GRADUATE PRELIMINARY EXAMINATION, Part A   Fall Semester 2022

1. Answer six of the nine problems each day. You will get no extra credit for attempting more than 6 problems.

2. The exam lasts 3 hours each day, including time to enter questions in gradescope.

3. Do not answer more than one question on any given piece of paper, as this will confuse the examiners.

4. Submit your answers is by taking pictures of them and uploading them to gradescope.

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6. In case of questions or unexpected problems during the prelim send email to the chair of the prelim committee at borcherds@berkeley.edu If a correction or announcement is needed during the exam it will be sent as an email to the address you use on gradescope for the prelim, so please keep an eye on this during the prelim.

Problem 1A.   *Score:*

The mean oscillation of a function f on an interval (a, b) is defined to be the average value of $|f(x) - \overline{f}|$ on (a, b), where $\overline{f}$ is the average value of f on (a, b). Calculate the mean oscillation of the function $f = \log x$ on the interval $(0, b)$.

Solution: Answer is $2/e$ by routine calculus.

Problem 2A.   *Score:*

Let I be an open interval, and let $f \colon I \to \mathbb{R}$ be a differentiable function. Assume that $f'(x) \geq 0$ for all $x \in I$, and that all zeroes of f' are isolated (this means that if $f'(x_0) = 0$ for some $x_0 \in I$, then there is some $\epsilon > 0$ such that $f'(x) \neq 0$ for all $x \in I$ such that $0 < |x - x_0| < \epsilon$).

Without using the Fundamental Theorem of Calculus, show that f is strictly increasing on I (the latter means that $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I).

Do not assume that f' is continuous.

Solution:

We first show that f is increasing on I; in other words, that if $x_1, x_2 \in I$ and $x_1 < x_2$ then $f(x_1) \leq f(x_2)$.

Let $x_1, x_2 \in I$ and assume that $x_1 < x_2$. Since f is differentiable on I, it is continuous on I, so f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Therefore, by the Mean Value Theorem, there is a $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \ .$$

But also $c \in I$, so $f'(x) \geq 0$, and we have

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \geq 0 \ ;$$

thus $f(x_2) \geq f(x_1)$.

To show that f is strictly increasing on I, let x_1, x_2 be as above. By the above, $f(x_1) \leq f(x_2)$. Suppose now, by way of contradiction, that $f(x_1) = f(x_2)$. Since f is weakly increasing, this implies that f is constant on $[x_1, x_2]$; hence $f' = 0$ on (x_1, x_2). This contradicts the assumption that all zeroes of f' are isolated, so $f(x_1) < f(x_2)$.

Thus f is strictly increasing on I.

Problem 3A.   *Score:*

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function from the reals to the reals, and let $X \subseteq \mathbb{R}$ be a subset of the real numbers. For each of the following statements either prove it is always true or give an example to show it can be false.

(a) If X is open then $f(X)$ is open.

(b) If X is closed then $f(X)$ is closed.

(c) If X is open then $f^{-1}(X)$ is open.

(d) If X is closed then $f^{-1}(X)$ is closed.

Solution:

(a) false with $f(x) = x^2$, $X = (-1, 1)$.

(b) false with $f(x) = 1/(1 + x^2)$, $X = \mathbb{R}$.

(c), (d) true.

Problem 4A.   *Score:*

How many zeros does the function $4z^{10} + e^z$ have in the open unit disc $\{z : |z| < 1\}$ of the complex plane? Prove that it has no multiple zeros.

Solution: 10 zeros by Rouche's theorem. To show there are no multiple zeros, just check that the function and its derivative have no zeros in common.

Problem 5A.   *Score:*

Let f be an analytic function on the real line, and define g by $g(s) = \int_0^1 f(t)t^s dt$ for $s > 0$. Prove that g can be extended to a meromorphic function on the whole complex plane. (Hint: first do the case $f(t) = t^n$.) Where would you expect to find the poles of g?

Solution:

This is easy if $f(t) = t^n$ for some fixed n by explicit calculation of $g(s) = t^{s+n+1}/(s+n+1)$, which has a pole at $-n-1$. Also observe that if f vanishes to order n at 0 then g is holomorphic for $\Re(s) > -n$. In general write f as the sum of a polynomial of degree n and a function h vanishing to order $n+1$ at 0 to see that g can be meromorphically continued to the region $\Re(s) > -n$ with poles possibly at $-1, -2, -3, ...$

Problem 6A.   *Score:*

Prove that the function taking A to $S = (I - A)(I + A)^{-1}$ is a bijection from orthogonal real matrices A whose eigenvalues are all different from -1 to skew-symmetric real matrices S.

Solution: If A is orthogonal ($AA^T = 1$) then $S^T = (1-A^{-1})(1+A^{-1})^{-1} = (A-1)(A+1)^{-1} = -S$ so S is skew symmetric ($S^T = -S$). The formula for S is well defined as A has no eigenvalues -1.

The inverse map is given by $A = (1 - S)(1 + S)^{-1}$ so if S is skew skew symmetric then $A^T = (1 + S)(1 - S)^{-1} = A^{-1}$ and A is orthogonal. The formula for A is well defined as S has eigenvalues that are imaginary (and therefore not -1).

Problem 7A.   *Score:*

Let V be the vector space of n by n matrices over a field, let A be an element of V, and let T be the linear transformation of V taking X to AX. Compute the determinant and trace of T in terms of the determinant and trace of A.

Solution: The trace is n times the trace of A, and the determinant is $\det(A)^n$.

Problem 8A.   *Score:*

Prove that any group of order 18 is a semidirect product of a normal subgroup of order 9 by a group of order 2, and use this to classify these groups.

Solution:

By the Sylow theorems, there is a subgroup of order 9, which must be normal as it has index 2, and a subgroup of order 2. So the group is a semidirect product of a normal subgroup of order 9 by a group of order 2.

The group of order 9 is either cyclic, in which case there are 2 ways the group of order 2 can act on it, or is a vector space over the field with 3 elements, in which case there are 3 ways a group of order 2 can ac on it as it has 2 eigenspaces whose dimensions sum to 2.

Problem 9A.   *Score:*

Let G be a finite group and H a subgroup not equal to G. Prove that there is an element of G not in any conjugate xHx^{-1} of H.

Solution: The total number of conjugates of H is $|G|/|N(H) \le |G|/|H|$, so the total number of non-identity elements in conjugates of H is at most $(|G|/|H|)(|H| - 1)$, which is less than the number $|G| - 1$ of non-identity elements of G if $|H| < |G|$. So there is a non-identity element of G not in any conjugate of H.

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GRADUATE PRELIMINARY EXAMINATION, Part B   Fall Semester 2022

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Problem 1B. *Score:*

Find continuous bounded functions f_n on the reals (for $n \geq 0$) such that all integrals and sums below are well defined and finite, but

$$\int_{x=0}^{\infty} \sum_{n=0}^{\infty} f_n(x)dx \quad \neq \quad \sum_{n=0}^{\infty} \int_{x=0}^{\infty} f_n(x)dx.$$

Solution: Pick $g(x)$ to be any continuous function vanishing outside $[0,1]$ with integral 1. Put $f_n(x) = g(x-n) - g(x-n-1)$. Then f_n has integral 0, but their sum is g which has integral 1.

Problem 2B.   *Score:*

Show that it is possible to find an uncountable set of subsets of the positive integers that is totally ordered by inclusion. ("Totally ordered by inclusion" means that given two of these subsets M, N either $M \subseteq N$ or $N \subseteq M$. For example, the subsets $\{1\} \subseteq \{1, 2\} \subseteq \{1, 2, 3\} \cdots$ are totally ordered by inclusion, but are only a countable set of subsets.)

Solution: We can use the rational numbers instead of the positive integers as these are both countable infinite sets. Then take the subsets to be the Dedekind cuts: for each real α we associate the set M_α of all rationals less than α.

Problem 3B.   *Score:*

Suppose that the functions $g, f_0, f_1, f_2, \ldots$ are smooth real valued functions on $\mathbb{R}$ such that they all vanish at 0 and $\lim_{n \to \infty} f_n'(x) = g'(x)$ for all x. (Here g' means the derivative of g.) Either prove or give a counter-example to the claim that $\lim_{n \to \infty} f_n(x) = g(x)$ for all x.

Solution:

The claim is false. Take $f_n(x)$ to be 0 for $x \le 0$ and 1 for $x > 1/n$, and g to be 0 everywhere.

Problem 4B.   *Score:*

Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function, and suppose that $f(z)/z \to 0$ as $|z| \to \infty$. Prove that f is constant.

Solution:

For all $R > 0$ let C_R denote the circle of radius R, centered at the origin. By Cauchy's differentiation formula,

$$f'(0) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(z)}{z^2} \, dz \; .$$

By assumption, for any $\epsilon > 0$ there is an R_ϵ such that $|f(z)/z| < \epsilon$ for all z such that $|z| > R_\epsilon$. Taking $R > R_\epsilon$, we then have

$$|f'(0)| \leq \frac{1}{2\pi} \cdot \frac{\epsilon}{R} \cdot 2\pi R = \epsilon \; ,$$

so letting $\epsilon \to 0$ we have $f'(0) = 0$.

Now the hypothesis on $f(z)/z$ is preserved by translation; i.e., for all $a \in \mathbb{C}$ we have that $f(z-a)/z \to 0$ as $|z| \to \infty$; hence $f'(a) = 0$ for all $a \in \mathbb{C}$. Therefore f is constant.

Problem 5B.   *Score:*

Let $a > 0$ be a constant $\neq 3$. Let C_a denote the positively oriented circle of radius a centered on the origin. Evaluate

$$\int_{C_a} \frac{z^3 + \sin z}{z^3(z-3)} \, dz \ .$$

(Your answer should be a function of a.)

Solution:
The integrand is analytic (holomorphic) everywhere except possibly at $z = 0$ and $z = 3$. We first compute the residues at those points.

Let $g(z)$ denote the integrand.

At $z = 3$ the integrand has at worst a simple pole, so the residue can be computed by extending $(z-3)g(z) = 1 + (\sin z)/z^3$ to a holomorphic function at $z = 3$ and evaluating it there. This gives

$$\operatorname{Res}_{z=3} g = 1 + \frac{\sin 3}{27} \ .$$

At $z = 0$, we write

$$
\begin{aligned}
g(z) &= \frac{1}{z-3} + \frac{\sin z}{z^3(z-3)} \\
&= \frac{1}{z-3} - \frac{\sin z}{3z^3(1 - z/3)} \\
&= \frac{1}{z-3} - \frac{1}{3}\left(z^{-2} - \frac{1}{6} + \dots\right)\left(1 - (z/3) + (z/3)^2 - \dots\right)
\end{aligned}
$$

The first term is holomorphic at $z = 0$, so it does not contribute to the residue. Therefore, the residue at $z = 0$ is the coefficient of z^{-1} in the second term, which is

$$\operatorname{Res}_{z=0} g = -\frac{1}{3} \cdot 1 \cdot \left(-\frac{1}{3}\right) = \frac{1}{9} \ .$$

Therefore, we have

$$\int_{C_a} \frac{z^3 + \sin z}{z^3(z-3)} \, dz = \begin{cases} \frac{2\pi i}{9} & 0 < a < 3 \ , \\ \frac{2\pi i(30 + \sin 3)}{27} & a > 3 \ . \end{cases}$$

Problem 6B.   *Score:*

Let A be the n by n complex matrix whose diagonal entries are all a and whose non-diagonal entries are all b. Find the eigenvalues of A and their multiplicities.

Solution: For all entries b the eigenvalues are 0 (multiplicity $n-1$) and nb (multiplicity 1). Adding $a-b$ times the identity matrix adds $a-b$ to the eigenvalues, so we get $a-b$ with multiplicity $n-1$ and $a+(n-1)b$ with multiplicity 1.

Problem 7B.   *Score:*

Prove that if A is a real n by n matrix then the series $\sum_{m=0}^{\infty} A^m/m!$ converges. (This means for each i, j, the sum of $(i, j)^{th}$ entries converges.)

Solution: Pick some norm on R^n and define a norm on matrices by $|A| = sup_{|v|=1}|Av|$. Then $|A^m| \leq |A|^m$, so the series $\sum |A^m/m!|$ converges. This implies converge of the entries of the matrices, as they are bounded by a constant times the norm.

Problem 8B.   *Score:*

Prove that $\sqrt{3} + \sqrt[3]{2}$ is irrational.

Solution:
    The numbers $\sqrt{3}$ and $\sqrt[3]{2}$ are roots of the irreducible (Eisenstein) polynomials $x^2 - 3$ and $x^3 - 2$ over $\mathbb{Q}$, respectively; therefore they have degrees 2 and 3 over $\mathbb{Q}$, respectively. In particular, $\mathbb{Q}(\sqrt{3}) \not\subseteq \mathbb{Q}(\sqrt[3]{2})$ (since $[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] \nmid [\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$).
    However, $\sqrt{3} + \sqrt[3]{2} \in \mathbb{Q}$ implies that both $\sqrt{3} + \sqrt[3]{2}$ and $\sqrt[3]{2}$ lie in $\mathbb{Q}(\sqrt[3]{2})$, therefore their difference $\sqrt{3}$ lies in $\mathbb{Q}(\sqrt[3]{2})$, contradiction.

Problem 9B.   *Score:*

Let G be a finite abelian group, written additively. Assume that for each positive integer n the set $\{a \in G : na = 0\}$ has at most n elements.
   (a). Prove that G is cyclic.
   (b). Deduce from this that the multiplicative group of every finite field is cyclic.

Solution:
   (a). We prove the contrapositive. Assume that G is not cyclic. By the structure theorem for finitely generated abelian groups, G can be uniquely represented as a product $(\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_s\mathbb{Z})$, with $n_1 \mid \cdots \mid n_s$ and $n_1, \ldots, n_s$ integers greater than 1.
   Since G is not cyclic, $s > 1$, so the set $\{a \in G : n_1 a = 0\}$ has $n_1^s > n_1$ elements.
   (b). Let $\mathbb{F}$ be a finite field. Since the polynomial $x^n - 1$ of degree n has at most n roots for each n, the group $\mathbb{F}^*$ satisfies the hypothesis of part (a), so $\mathbb{F}^*$ is cyclic.