on the open disk centered at 0 and of radius 3/2.
- ____: If two holomorphic functions $f, g : \mathbb{C} \to \mathbb{C}$ agree on infinitely many values, then they are equal.
- ____: If a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is such that $f(U)$ is bounded for an unbounded $U \subset \mathbb{C}$, then f is constant.
- ____: If $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is holomorphic, then there exist holomorphic functions $g, h : \mathbb{C} \to \mathbb{C}$ such that $f(z) = g(z) + h(1/z)$.
- ____: If $U, V \subset \mathbb{C}$ are conformally equivalent subsets of $\mathbb{C}$, then there exists a linear fractional transformation ϕ with $V = U(\phi)$.

(2) What is the residue at i of $\frac{z^2-1}{z^2+1}$? ____________

(3) Consider the Riemann ζ function $\zeta(z) = \sum_{n\geq 1} n^{-z}$. The domain of convergence of this series is $\{z \in \mathbb{C} :$ ____________ $\}$.

(4) Consider an analytic continuation F of ζ to a neighbourhood of 1. The residue of $F(z)$ at $z = 1$ is ____________.

(5) The first three terms of the Laurent series expansion of $\frac{e^z+1}{e^z-1}$ around $z = 0$ are ____________.

(6) How many zeros does the function $\exp(\frac{z}{1-z}) - 1$ have in the open unit disk? ____________

(7) Let $f(z) = \frac{az+b}{cz+d}$ be a linear fractional transformation with $a, b, c, d \in \mathbb{R}$ and $ad - bc$ negative. Where is the upper half plane mapped to? ____________

(8) Evaluate, for $\alpha < 1$, the integral

$$\int_0^\infty \frac{x^\alpha}{1+x^2} dx = \text{____________}.$$

1. Elliptic functions

Fix once and for all $\omega_1, \omega_2 \in \mathbb{C}$ with $\Im(\omega_1/\omega_2) > 0$. The *lattice* $\mathcal{L}$ is the set $\{n_1\omega_1 + n_2\omega_2 | n_1, n_2 \in \mathbb{Z}\}$. An $\mathcal{L}$*-elliptic function* is a meromorphic function $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ with $f(z) = f(z + \omega)$ for all $z \in \mathbb{C}, \omega \in \mathcal{L}$. A *fundamental parallelogram* is, for a given $\alpha \in \mathbb{C}$, the set $P_\alpha = \{\alpha + t_1\omega_1 + t_2\omega_2 | t_1, t_2 \in [0, 1]\}$.

Prove each of the following assertions in **at most one line**. If you refer to a theorem we saw in class, quote it by name or contents. Let f be $\mathcal{L}$-elliptic, and fix a fundamental parallelogram P_α.

(1) If f is elliptic and has no poles on ∂P_α, then $\sum_{z \in P_\alpha} \mathrm{res}_z(f) = 0$. *(hint: integrate)*

(2) If f has no poles in P_α, then f is constant.

(3) f has at least 2 poles in P_α, counting multiplicities.

(4) The set $S = \{a_i\}$ of poles and zeroes of f in P_α is finite. *(hint: compactness of P_α)*

(5) Let m_i be the multiplicity of a_i, positive for zeroes and negative for poles. Then $\sum_{a_i \in S} m_i = 0$. *(hint: consider f'/f)*

(6) $\sum_{a_i \in S} a_i m_i \in \mathcal{L}$. *(hint: consider zf'/f)*

Define the *Weierstrass $\wp$-function* by $\wp(z) = \dfrac{1}{z^2} + \displaystyle\sum_{\omega \in \mathcal{L}, \omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$.

(7) Show that $\wp(z)$ is meromorphic, with a double pole at each $\omega \in \mathcal{L}$ and no other pole. *(hint: locally uniform convergence)*

(8) Show that $\wp'(z)$ is elliptic, and odd, i.e. $\wp'(-z) = -\wp'(z)$.

(9) Show that $\wp(z)$ is elliptic, and even, i.e. $\wp(-z) = \wp(z)$.

(10) If f is elliptic, then it is a rational function of $\wp$ and $\wp'$. *(hint: using $\wp'(z)$ reduce to $f(z)$ even; then compare $f(z)$ with $\prod_{a_i \in S}(\wp(z) - \wp(a_i))^{m_i}$ for appropriate $m_i \in \mathbb{Z}$ and $a_i \in P_\alpha$)*

(11) Define $e_1 = \wp(\omega_1/2)$, $e_2 = \wp(\omega_2/2)$ and $e_3 = \wp((\omega_1 + \omega_2)/2)$. Then

$$(\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$