INTRODUCTION TO CHOW FORMS

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1. Introduction

This article gives an elementary introduction to the theory of Chow forms. It is based on a sequence of four lectures given at the conference on "Invariant Methods in Discrete and Computational Geometry", Curacao, Netherlands Antilles, June 1994. We thank the organizer, Neil White, for giving us the opportunity to participate and present this material. This work was supported in part by the National Science Foundation and the David and Lucile Packard Foundation.

The Chow form is a device for assigning invariant "geometric" coordinates to any subvariety of projective space. It was introduced by Cayley [5] for curves in 3-spaces and later generalized by Chow and van der Waerden [6]. Geometric operations in terms of Chow forms extend the operations of the Cayley-Grassmann algebra for linear subspaces. This indicates the use of Chow forms as a computational tool for projective geometry.

Most of the results presented here are classical and well known, proofs are either sketched or omitted, and details can be found in the references. An exception is the algorithm for computing the join in §4, which will appear in the Ph. D. thesis of the first author. We refer to the recent book of Gel'fand, Kapranov and Zelevinsky [13] for an excellent exposition of Chow forms in the context of elimination theory. A comprehensive treatment of Chow forms will appear in the forthcoming monograph by Gaeta [11].

1.1. DEFINITION OF THE CHOW FORM

We first recall the definition of Plücker coordinates: Given a d-dimensional linear subspace L of n-dimensional complex projective space $\mathbf{P}^n$, we can write L as the intersection of $n-d$ hyperplanes. Each hyperplane corre-

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sponds to a point in the dual projective space, and we write the coordinates of these points as the rows of an $(n-d) \times (n+1)$ matrix M. Left multiplication gives a $GL(n-d, \mathbf{C})$-action on the hyperplanes that preserves the subspace L. The invariants of this action are the maximal minors of M, and these minors determine L uniquely. Conversely, L determines the vector of minors up to multiplication by a nonzero constant. Thus we can represent L by the projective vector of maximal minors of M, which we call the **(primal) Plücker coordinates** or **brackets** of L. The set of all d-dimensional linear subspaces of $\mathbf{P}^n$, thus coordinatized, is called the **Grassmannian** and is denoted by $G(d, n)$.

Notational conventions: We number the columns of M from 0 to n, and we specify a given maximal minor by writing the indices of the columns involved between square brackets. We order the brackets from $[0, 1, \ldots, n-d-1]$ to $[d+1, d+2, \ldots, n]$ lexicographically. For example, the Plücker coordinates of a line in $\mathbf{P}^3$ will be written in the form $([01] : [02] : [03] : [12] : [13] : [23])$.

The subspace L can also be written as the span of $d+1$ points. This gives a $(d+1) \times (n+1)$ matrix N with the same uniqueness properties as M. The maximal minors of N are the **dual** Plücker coordinates. We denote them by double brackets $[[i_0, i_1, \ldots, i_d]]$. Primal and dual coordinates with complementary index sets are the same up to a sign change.

We now replace the linear space L by an arbitrary **projective variety**

$$X \quad = \quad \{\mathbf{x} \in \mathbf{P}^n \,:\, f_1(\mathbf{x}) = \cdots = f_r(\mathbf{x}) = 0\},$$

where the f_i are homogeneous polynomials in $k[x_0, x_1, \ldots, x_n]$ and k is a subfield of the complex numbers. Typically, k is the field of rational numbers. We suppose that X is **irreducible**, that is, X is not the union of two proper subvarieties, and that the dimension of X is d. See [7], [15], [22] for the definition of "dimension" and other basic concepts in algebraic geometry.

Let L be an $(n-d-1)$-dimensional linear subspace of $\mathbf{P}^n$. If L is chosen generically, then $X \cap L$ is empty. Let Y be the set of all $(n-d-1)$-dimensional linear subspaces L of $\mathbf{P}^n$ such that $X \cap L$ is nonempty.

Theorem 1.1 *The set Y is an irreducible hypersurface in the Grassmannian $G(n-d-1, n)$.*

For a proof see e.g. Section 3.2.B in [13]. It is known that every hypersurface in the Grassmannian is defined by a single polynomial equation. The defining irreducible polynomial of Y is denoted $\mathcal{R}_X$ and called the **Chow form** of X. We can express $\mathcal{R}_X$ as a polynomial in brackets $[i_0, i_1, \ldots, i_d]$.

While this representation is not unique, due to the **syzygies** among the brackets (see e.g. §3.1 in [24]), the Chow form itself is unique up to multiplication by a nonzero constant. We shall see in Section 3.3 that $\mathcal{R}_X$ determines X uniquely. The coefficients of the Chow form are called the **Chow coordinates** of the projective variety X.

1.2. EXAMPLES

We first show that the Chow form is indeed a generalization of Plücker coordinates. Let X denote a fixed line in $\mathbf{P}^3$ passing through the points $(a_0 : a_1 : a_2 : a_3)$ and $(b_0 : b_1 : b_2 : b_3)$. If L is a variable line, given by its primal Plücker coordinates $([01] : [02] : [03] : [12] : [13] : [23])$, then $X \cap L \neq \emptyset$ if and only if the following linear bracket polynomial vanishes:

$$\mathcal{R}_X \quad = \quad (a_0b_1 - a_1b_0)[01] + (a_0b_2 - a_2b_0)[02] + \cdots + (a_2b_3 - a_3b_2)[23].$$

This is the Chow form of the line X. We note that the coefficient of a primal bracket of L in $\mathcal{R}_X$ is the dual bracket of X with the same indices. This observation generalizes to arbitrary dimensions.

Proposition 1.1 *The Chow coordinates of a linear subspace X of $\mathbf{P}^n$ are the dual Plücker coordinates of X with the same indices.*

Our first non-linear example is the **twisted cubic curve**

$$X \quad = \quad \{ (s^3 : s^2t : st^2 : t^3) \in \mathbf{P}^3 \ : \ (s:t) \in \mathbf{P}^1 \}.$$

This curve is the intersection of three quadratic surfaces:

$$X \ = \ \{ (x_0 : x_1 : x_2 : x_3) \in \mathbf{P}^3 \ : \ x_0x_2 - x_1^2 = x_0x_3 - x_1x_2 = x_1x_3 - x_3^2 = 0 \}.$$

Consider a variable line L in $\mathbf{P}^3$, presented as the intersection of two planes:

$$L \quad = \quad kernel \begin{pmatrix} u_0 & u_1 & u_2 & u_3 \\ v_0 & v_1 & v_2 & v_3 \end{pmatrix}.$$

The line L meets the curve X if and only if

$$\exists\, (s:t) \in \mathbf{P}^1 \ : \ u_0s^3 + u_1s^2t + u_2st^2 + u_3t^3 \ = \ v_0s^3 + v_1s^2t + v_2st^2 + v_3t^3 \ = \ 0.$$

This condition is equivalent to the vanishing of the **Sylvester resultant**:

$$\mathcal{R}_X = \det \begin{pmatrix} u_0 & u_1 & u_2 & u_3 & 0 & 0 \\ 0 & u_0 & u_1 & u_2 & u_3 & 0 \\ 0 & 0 & u_0 & u_1 & u_2 & u_3 \\ v_0 & v_1 & v_2 & v_3 & 0 & 0 \\ 0 & v_0 & v_1 & v_2 & v_3 & 0 \\ 0 & 0 & v_0 & v_1 & v_2 & v_3 \end{pmatrix} = \det \begin{pmatrix} [01] & [02] & [03] \\ [02] & [03] + [12] & [13] \\ [03] & [13] & [23] \end{pmatrix}.$$

The 3×3-determinant in terms of brackets $[ij] = u_i v_j - u_j v_i$ is called the **Bezout resultant.** Its expansion equals

$$\begin{aligned}\mathcal{R}_X &= -[03]^3 - [03]^2[12] + 2[02][03][13] - [01][13]^2 \\ &\quad - [02]^2[23] + [01][03][23] + [01][12][23].\end{aligned}$$

Proposition 1.2 *The Chow form of the twisted cubic curve is the Bezout resultant of two cubic binary forms.*

In Section 3.2 we shall generalize this example by demonstrating a method for computing the Chow forms of parametrized varieties. These Chow forms can be viewed as "generalized resultants" (cf. [18]).

In the definition of the Chow form $\mathcal{R}_X$ it had been assumed that X is an irreducible variety. If $X = X_1 \cup X_2 \cup \cdots \cup X_r$, where X_i is irreducible and appears with "multiplicity" m_i in X, then we define the Chow form of X as

$$\mathcal{R}_X := \mathcal{R}_{X_1}^{m_1} \mathcal{R}_{X_2}^{m_2} \cdots \mathcal{R}_{X_r}^{m_r}.$$

In this situation $\mathcal{R}_X$ has coefficients in k, the field of definition of X, while the factors $\mathcal{R}_{X_i}$ have coefficients in some algebraic field extension k' of k. In particular, it is possible that the Chow form $\mathcal{R}_X$ is irreducible over k.

Here is an example with $k = \mathbf{Q}$ and $k' = \mathbf{Q}(\omega)$, where ω is a primitive cube root of unity. Consider the point set $\{(1 : \omega : \omega^2), (1 : \omega^2 : \omega)\}$ in the projective plane. It is irreducible over k. Its Chow form is $[0]^2 - [0][1] + [1]^2 - [0][2] - [1][2] + [2]^2$, which is irreducible over k but factors over k' as $([0] + \omega[1] + \omega^2[2]) \cdot ([0] + \omega^2[1] + \omega[2])$.

The Chow form of any point set $X = \{p_1, p_2, \ldots, p_r\} \subset \mathbf{P}^n$ is given by $\mathcal{R}_X = \prod_{i=1}^r \langle U, p_i \rangle$, where $U = (u_0 : u_1 : \cdots : u_n)$ is a variable hyperplane. This instance of the Chow form is also known as the **U-resultant**. It is generally harder to compute the Chow form (or U-resultant) of a point set from its defining ideal than to compute it from the coordinates of the points. In the above example, the defining ideal is $(x_0 + x_1 + x_2, x_1^2 + x_1 x_2 + x_2^2)$.

One other extreme case deserves attention. A hypersurface X in $\mathbf{P}^n$ is the zero set of a single homogeneous polynomial $F(x_0, x_1, \ldots, x_n)$. The Chow form of X is the same as F but rewritten in primal brackets:

$$\mathcal{R}_X = F([12\ldots n], -[023\ldots n], \cdots, (-1)^n[01\ldots(n-1)]).$$

For example, the Chow form of the Fermat cubic surface $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$ in $\mathbf{P}^3$ is $[123]^3 - [023]^3 + [013]^3 - [012]^3$. There are exactly twenty-seven lines lying on the Fermat cubic in $\mathbf{P}^3$. The Chow form of their union is a bracket polynomial of degree 27 with rational coefficients. It factors over $\mathbf{Q}(\omega)$ as

$$\begin{aligned}\textstyle\prod_{i=0}^2 \prod_{j=0}^2 &([13] - \omega^i[14] - \omega^j[23] + \omega^{i+j}[24]) \times \\ &\times ([12] - \omega^i[14] + \omega^j[23] + \omega^{i+j}[34]) \times \\ &\times ([12] - \omega^i[13] + \omega^j[24] - \omega^{i+j}[34]).\end{aligned}$$

Each linear factor is the Chow form of one of the 27 lines.

We close our list of examples with two other important classes of Chow forms. If the variety X is defined by monomial equations, then its Chow form $\mathcal{R}_X$ is a **bracket monomial**. Conversely, every bracket monomial arises from a monomial scheme in this way; see [18] and [23] for details. Given integers $n \geq m \geq 2$, consider the variety of all $n \times m$-matrices of rank at most $m-1$. The Chow form of this determinantal variety is the **hyperdeterminant** of format $m \times n \times (n-m+1)$; see [12] and [25].

2. Geometric Operations

In this section we show how certain basic geometric operations with projective varieties can be carried out using Chow forms. An important such operation is the intersection of varieties. The intersection operation is very difficult in general, and we treat here only some easy cases. The general case will be addressed later in Section 4 in the context of joins.

2.1. INTERSECTIONS

Let X be an irreducible projective variety and L_1 a general linear subspace. The following formula expresses the Chow form of their intersection in terms of the Chow form of X and the Plücker coordinates of L_1.

Proposition 2.1 *Let* $s = \mathrm{codim}(L_1) + \mathrm{codim}(X) - 1$. *Then*

$$\mathcal{R}_{X \cap L_1}(L_2) \quad = \quad \mathcal{R}_X(L_1 \cap L_2) \tag{2.1}$$

for all linear subspaces L_2 of dimension s which are transverse to L_1.

Proof: Both $\mathcal{R}_{X\cap L_1}(\,\cdot\,)$ and $\mathcal{R}_X(L_1 \cap \cdot\,)$ are squarefree polynomial functions on $G(s,n)$ having the same degree. To show that they are equal it suffices to note that they have the same zero set:

$$\mathcal{R}_{X\cap L_1}(L_2) = 0 \iff X \cap L_1 \cap L_2 \neq \emptyset \iff \mathcal{R}_X(L_1 \cap L_2) = 0. \quad \square$$

When applying this formula, one uses the fact that the (primal) Plücker coordinates of $L_1 \cap L_2$ are the exterior product of the Plücker coordinates of L_1 with those of L_2. For instance, consider the intersection of the twisted cubic with the hyperplane $x_0 + x_1 + x_2 + x_3 = 0$. By substituting its coefficient vector $(1,1,1,1)$ into the last row of the matrix M, we see that each rank 2 bracket $[ij]$ is to be replaced by the difference of rank 1 brackets $[i] - [j]$. Hence the Chow form of this intersection is the determinant of the matrix

$$\begin{pmatrix} [0]-[1] & [0]-[2] & [0]-[3] \\ [0]-[2] & [0]+[1]-[2]-[3] & [1]-[3] \\ [0]-[3] & [1]-[3] & [2]-[3] \end{pmatrix}.$$

This determinant factors over the complex numbers as

$$-([0]-[1]+[2]-[3])\cdot([0]+i[1]-[2]-i[3])\cdot([0]-i[1]-[2]+i[3]).$$

We conclude that the intersection is the point set

$$\{(1:-1:1:-1),(1:i:-1:-i),(1:-i:-1:i)\} \quad \subset \quad \mathbf{P}^3.$$

Consider the special case $s = n-1$, where $L := L_1$ is a complementary linear subspace to X and $U := L_2$ is a hyperplane. Then the intersection is a finite point set, say, $X \cap L = \{p_1(L), p_2(L), \ldots, p_r(L)\}$. The coordinates of the p_i are algebraic (but not rational) functions in the brackets of L. Proposition 2.1 implies the following factorization of the Chow form over a suitable field extension:

$$\mathcal{R}_X(L \cap U) \quad = \quad \prod_{i=1}^{r} \langle U, p_i(L)\rangle. \tag{2.2}$$

This formula plays an important role in the original work of Chow and van der Waerden [6]; a refinement appears in [21]. Caveat: It is generally impossible to carry out the factorization (2.2) using a computer algebra system because $\mathcal{R}_X$ is irreducible as a polynomial over k in the coordinates of U and the brackets of L.

The number r of intersection points with a generic complementary linear subspace is called the **degree** of the projective variety X.

Corollary 2.1 *The degree of $\mathcal{R}_X$ in brackets equals the degree of X.*

Proof: The bracket degree of $\mathcal{R}_X$ and $\mathcal{R}_{X\cap L_1}$ coincide by Proposition 2.1. For $s = n-1$ this degree equals $r = \deg(X)$. □

We next consider the intersection of X with a hypersurface Y.

Proposition 2.2 *If Y is a hypersurface in $\mathbf{P}^n$ defined by $Q(\mathbf{x}) = 0$, then*

$$\mathcal{R}_{X\cap Y}(L) \quad = \quad \prod_{i=1}^{r} Q(p_i(L)) \qquad \textit{for all} \quad L \in G(\mathrm{codim}(X), n).$$

Proof: Both the right hand polynomial and the left hand polynomial vanish if and only if $Y \cap X \cap L \neq \emptyset$. □

In order to evaluate this formula for the Chow form of $X \cap Y$, we need the theory of **multisymmetric functions**; see e.g. [17] or [20]. This theory allows us to compute with the point set $\{p_1(L), \ldots, p_r(L)\}$ without having to deal with the individual points. Every multisymmetric function can be represented (up to an extraneous monomial factor) as a polynomial in the elementary multisymmetric functions, which are the coefficients of (2.2) as a polynomial in U. In particular, the multisymmetric function $\prod_{i=1}^{r} Q(p_i(L))$ has such a representation in terms of the U-coefficients of (2.2). That gives the Chow form $\mathcal{R}_{X\cap Y}$ as a polynomial in the brackets of L.

The intersection of the twisted cubic X and the Fermat cubic surface Y consists of nine points. Its Chow form may be evaluated by this method. First one finds the elementary multisymmetric functions of the three intersection points $(p_{i0} : p_{i1} : p_{i2} : p_{i3})$ of the twisted cubic and a hyperplane. This amount to writing the Sylvester resultant as a polynomial in $v_0, \ldots, v_3$:

$$\mathcal{R}_X \quad = \sum_{i_0+i_1+i_2+i_3=3} e_{i_0i_1i_2i_3}(u_0,u_1,u_2,u_3)\cdot v_0^{i_0}v_1^{i_1}v_2^{i_2}v_3^{i_3} \quad = \quad \prod_{i=1}^{3}(\sum_{j=0}^{3} p_{ij}v_j).$$

We then form the following multisymmetric function of degree 9,

$$\prod_{i=1}^{3} Q(p_i) \quad = \quad \prod_{i=1}^{3}(p_{i0}^3+p_{i1}^3+p_{i2}^3+p_{i3}^3),$$

and we rewrite it as a polynomial in the elementary multisymmetric functions $e_{i_0i_1i_2i_3}(u_0,u_1,u_2,u_3)$. This can be done using the MAPLE package "ms", which was written by the first author and is available upon request. Replacing each variable u_i by a bracket $[i]$, the final result factors over $\mathbf{Q}$ as

$$\begin{aligned}
&-([0]-[1]+[2]-[3])([0]^2-2[0][2]+[1]^2+[2]^2-2[1][3]+[3]^2)\\
&\times([0]^2+[0][1]-[0][2]+[1]^2+[1][2]+[2]^2-2[0][3]-[1][3]+[2][3]+[3]^2)\\
&\times([0]^4+2[0]^3[2]-[0]^2[1]^2+3[0]^2[2]^2-4[0][1]^2[2]+2[0][2]^3+[1]^4-\\
&\quad -[1]^2[2]^2+[2]^4+2[0]^2[1][3]-4[0][1][2][3]+2[1]^3[3]-4[1][2]^2[3]+\\
&\quad +2[0]^2[3]^2+2[0][2][3]^2+3[1]^2[3]^2-[2]^2[3]^2+2[1][3]^3+[3]^4)
\end{aligned}$$

2.2. PROJECTIONS

Given a point $p \notin X$, consider the cone of X over p, which is the union of all lines through p which meet X. Its Chow form satisfies the equation

$$\mathcal{R}_{\mathrm{cone}(p,X)}(L) \quad = \quad \mathcal{R}_X(\mathrm{span}(p,L)) \qquad (2.3)$$

for linear subspaces L with $\dim(L)+\dim(X) = n-2$. To prove (2.3) it suffices to observe that $\mathrm{cone}(p,X)\cap L \neq \emptyset$ if and only if $X\cap\mathrm{span}(p,L) \neq \emptyset$. Indeed, both conditions are equivalent to the existence of points $x \in X$ and $l \in L$ such that x, p, and l lie on a line.

We can compute the Plücker coordinates of span(p, L) using the formula

$$[i\,j\ldots k] \quad = \quad \sum_{l=0}^{n} p_l\cdot[\,l\,i\,j\ldots k],$$

where the bracket on the left-hand side belongs to $\mathrm{span}(p, L)$ and the bracket on the right-hand side belongs to L. The proof is left as an exercise for the reader. (Expand the dual bracket with respect to p, then dualize to get the formula for primal brackets.) For instance, the Chow form of the cone of the twisted cubic over $(0:1:1:0)$ is the determinant of the matrix

$$\begin{pmatrix} [012] & -[012] & -[013]-[023] \\ -[012] & -[013]-[023] & -[123] \\ -[013]-[023] & -[123] & [123] \end{pmatrix} \tag{2.4}$$

which is $([013]+[023])^3-[012][123]([012]+3[013]+3[023]+[123])$, the Chow form of the surface $(x_1-x_2)^3-x_0x_3(x_0-3x_1+3x_2-x_3)=0$.

Given a point $p \notin X$ and a hyperplane H not containing p, then the projection of X onto H with center p is the variety $Y := H \cap \mathrm{cone}(p, X)$. Thus the Chow form $\mathcal{R}_Y$ of such a projection can be computed from $\mathcal{R}_X$ by combining formulas (2.3) and (2.1).

For instance, let Y be the projection of the twisted cubic curve X onto the plane $H = \{x_0+x_1+x_2+x_3=0\}$ with center at $p=(0:1:1:0)$. Thus Y is a planar cubic curve. Viewed as a curve in $\mathbf{P}^3$, its Chow form is gotten from the determinant of (2.4) by the substitutions

$$[ijk] \mapsto [ij]-[ik]+[jk], \qquad (0 \le i < j < k \le 3).$$

This corresponds to substituting the vector $(1,1,1,1)$ into the last row of the matrix M, whose kernel is $\mathrm{span}(p, L)$.

Consider the isomorphism from the plane $H \subset \mathbf{P}^3$ onto $\mathbf{P}^2$ given by the map $(x_0-x_1 : x_1-x_2 : x_2-x_3)$. We can calculate the brackets in $\mathbf{P}^3$ directly in terms of the brackets in $\mathbf{P}^2$ by examining the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ m_{10} & -m_{10}+m_{11} & -m_{11}+m_{12} & -m_{12} \\ m_{20} & -m_{20}+m_{21} & -m_{21}+m_{22} & -m_{22} \end{pmatrix}.$$

The 2×2 minors of the bottom rows are $[01]$, $-[01]+[02]$, $-[02]$, $[01]-[02]+[12]$, $[02]-[12]$, and $[12]$, so the 3×3 minors are $[012]=3[01]-2[02]+[12]$, $[013]=[01]+2[02]-[12]$, $[023]=-[01]+2[02]+[12]$, and $[123]=[01]-2[02]+3[12]$. We substitute these into the Chow form of the cone over the twisted cubic to obtain the Chow form of Y as a cubic curve in $\mathbf{P}^2$:

$$\begin{aligned} -4(3[01]^3 &- 2[01]^2[02] + 13[01]^2[12] - 12[01][02]^2 + 4[01][02][12] \\ &+ 13[01][12]^2 - 8[02]^3 - 12[02]^2[12] - 2[02][12]^2 + 3[12]^3). \end{aligned}$$

This is the Chow form of the planar cubic curve defined by

$$3x_0^3+2x_0^2x_1+13x_0^2x_2-12x_0x_1^2-4x_0x_1x_2+13x_0x_2^2+8x_1^3-12x_1^2x_2+2x_1x_2^2+3x_2^3.$$

2.3. LINEAR TRANSFORMATIONS AND TORIC DEFORMATIONS

The group $GL(n+1,k)$ acts naturally on the vector space k^{n+1} and its projectivization $\mathbf{P}^n$. In particular, for each variety $X \subset \mathbf{P}^n$ and each linear transformation $g \in GL(n+1,k)$, the image $g(X)$ is a projective variety isomorphic to X. There is an induced action on the vector space $(\bigwedge^{n-d}(k^{n+1})^*)^* = \bigwedge^{n-d} k^{n+1}$, which has as its basis the set of functions that take an $(n-d)$-dimensional linear subspace to one of its brackets. This action extends to the projectivization as well. Clearly, the Grassmannian of $(n-d-1)$-dimensional subspaces of $\mathbf{P}^n$ is invariant under this action. Similarly, $GL(n+1,k)$ acts naturally on $S^{\deg X} \bigwedge^{n-d}(k^{n+1})$, the space of bracket polynomials of degree the degree of X. The Chow form $\mathcal{R}_X$ is a point in this vector space, and we write $g \circ \mathcal{R}_X$ for the image of this point under g. It turns out that passing to the Chow form is an equivariant operation:

Proposition 2.3 *If X is any projective variety in $\mathbf{P}^n$ and g any linear transformation in $GL(n+1,k)$, then $\mathcal{R}_{g(X)} = g \circ \mathcal{R}_X$.*

Proof: We have the following identity of subsets in the Grassmannian:

$$\{L : gX \cap L \neq \emptyset\} = \{L : X \cap g^{-1}L \neq \emptyset\} = g \circ \{L' : X \cap L' \neq \emptyset\}. \quad \square$$

To compute the action of an $(n+1)\times(n+1)$-matrix g on the brackets, one takes the transpose of the matrix of its $(n-d)\times(n-d)$ minors. Its action on the Chow form of X can be computed by acting on the brackets and distributing the products. For instance, consider the action of the matrix

$$g = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

on the Chow form of the twisted cubic. Its matrix of minors is

$$\wedge^2 g = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore, the image of the Chow form of the twisted cubic curve is

$$\det \begin{pmatrix} [01] & [01]+[02] & [02]+[03] \\ [01]+[02] & [01]+2[02]+[03]+[12] & [02]+[03]+[12]+[13] \\ [02]+[03] & [02]+[03]+[12]+[13] & [12]+[13]+[23] \end{pmatrix}.$$

The matrix g acts on $\mathbf{P}^3$ by sending the point $(x_0 : x_1 : x_2 : x_3)$ to the point $(x_0+x_1 : x_1+x_2 : x_2+x_3 : x_3)$, so the determinant above is the Chow form of the curve parametrized by $(x_0^3 + x_0^2x_1 : x_0^2x_1 + x_0x_1^2 : x_0x_1^2 + x_1^3 : x_1^3)$.

An important subgroup of $GL(n+1,k)$ is the group $(k^*)^{n+1}$ of invertible diagonal matrices. This abelian group is called the $(n+1)$-dimensional **(algebraic) torus.** Given a lattice vector $w = (w_0, w_1, \ldots, w_n) \in \mathbf{Z}^{n+1}$ and a scalar $t \in k^*$, we define the diagonal matrix

$$w(t) \quad := \quad \mathrm{diag}(t^{w_0}, t^{w_1}, \ldots, t^{w_n}).$$

The map $k^* \to (k^*)^{n+1}$, $t \mapsto w(t)$ is a one-parameter subgroup of the torus.

The image $w(t) \circ \mathcal{R}_X = \mathcal{R}_{w(t)X}$ of the Chow form of X under the diagonal matrix $w(t)$ is computed by scaling each bracket as follows:

$$[i_0 i_1 \ldots i_d] \quad \mapsto \quad t^{w_{i_0}+w_{i_1}+\cdots+w_{i_d}} \cdot [i_0 i_1 \ldots i_d]. \tag{2.5}$$

In terms of the vanishing ideals of X and $w(t) \circ X$, this action looks like

$$\mathcal{I}(w(t) \circ X) \quad = \quad \langle\, f(x_0 t^{-w_0}, x_1 t^{-w_1}, \ldots, x_n t^{-w_n}) \;:\; f \in \mathcal{I}(X) \,\rangle. \tag{2.6}$$

We can now pass to the limit as t goes to infinity to get the **toric deformation** $lim_{t\to\infty} w(t)X$. This means that in (2.6) we pass to the initial ideal $in_{-w}(I(X))$ with respect to the weights $-w$. Algebraically, this amounts to a Gröbner basis computation. On the level of Chow forms, the toric deformation is simply given as the leading form of $\mathcal{R}_{w(t)X}$ with respect to the weights (2.5). For details and applications of toric deformations of Chow forms see [13],[18],[23]. A related object introduced in [18] is the **Chow polytope** associated with variety X. This is a convex polytope whose faces are in bijection with all possible toric deformations of the Chow form $\mathcal{R}_X$.

For example, the initial ideal of the twisted cubic curve X with respect to the $-w = (3,1,0,0)$ equals $(x_0x_2, x_0x_3, x_1x_3) = (x_0,x_1) \cap (x_0,x_3) \cap (x_2,x_3)$. The leading form of the Chow form $\mathcal{R}_X$ with respect to $w = (-3,-1,0,0)$ is the bracket monomial [01][12][23]. This is the Chow form of three coordinate lines in $\mathbf{P}^3$. We conclude that the one-parameter subgroup $w(t)$ effects a toric deformation of the twisted cubic into a union of three distinct lines.

3. Computational Aspects

Projective varieties appearing in computational problems are usually represented "implicitly" as the solution set of a system of polynomial equations, or "parametrically" as the image of a polynomial mapping. While the implicit representation exists for all varieties, only few varieties admit a rational parametrization. It is a standard problem in computer algebra to pass back and forth between both representations.

In this section we present algorithms for the transition between the Chow form of a projective variety X and its implicit and parametric representations. We assume that the reader is familiar with computational algebraic geometry at the level of the text book by Cox-Little-O'Shea [7].

3.1. FROM EQUATIONS TO THE CHOW FORM

We suppose that X is an irreducible projective variety, which is presented by a finite set of generators for the corresponding homogeneous prime ideal $I = \mathcal{I}(X)$ in $k[x_0, x_1, \ldots, x_n]$. The following algorithm computes the Chow form $\mathcal{R}_X$ in primary Plücker coordinates.

`Step 0`: Compute $d = \dim(X)$, e.g., by computing a Gröbner basis for I.
`Step 1`: Add $d+1$ linear forms $\ell_i = u_{i0}x_0 + u_{i1}x_1 + \cdots + u_{in}x_n$ with indeterminate coefficients, and consider the ideal

$$J \; = \; I + (\ell_0, \ell_1, \ldots, \ell_d) \;\subset\; k[\,x_i, u_{ji} : i = 0, \ldots, d, j = 0, \ldots, n\,] \; =: \; S.$$

`Step 2`: Replace J by the ideal

$$J' \quad := \quad (J : (x_0, \ldots, x_n)^\infty) \quad = \quad \{\, f \in S \mid \forall i \, \exists d_i : x_i^{d_i} f \in J \,\}.$$

This is the **saturation** of J with respect to the irrelevant maximal ideal in $k[x_0, \ldots, x_n]$. It can be computed using Gröbner bases; see pp. 195-6 in [7].
`Step 3`: Compute the elimination ideal $J' \cap k[u_{00}, u_{01}, \ldots, u_{dn}]$. This ideal is principal; let $R(u_{00}, \ldots, u_{dn})$ be its principal generator.
`Step 4`: Rewrite the polynomial R in terms of brackets $[j_0 j_1 \ldots j_d]$, e.g. by using the straightening algorithm or the subduction technique presented in Algorithm 3.2.8 of [24]. The result is the Chow form $\mathcal{R}_X$.

Here is the geometric interpretation of this algorithm: In step 1 we form the natural incidence correspondence $\{(\mathbf{x}, L) : \mathbf{x} \in X \cap L\}$ between (the cone over) X and (the cone over) the Grassmannian. In step 2 we remove trivial solutions with all x-coordinates zero, for which there is no point in $\mathbf{P}^n$. In steps 3 and 4 we project the incidence correspondence onto the Grassmannian.

The same algorithm works not only for prime ideals but for all unmixed homogeneous ideals. As an example consider the ideal $I = (x_1^2 - x_0x_2, x_2^2 - x_1x_3)$. Its variety consists of the twisted cubic and the line $x_1 = x_2 = 0$.

To this ideal we add the generic linear forms $u_{00}x_0 + u_{01}x_1 + u_{02}x_2 + u_{03}x_3$ and $u_{10}x_0 + u_{11}x_1 + u_{12}x_2 + u_{13}x_3$. We saturate with respect to the maximal ideal (x_0, x_1, x_2, x_3) and eliminate the x-variables. The result is the resultant of the two given quadrics and the two general linear forms, which is a homogeneous polynomial R of bidegree $(4,4)$ in the u_{ij}. To obtain a polynomial in brackets, we may take the ideal generated by the previous result R together with the polynomials $u_{0i}u_{1j} - u_{0j}u_{1i} - [ij]$ for $0 \leq i < j \leq 3$, and eliminate the u-variables. The output is the Bezout resultant (see §1.2) times the Chow form of the line $x_1 = x_2 = 0$, which is $[03]$. Naturally, this factorization may not be apparent in the output, because all bracket polynomials are to be understood modulo the syzygy $[01][23] - [02][13] + [03][12]$.

A complexity analysis of our problem has been undertaken by Caniglia in [3]. Using a somewhat different method, he shows "how to compute the Chow form of an unmixed polynomial ideal in single exponential time".

3.2. FROM A PARAMETRIZATION TO THE CHOW FORM

Suppose we are given a parametrically presented projective variety:

$$X = \{ (f_0(\mathbf{t}) : f_1(\mathbf{t}) : \cdots : f_n(\mathbf{t})) \in \mathbf{P}^n \ : \ \mathbf{t} \in \mathbf{P}^d \}.$$

The input data $f_0, f_1, \ldots, f_n$ are homogeneous polynomials of the same degree in the variables $\mathbf{t} = (t_0, t_1, \ldots, t_d)$. The following algorithm computes the Chow form $\mathcal{R}_X$. For simplicity we assume $\dim(X) = d$.

`Step 1:` Consider the ideal

$$J = \langle u_{i0}f_0(\mathbf{t}) + u_{i1}f_1(\mathbf{t}) + \cdots + u_{in}f_n(\mathbf{t}) \ : \ i = 0, 1, \ldots, d \rangle$$

in $k[t_i, u_{ij} \ : i = 0, \ldots, d, j = 0, \ldots, n]$.

`Step 2:` Replace J by $J' = (J : (f_0, \ldots, f_n)^\infty)$, its saturation with respect to the base point locus of the parametrization.

`Step 3,4:` Same as in Section 3.1.

As an example we consider the parametrization $(f_0 : f_1 : f_2 : f_3)$ of the Fermat cubic surface, where

$$\begin{aligned}
f_0(\mathbf{t}) &= \omega t_1^2 t_2 - \omega t_0 t_2^2 - t_0^2 t_1 + t_1^2 t_2, \\
f_1(\mathbf{t}) &= \omega t_1^2 t_2 - \omega t_0 t_2^2 + t_0^2 t_1 - t_0 t_2^2, \\
f_2(\mathbf{t}) &= \omega t_0 t_1^2 - \omega t_1 t_2^2 + t_0 t_1^2 - t_0^2 t_2, \\
f_3(\mathbf{t}) &= \omega t_0 t_1^2 - \omega t_1 t_2^2 + t_0^2 t_2 - t_1 t_2^2.
\end{aligned}$$

We take the ideal generated by the three polynomials $u_{00}f_0+u_{01}f_1+u_{02}f_2+u_{03}f_3$, $u_{10}f_0+u_{11}f_1+u_{12}f_2+u_{13}f_3$, and $u_{20}f_0+u_{21}f_1+u_{22}f_2+u_{23}f_3$, we saturate it with respect to the ideal (f_0, f_1, f_2, f_3), and then we eliminate the **t**-variables. As the result we get the expansion of the Chow form $\mathcal{R}_X = [123]^3 - [023]^3 + [013]^3 - [012]^3$ into coordinates u_{ij}.

An important special case occurs when $f_0, \ldots, f_n$ are monomials of the same degree, say, $f_i(\mathbf{t}) = \mathbf{t}^{a_i}$, where $\mathcal{A} = \{a_0, a_1, \ldots, a_n\} \subset \mathrm{N}^{d+1}$. The variety $X_\mathcal{A} = \{(\mathbf{t}^{a_0} : \cdots : \mathbf{t}^{a_n}) \in \mathbf{P}^n : \mathbf{t} \in \mathbf{P}^d\}$ is called the toric variety defined by the set $\mathcal{A}$. Its Chow form $\mathcal{R}_{X_\mathcal{A}}$ is the **A-resultant**, which plays a central role in sparse elimination theory; see [23],[13]. For example, if $\mathcal{A} = \{(3,0),(2,1),(1,2),(0,3)\}$ then the toric variety $X_\mathcal{A}$ is the twisted cubic curve and the $\mathcal{A}$-resultant is the resultant of two binary cubic forms.

3.3. FROM THE CHOW FORM TO EQUATIONS

Given the Chow form of a projective variety X, we can recover a set of equations that define X set-theoretically, by virtue of the following result.

Proposition 3.1 *A point $p \in \mathbf{P}^n$ lies in X if and only if $X \cap \mathrm{span}(p, L) \neq \emptyset$ for all $(n-d-2)$-dimensional linear subspaces L.*

Proof: We must prove the "if"-direction. Suppose $X \cap \mathrm{span}(p, L) \neq \emptyset$ for a generic subspace $L \in G(n-d-2, n)$. Project from p onto a hyperplane H. Then the image of $X \cap \mathrm{span}(p, L)$ under the projection is the intersection of the images of X and L. But this intersection is empty because the image of L is generic and has codimension $d+1$ in H. Since $X \cap \mathrm{span}(p, L)$ is nonempty, it must be $\{p\}$. □

For each fixed $L \in G(n-d-2, n)$, the expression $\mathcal{R}_X(\mathrm{span}(p, L))$ is a polynomial in $p = (p_0, p_1, \ldots, p_n)$ of degree $\deg(X)$. These polynomials are called the **Chow equations**. By Proposition 3.1, they define X set-theoretically. We may also replace the Chow equations by the following finite set of equivalent equations: Expand $\mathcal{R}_X(\mathrm{span}(p, L))$ as a polynomial in Plücker coordinates of L. The coefficients are polynomials in p of degree $\deg(X)$. They span the same k-vector space as the Chow equations, and hence they define the same variety, namely X.

In practice this computation is best carried out by first rewriting $\mathcal{R}_X$ in terms of dual Plücker coordinates. For instance, if $\mathcal{R}_X$ is the Chow form of a curve in $\mathbf{P}^3$, given in (primal) brackets as before, then we convert them to dual brackets by replacing $([01],[02],[03],[12],[13],[23])$ by $([[23]], -[[13]], [[12]], [[03]], -[[02]], [[01]])$. Thereafter we substitute $x_iy_j - x_jy_i$ for each dual bracket $[[ij]]$ and expand the result as a polynomial in the y-variables. The coefficient polynomials in the x-variables are the Chow equations for X.

The Chow form of the twisted cubic curve in **dual** brackets equals

$$\det \begin{pmatrix} [[23]] & -[[13]] & [[12]] \\ -[[13]] & [[12]] + [[03]] & -[[02]] \\ [[12]] & -[[02]] & [[01]] \end{pmatrix}.$$

After the substitutions $[[ij]] \mapsto x_i y_j - x_j y_i$ this equals

$$\begin{gathered} (x_3 x_2^2 - x_3^2 x_1) y_0^2 y_2 + (x_2 x_3 x_1 - x_2^3) y_0^2 y_3 + (x_3^2 x_1 - x_3 x_2^2) y_0 y_1^2 \\ + (x_3^2 x_0 - x_2 x_3 x_1) y_0 y_1 y_2 + (3 x_2^2 x_1 - 2 x_3 x_1^2 - x_2 x_3 x_0) y_0 y_1 y_3 \\ + (x_3 x_1 x_0 + 2 x_2^2 x_0 - 3 x_2 x_1^2) y_0 y_2 y_3 + (x_1^3 - x_2 x_1 x_0) y_0 y_3^2 \\ + (x_2^3 - x_3^2 x_0) y_1^3 + (3 x_2 x_3 x_0 - 3 x_2^2 x_1) y_1^2 y_2 + (2 x_3 x_1 x_0 - 2 x_2^2 x_0) y_1^2 y_3 \\ + (3 x_2 x_1^2 - 3 x_3 x_1 x_0) y_1 y_2^2 + (x_2 x_1 x_0 - x_3 x_0^2) y_1 y_2 y_3 + (x_2 x_0^2 - x_1^2 x_0) y_1 y_3^2 \\ + (2 x_3 x_1^2 - 2 x_2 x_3 x_0) y_0 y_2^2 + (x_3 x_0^2 - x_1^3) y_2^3 + (x_1^2 x_0 - x_2 x_0^2) y_2^2 y_3 \end{gathered}$$

The coefficients are defining polynomials for the twisted cubic curve.

A natural question for an algebraic geometer is whether the Chow equations actually define their variety X scheme-theoretically. The following beautiful answer has been given by Fabrizio Catanese in [4]:

Theorem 3.1 *The Chow equations of an irreducible projective variety X define X scheme-theoretically if and only if X is smooth or a hypersurface.*

We invite the reader to verify that the ideal of the twisted cubic is the saturation of the Chow equations with respect to the ideal (x_0, x_1, x_2, x_3).

4. The Join of Two Varieties

In this section we address the problem of computing the ruled join of two projective varieties in terms of Chow forms. This material is drawn from the forthcoming Ph. D. thesis of the first author.

4.1. DEFINITION OF THE JOIN MAP

Let two varieties $X \subset \mathbf{P}^m$ and $Y \subset \mathbf{P}^n$ be given. We embed $\mathbf{P}^m$ and $\mathbf{P}^n$ into $\mathbf{P}^{m+n+1}$ by sending the point $(x_0 : \cdots : x_m)$ in $\mathbf{P}^m$ to the point $(x_0 : \cdots : x_m : 0 : \cdots : 0)$ in $\mathbf{P}^{m+n+1}$ and sending the point $(x_0 : \cdots : x_n)$ in $\mathbf{P}^n$ to the point $(0 : \cdots : 0 : x_0 : \cdots : x_n)$. The **ruled join** $X \# Y$ is the union of all lines in $\mathbf{P}^{m+n+1}$ joining a point of X to a point of Y. The ideal of equations defining $X \# Y$ is simply the union of the equations for X and the equations for Y, taken in disjoint sets of variables.

This construction appears in intersection theory as follows. The ruled join of a projective space $\mathbf{P}^n$ with itself has a diagonal subspace, defined by $x_0 = x_{n+1}, x_1 = x_{n+2}, \ldots, x_n = x_{2n+1}$. Intersecting the ruled join of two subvarieties X and Y of $\mathbf{P}^n$ with this subspace gives the intersection $X \cap Y$

of the two varieties. This method of computing intersections is classical, but it appears frequently in the current literature (see e.g. [8],[10]). It is important for us because, once we have computed the Chow form $\mathcal{R}_{X\#Y}$ of the join, then we can find the Chow form $\mathcal{R}_{X\cap Y}$ of $X \cap Y$ by simply intersecting with the diagonal subspace as in Proposition 2.1.

The Chow forms of all unions (with "multiplicity") of subvarieties of $\mathbf{P}^n$ of dimension d and degree r form a subvariety of $\mathbf{P}(S^r \bigwedge^{n-d} k^{n+1})$, called the **Chow variety** $C(n,d,r)$. In general, the Chow variety is singular, reducible, and of mixed dimension (see [22] for a first introduction to Chow varieties). We can define a map

$$\# : C(m,d,r) \times C(n,e,s) \to C(m+n+1, d+e+1, rs)$$

that takes a pair of Chow forms $(\mathcal{R}_X, \mathcal{R}_Y)$ to the Chow form $\mathcal{R}_{X\#Y}$ of the join of the corresponding varieties. Is this map regular? This question was posed by Friedlander and Mazur in [9] (they show that the map is continuous and that its graph is an algebraic variety) and answered in the affirmative by Barlet in [1]. Barlet's proof is complex-analytic, and the first author has been trying to discover an algebraic proof. We will describe an algorithm to compute the Chow form of the ruled join that uses only algebraic functions. Using this algorithm, it can be shown that the join map is rational in characteristic $p > rs$. We hope that this method can be used to extend Barlet's result to large prime characteristic also.

4.2. COMPUTING THE CHOW FORM OF THE JOIN

We first give a geometric description of the algorithm:

Algorithm 4.1 *JoinChow*$(\mathcal{R}_X, \mathcal{R}_Y, m, n, d, e, r, s)$

Input:
$\mathcal{R}_X$ is the Chow form of $X \subset \mathbf{P}^m$; $d = \dim X$; $r = \deg X$.
$\mathcal{R}_Y$ is the Chow form of $Y \subset \mathbf{P}^n$; $e = \dim Y$; $s = \deg Y$.
Output:
JoinChow$(\mathcal{R}_X, \mathcal{R}_Y, m, n, d, e, r, s) = \mathcal{R}_{X\#Y}$, the Chow form of the join.
Step 0: If $d = 0$ then $X = \{p_1, \ldots, p_r\}$. In this case, let H be a generic hyperplane in $\mathbf{P}^m$, compute $p_1, \ldots, p_r$ by factoring

$$\mathcal{R}_X(H) = \prod_{i=1}^{r} \langle H, p_i \rangle$$

over an algebraic extension of k, and output $\prod_{i=1}^{r} \mathcal{R}_{\mathrm{cone}(p_i,Y)}$. We are done.
Step 1: We may now assume that $d > 0$. Let H be a generic hyperplane in $\mathbf{P}^m$ and compute

$$\mathcal{R}_{(X\cap H)\#Y} = \mathrm{JoinChow}(\mathcal{R}_{X\cap H}, \mathcal{R}_Y, m, n, d-1, e, r, s).$$

Step 2: Let L be a generic $(m+n-d-e)$-dimensional linear subspace of $\mathbf{P}^{m+n+1}$. Then $(X\#Y)\cap L = \{p_1(L),\ldots,p_{rs}(L)\}$, and

$$\mathcal{R}_{X\#Y}\langle H'\cap L\rangle \;=\; \mathcal{R}_{(X\#Y)\cap L}(H') \;=\; \prod_{i=1}^{rs}\langle H', p_i(L)\rangle\,,$$

where H' is a hyperplane in $\mathbf{P}^{m+n+1}$.

Step 3: For $1\le i\le rs$ let $q_i(L)$ be the projection of $p_i(L)$ onto its first $m+1$ coordinates. Compute $q_1(L),\ldots,q_{rs}(L)$ by factoring

$$\mathcal{R}_{(X\cap H)\#Y}(L) = \prod_{i=1}^{rs}\langle H, q_i(L)\rangle$$

over an algebraic extension field k' of $k(L)$. Here H is a hyperplane in $\mathbf{P}^m$.

Proposition 4.1 *Using notation as above, we have*

$$\{p_i(L)\} \;=\; \mathrm{cone}(q_i(L),Y)\cap L \qquad \textit{for}\quad 1\le i\le rs.$$

Proof: We know that

$$p_i(L)\in (q_i(L)\#\mathbf{P}^n)\cap(X\#Y)\cap L = \mathrm{cone}(q_i(L),Y)\cap L.$$

Conversely let $p\in\mathrm{cone}(q_i(L),Y)\cap L$. Since $p\in(X\#Y)\cap L$, there exists an index j such that $p=p_j(L)$. Since L is generic, the points $q_1(L),\ldots,q_{rs}(L)$ are distinct. Therefore, since $p\in q_i(L)\#\mathbf{P}^n$, we must have $j=i$. □

For $i=1$ to rs do

Step 4: Unfortunately, the dimension of L is too small to allow us to calculate the Chow form of $\{p_i(L)\}$ by using formulas (2.1) and (2.3). Therefore, let L' be an $(m+n-e)$-dimensional linear subspace of $\mathbf{P}^{m+n+1}$, generic modulo the condition $L\subset L'$. Then $\mathrm{cone}(q_i(L),Y)\cap L' = \{t_{i1}(L,L'),\ldots,t_{is}(L,L')\}$, and we can compute $\mathcal{R}_{\mathrm{cone}(q_i(L),Y)\cap L'}$ using formulas (2.1) and (2.3).

Step 5: Compute the points $t_{i1},\ldots,t_{is}$ by factoring

$$\mathcal{R}_{\mathrm{cone}(q_i(L),Y)\cap L'}(H') \;=\; \prod_{j=1}^{s}\langle H', t_{ij}(L,L')\rangle$$

over an algebraic extension k'' of $k(L,L')$, where H' is a generic hyperplane in $\mathbf{P}^{m+n+1}$. Since L' is generic, the points $t_{i1},\ldots,t_{is}$ are distinct. By Proposition 4.1, p_i is the unique point t_{ij} that lies on L. Thus we have computed p_i.

Step 6: Now that we know the points $p_1(L), \dots, p_{rs}(L)$, we can compute

$$\mathcal{R}_{X\#Y}(H' \cap L) = \prod_{i=1}^{rs} \langle H', p_i(L) \rangle ,$$

where H' is a generic hyperplane in $\mathbf{P}^{m+n+1}$. Express $\mathcal{R}_{X\#Y}$ in terms of the Plücker coordinates of $H' \cap L$. Output the result. □

Since the factorizations in steps 3 and 5 must be performed over the algebraic extensions k' and k'' of function fields over the ground field k, most computer algebra programs will be unable to produce a factorization. In this case, it is necessary to perform computations over the ground field instead. We will give another description of the join algorithm which explains how to simulate the factorizations and calculations in the first version by performing calculations in polynomial rings over the ground field only.

Algorithm 4.2 *JoinChow*$(\mathcal{R}_X, \mathcal{R}_Y, m, n, d, e, r, s)$

Input and output are unchanged.
Step 0: If $d = 0$ then let $p_1, \dots, p_r$ be points in $\mathbf{P}^m$ with indeterminate coordinates, rewrite $\prod_{i=1}^{r} \mathcal{R}_{\mathrm{cone}(p_i,Y)}$ in terms of the elementary multisymmetric functions of the points $p_1, \dots, p_r$, and substitute the coefficients of $\mathcal{R}_X$ for the elementary multisymmetric functions of the points $p_1, \dots, p_r$. Output the result. We are done.
Step 1: We may now assume that $d > 0$. Let H be a generic hyperplane in $\mathbf{P}^m$, compute $\mathcal{R}_{X\cap H}$ using Proposition 2.1, and compute recursively

$$\mathcal{R}_{(X\cap H)\#Y} = \mathrm{JoinChow}(\mathcal{R}_{X\cap H}, \mathcal{R}_Y, m, n, d-1, e, r, s).$$

Step 2: Let L be an $(m+n-d-e)$-dimensional linear subspace of $\mathbf{P}^{m+n+1}$ with indeterminate Plücker coordinates.
Step 3: Let q be a point in $\mathbf{P}^m$ with indeterminate coordinates.
Step 4: Let L' be an $(m+n-e)$-dimensional linear subspace of $\mathbf{P}^{m+n+1}$ with indeterminate Plücker coordinates, subject to condition $L \subset L'$. This inclusion translates into a set of bilinear polynomial equations in the Plükker coordinates of L and L'. These equations can be precomputed from the values of m, n, d, and e. Compute $\mathcal{R}_{\mathrm{cone}(q,Y)\cap L'}$ by the methods of §2.
Step 5: Let H' be a generic hyperplane in $\mathbf{P}^{m+n+1}$, and let p be a point in $\mathbf{P}^{m+n+1}$ with coordinates subject to the conditions that $p \in L$, which gives bilinear equations in the coordinates of p and L, and $p \in \mathrm{cone}(q, Y) \cap L'$, which we represent by a system of equations constructed as follows: Let f be a polynomial with indeterminate coefficients that has degree $s-1$ in the coefficients of H'; set

$$\mathcal{R}_{\mathrm{cone}(q,Y)\cap L'}(H') = \langle H', p \rangle * f(H'); \tag{4.1}$$

take coefficients of both sides as polynomials in H'; and eliminate the coefficients of f from the ideal generated by the coefficient polynomials. Together the two conditions imply that $p \in \text{cone}(q, Y) \cap L$, which means that, if $q = q_i$, then $p = p_i$. By rescaling p, we may assume that its first $m + 1$ coordinates are the same as the $m + 1$ coordinates of q. This is appropriate because q is the projection of p onto $\mathbf{P}^m$.

We can partially precompute the relations coming from equation 1 from the values of m, n, and s by replacing the left hand side of equation 1 by the Chow form of a generic point set of degree s, which is a degree s polynomial in H' with indeterminate coefficients. Then we can substitute the coefficients of $\mathcal{R}_{\text{cone}(q,Y)\cap L'}$ in place of the indeterminate coefficients. The coordinates of p can now be expressed as rational functions of q and L. (This requires a Galois theory argument using the fact that p does not depend on the choice of L'.) We write $p = p(q, L)$.

`Step 6:` Let $q_1, \ldots, q_{rs}$ be points in $\mathbf{P}^m$ with indeterminate coordinates. Rewrite

$$\prod_{i=1}^{rs} \langle H', p(q_i, L) \rangle$$

in terms of the elementary multisymmetric functions of these points.

`Step 3':` Substitute the coefficients of $\mathcal{R}_{(X\cap H)\#Y}(L)$ as a polynomial in H for the elementary multisymmetric functions of the points $q_1, \ldots, q_{rs}$ in the previous result.

`Step 6':` Express the result in terms of the Plücker coordinates of $H' \cap L$. Output the resulting bracket polynomial. □

The algebraic description of the algorithm, while shorter, tends to bury its geometric content. However, this description is much more useful for the purpose of actually computing the join in this manner. In principle, the algorithm can be used to compute the join map for any m, n, d, e, r, and s. As it stands, this algorithm is not very practical for actually computing the join maps, but, hopefully, it will shed some light on their properties.

4.3. APPLYING THE ALGORITHM TO AN EXAMPLE

We can compute the Chow form of the join of two special plane conics (imaginary circles) by this method. The input data are:

$$\begin{array}{ccccl} \mathcal{R}_X & = & \mathcal{R}_Y & = & [01]^2 + [02]^2 + [12]^2, \\ m & = & n & = & 2, \\ d & = & e & = & 1, \quad \text{and} \\ r & = & s & = & 2. \end{array}$$

Since $d > 0$, we skip step 0 and proceed to step 1.

`Step 1:` Letting $H = V(h_0x_0 + h_1x_1 + h_2x_2)$, we have

$$\begin{aligned}\mathcal{R}_{X\cap H} &= (h_0[1] - h_1[0])^2 + (h_0[2] - h_2[0])^2 + (h_1[2] - h_2[1])^2 \\ &= (h_1^2 + h_2^2)[0]^2 - 2h_0h_1[0][1] + (h_0^2 + h_2^2)[1]^2 - 2h_0h_2[0][2] - \\ &\quad - 2h_1h_2[1][2] + (h_0^2 + h_1^2)[2]^2.\end{aligned}$$

We now call the algorithm recursively to compute $\mathcal{R}_{(X\cap H)\#Y}$.

`Step 0:` Rewrite the Chow form of Y using new indices: $\mathcal{R}_Y = [34]^2 + [35]^2 + [45]^2$. Given a point $x = (x_0 : x_1 : x_2)$ in $\mathbf{P}^2$, we have

$$\begin{aligned}\mathcal{R}_{\mathrm{cone}(x,Y)} &= (x_0[034] + x_1[134] + x_2[234])^2 + \\ &+ (x_0[035] + x_1[135] + x_2[235])^2 + \\ &+ (x_0[045] + x_1[145] + x_2[245])^2.\end{aligned}$$

We now consider two indeterminate points p_1 and p_2 in $\mathbf{P}^2$, and we rewrite the expression $\mathcal{R}_{\mathrm{cone}(p_1,Y)}\mathcal{R}_{\mathrm{cone}(p_2,Y)}$ in terms of the elementary multisymmetric functions of p_1 and p_2. We then substitute the coefficients of $\mathcal{R}_{X\cap H}$ for the elementary multisymmetric functions of p_1 and p_2. This gives us

$$\mathcal{R}_{(X\cap H)\#Y} = h_0^4[134]^4 + 2h_0^4[134]^2[135]^2 + h_0^4[135]^4 + (420 \text{ more terms}).$$

`Steps 2, 3, 4:` Let L be a 2-dimensional linear subspace of $\mathbf{P}^5$ with indeterminate Plücker coordinates $(l_{012} : l_{013} : \cdots : l_{345})$. Let q be a point in $\mathbf{P}^2$ with indeterminate coefficients $(q_0 : q_1 : q_2)$. Let L' be a 3-dimensional linear subspace of $\mathbf{P}^5$ with indeterminate Plücker coordinates $(l'_{01} : l'_{02} : \cdots : l'_{45})$ subject to the quadratic relations induced by $L \subset L'$. We have already computed $\mathcal{R}_{\mathrm{cone}(x,Y)}$, so to compute $\mathcal{R}_{\mathrm{cone}(q,Y)\cap L'}$, we simply perform the substitutions $x_i \mapsto q_i$ and $[ijk] \mapsto l'_{jk}[i] - l'_{ik}[j] + l'_{ij}[k]$.

`Step 5:` Let $H' = (h'_0 : \cdots : h'_5)$ be an indeterminate hyperplane in $\mathbf{P}^5$. Let v_i, e_{ij}, $0 \le i \le j \le 5$, be indeterminates. Let $p = (p_0 : \cdots : p_5)$ be an indeterminate point in $\mathbf{P}^5$ subject to the 15 bilinear relations $\sum_{k=0}^{5} l_{ijk}p_k$, where $0 \le i < j \le 5$, and to the relations obtained by taking coefficients of

$$\sum_{i=0}^{5}\sum_{j=i}^{5} e_{ij}h'_ih'_j = \left(\sum_{i=0}^{5} h'_ip_i\right)\left(\sum_{i=0}^{5} h'_iv_i\right)$$

as a polynomial in H' and eliminating the variables $v_0, \ldots, v_5$. There are 490 irredundent relations belonging to 12 symmetry classes with respect to the action of S_6 on the indices. Representatives of these classes are:

$$p_1^2e_{00} - p_0p_1e_{01} + p_0^2e_{11}$$
$$e_{02}^2e_{11} - e_{01}e_{02}e_{12} + e_{00}e_{12}^2 + e_{01}^2e_{22} - 4e_{00}e_{11}e_{22}$$
$$p_2^2e_{01} - p_1p_2e_{02} - p_0p_2e_{12} + 2p_0p_1e_{22}$$
$$p_2e_{01}^2 - p_1e_{01}e_{02} - 4p_2e_{00}e_{11} + 2p_0e_{02}e_{11} + 2p_1e_{00}e_{12} - p_0e_{01}e_{12}$$
$$e_{02}e_{03}e_{12} - e_{02}^2e_{13} - 2e_{01}e_{03}e_{22} + 4e_{00}e_{13}e_{22} + e_{01}e_{02}e_{23} - 2e_{00}e_{12}e_{23}$$
$$p_2e_{01}e_{03} - p_1e_{02}e_{03} - 2p_2e_{00}e_{13} + p_0e_{02}e_{13} + 2p_1e_{00}e_{23} - p_0e_{01}e_{23}$$
$$p_2p_3e_{01} - p_1p_2e_{03} - p_0p_3e_{12} + p_0p_1e_{23}$$
$$e_{03}e_{04}e_{12} - e_{02}e_{03}e_{14} - e_{01}e_{04}e_{23} + 2e_{00}e_{14}e_{23} + e_{01}e_{02}e_{34} - 2e_{00}e_{12}e_{34}$$
$$\begin{gathered} p_3e_{04}e_{12} - p_2e_{03}e_{14} - p_1e_{04}e_{23} + p_0e_{14}e_{23} - \\ - p_3e_{01}e_{24} + p_1e_{03}e_{24} + p_2e_{01}e_{34} - p_0e_{12}e_{34} \end{gathered}$$
$$p_4e_{03}e_{12} - p_2e_{03}e_{14} - p_4e_{01}e_{23} + p_0e_{14}e_{23} + p_2e_{01}e_{34} - p_0e_{12}e_{34}$$
$$\begin{gathered} e_{04}e_{15}e_{23} - e_{03}e_{14}e_{25} - e_{02}e_{15}e_{34} + e_{01}e_{25}e_{34} - \\ - e_{04}e_{12}e_{35} + e_{02}e_{14}e_{35} + e_{03}e_{12}e_{45} - e_{01}e_{23}e_{45} \end{gathered}$$
$$e_{05}e_{14}e_{23} - e_{03}e_{14}e_{25} - e_{05}e_{12}e_{34} + e_{01}e_{25}e_{34} + e_{03}e_{12}e_{45} - e_{01}e_{23}e_{45}$$

Then we substitute the coefficients of

$$\mathcal{R}_{\mathrm{cone}(q,Y)\cap L'} = \sum_{i=0}^{5}\sum_{j=i}^{5} e_{ij}[i][j]$$

into the precomputed relations, and set $p_i = q_i$ for $i = 0, 1, 2$. We now consider the ideal of all relations constructed in Step 5. In this ideal we find polynomials which express the coordinates of p as rational functions of the coordinates of q and the Plücker coordinates of L.

In our example we have the relations

$$\begin{aligned} p_3 &= -(q_0l_{045} + q_1l_{145} + q_2l_{245})/l_{345} \\ p_4 &= (q_0l_{035} + q_1l_{135} + q_2l_{235})/l_{345} \\ p_5 &= -(q_0l_{034} + q_1l_{134} + q_2l_{234})/l_{345} \end{aligned}$$

Step 6: Now let q_1, q_2, q_3, and q_4 be points in $\mathbf{P}^2$ with indeterminate coordinates. We rewrite

$$\prod_{i=1}^{4} \langle H', p(q_i, L)\rangle$$

in terms of the elementary multisymmetric functions of q_1, q_2, q_3, and q_4 to obtain

$$\frac{1}{l_{345}^4}\left(e_{0000}h_5^4l_{034}^4 - e_{0000}h_4h_5^3l_{034}^3l_{035} + e_{0000}h_4^2h_5^2l_{034}^2l_{035}^2 + (1362 \text{ terms})\right)$$

where e_{ijkl} is the symmetrized sum

$$\sum (q_1)_i (q_2)_j (q_3)_k (q_4)_l.$$

`Step 3'`: We substitute the coefficients of $\mathcal{R}_{(X\cap H)\#Y}(L)$ as a polynomial in H for the elementary symmetric functions of q_1, q_2, q_3, and q_4. This gives a polynomial of degree 4 in H' and degree 4 in the Plücker coordinates of L:

$$h_4^4 l_{013}^4 + 2h_4^2 h_5^2 l_{013}^4 + h_5^4 l_{013}^4 - 4h_3 h_4^3 l_{013}^3 l_{014} + (5280 \text{ more terms})$$

`Step 6'`: We express this polynomial as a function of the Plücker coordinates of $H' \cap L$, and we obtain $\mathcal{R}_{X\#Y}$, a polynomial of degree 4 in rank 4 primal brackets.

We summarize the result of this computation. The join $X\#Y$ is the threefold in $\mathbf{P}^5$ defined by the two equations

$$x_0^2 + x_1^2 + x_2^2 \;=\; x_3^2 + x_4^2 + x_5^2 \;=\; 0.$$

Its Chow form $\mathcal{R}_{X\#Y}$ is the resultant of these equations and four general linear forms $\ell_i = u_{i0}x_0 + u_{i1}x_1 + \cdots + u_{i5}x_5$, $i = 1,2,3,4$. Here is a formula for $\mathcal{R}_{X\#Y}$ in terms of the maximal minors of the 4×6-matrix (u_{ij}):

$$\begin{gathered}
[0134]^4 + 2[0134]^2[0135]^2 + [0135]^4 + 2[0134]^2[0145]^2 + 2[0135]^2[0145]^2 \\
+ [0145]^4 + 2[0134]^2[0234]^2 + [0234]^4 + 4[0134]^2[0235]^2 + 2[0135]^2[0235]^2 \\
+ 2[0234]^2[0235]^2 + [0235]^4 + 4[0134]^2[0245]^2 + 4[0135]^2[0245]^2 \\
+ 2[0145]^2[0245]^2 + 2[0234]^2[0245]^2 + 2[0235]^2[0245]^2 + [0245]^4 \\
- 4[0123][0134][0235][0345] - 4[0124][0134][0245][0345] \\
- 4[0125][0135][0245][0345] - 2[0123]^2[0345]^2 - 2[0124]^2[0345]^2 \\
- 2[0125]^2[0345]^2 + 2[0134]^2[1234]^2 + 2[0234]^2[1234]^2 + [1234]^4 \\
+ 4[0134]^2[1235]^2 + 2[0135]^2[1235]^2 + 4[0234]^2[1235]^2 + 2[0235]^2[1235]^2 \\
+ 2[1234]^2[1235]^2 + [1235]^4 + 4[0134]^2[1245]^2 + 4[0135]^2[1245]^2 \\
+ 2[0145]^2[1245]^2 + 4[0234]^2[1245]^2 + 4[0235]^2[1245]^2 + 2[0245]^2[1245]^2 \\
+ 2[1234]^2[1245]^2 + 2[1235]^2[1245]^2 + [1245]^4 \\
- 4[0123][0134][1235][1345] - 4[0124][0134][1245][1345] \\
- 4[0125][0135][1245][1345] - 2[0123]^2[1345]^2 - 2[0124]^2[1345]^2 \\
- 2[0125]^2[1345]^2 - 4[0123][0234][1235][2345] \\
- 4[0124][0234][1245][2345] - 4[0125][0235][1245][2345] \\
- 2[0123]^2[2345]^2 - 2[0124]^2[2345]^2 - 2[0125]^2[2345]^2
\end{gathered}$$

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