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On the Number of Real Roots of a Sparse Polynomial System

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Abstract. We estimate the asymptotic number of real roots of a system of polynomial equations under a multiplicative deformation which preserves the Newton polytope.

1 Introduction

We consider a system of d polynomial equations in d variables $\mathbf{x} = (x_1, \ldots, x_d)$:

$$\begin{array}{rclcl} f_1(\mathbf{x}) & = & c_{11}\mathbf{x}^{a_1} + c_{12}\mathbf{x}^{a_2} + \ldots + c_{1n}\mathbf{x}^{a_n} & = & 0 \\ \ldots & & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots & & \\ f_d(\mathbf{x}) & = & c_{d1}\mathbf{x}^{a_1} + c_{d2}\mathbf{x}^{a_2} + \ldots + c_{dn}\mathbf{x}^{a_n} & = & 0 \end{array} \tag{1.1}$$

where $\mathcal{A} = \{a_1, \ldots, a_n\} \subset \mathbf{N}^d$, $\mathbf{x}^{a_i} = x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_d^{a_{id}}$, and the coefficients c_{ij} are real numbers. The convex polytope $Q = conv(\mathcal{A})$ in $\mathbf{R}^d$ is called the *Newton polytope* of the system (1.1). It has the *normalized volume* $v(\mathcal{A}) := vol(Q) \cdot d!$.

Theorem 1.1 *(Koushnirenko [1975]) For all coefficient matrices (c_{ij}) in a dense subset $\mathcal{U}_\mathcal{A}$ of $\mathbf{R}^{d\times n}$, the system (1.1) has $v(\mathcal{A})$ distinct zeros in the complex torus $(\mathbf{C}^*)^d$.*

Bernstein [1975] gives an extension of Theorem 1.1 to *mixed systems*, where each polynomial $f_i(\mathbf{x})$ has a different set of exponent vectors $\mathcal{A}_i \subset \mathbf{N}^d$. Here we restrict ourselves to the unmixed case, and we assume $dim(Q) = d$, or, equivalently, $v(\mathcal{A}) > 0$. All our results can be extended to mixed systems using the techniques in Pedersen and Sturmfels [1993], Sect. 7. The dense subset $\mathcal{U}_\mathcal{A} \subset \mathbf{R}^{d\times n}$ appearing in Theorem 1.1 is Zariski-open, i.e., $\mathcal{U}_\mathcal{A}$ is the set of non-zeros of a finite system of

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polynomials. This system consists of the face resultants appearing in Pedersen and Sturmfels [1993], Theorem 3.1, plus a certain discriminant which rules out multiple roots.

This note is concerned with following question: *How many of the $v(\mathcal{A})$ complex zeros are real zeros?* Let $\rho(\mathcal{A})$ denote the maximum number of real roots $\mathbf{x} \in (\mathbf{R}^*)^d$ of (1.1) as (c_{ij}) ranges over $\mathcal{U}_{\mathcal{A}}$. Similarly we write $\rho_+(\mathcal{A})$ for the maximum number of real positive roots $\mathbf{x}$ in $(\mathbf{R}_+)^d$. Counting the 2^d orthants in $\mathbf{R}^d$, we get the obvious inequality

$$\rho(\mathcal{A}) \quad \leq \quad 2^d \cdot \rho_+(\mathcal{A}). \tag{1.2}$$

The theory of *Fewnomials*, due to Khovanskii [1991], shows that the number of real roots is generally much smaller than the number of complex roots. There exists an upper bound for $\rho(\mathcal{A})$ which depends only on the dimension d and the number of terms $n = |\mathcal{A}|$.

Theorem 1.2 *(Khovanskii [1980], [1991]) The number of positive real roots of (1.1) satisfies*

$$\rho_+(\mathcal{A}) \quad \leq \quad 2^{n(n-1)/2} \cdot (d+1)^n. \tag{1.3}$$

This theorem raises the following natural question.

Problem 1.3 *Find lower bounds and more precise upper bounds for $\rho(\mathcal{A})$, the maximum number of real roots, in terms of the combinatorial structure of the configuration $\mathcal{A}$.*

Very little is known at present, as is witnessed by a challenging little example.

Example 1.4 *What is the maximum number of real roots of a bivariate system with five terms?* In other words, determine the maximum $\rho(2,5)$ of the integers $\rho(\mathcal{A})$, where $\mathcal{A}$ runs over all five element subsets of $\mathbf{N}^2$.

Let $\mathcal{A} = \{(0,0),(2,0),(4,0),(0,2),(0,4)\}$. This configuration is the support of

$$(x^2-4)^2 + (y^2-3)^2 - 13 \quad = \quad (x^2-4)^2 - (y^2-3)^2 - 5 \quad = \quad 0. \tag{1.4}$$

The Newton polytope $Q = conv(\mathcal{A})$ is a triangle with area 8, therefore $v(\mathcal{A}) = 2 \cdot 8 = 16 \geq \rho(\mathcal{A})$. The specific system (1.4) has all 16 roots real, and therefore $\rho(\mathcal{A}) = 16$.

The resulting inequality $\rho(2,5) \geq 16$ is the best lower bound for $\rho(2,5)$ known to me at present. There is an embarrassingly wide gap to the upper bound from Khovanskii's Theorem 1.2. Combining (1.2) and (1.3), we have $\rho(2,5) \leq 2^2 \cdot 2^{10} \cdot 3^5 = 995{,}328$. □

In this note we venture a first step in the attack on Problem 1.3. We fix $\omega = (\omega_1, \ldots, \omega_n) \in \mathbf{N}^n$, and we consider the perturbed system

$$\begin{aligned} f_1(t; x_1, \ldots, x_n) &= c_{11}t^{\omega_1}\mathbf{x}^{a_1} + c_{12}t^{\omega_2}\mathbf{x}^{a_2} + \ldots + c_{1n}t^{\omega_n}\mathbf{x}^{a_n} = 0 \\ \ldots \quad \ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ f_d(t; x_1, \ldots, x_n) &= c_{d1}t^{\omega_1}\mathbf{x}^{a_1} + c_{d2}t^{\omega_2}\mathbf{x}^{a_2} + \ldots + c_{dn}t^{\omega_n}\mathbf{x}^{a_n} = 0 \end{aligned} \tag{1.5}$$

as the real parameter t tends to zero.

Our main result (Theorem 2.2) gives precise bounds for the number of real roots of (1.5) for sufficiently small $t > 0$. As a corollary we get a lower bound for $\rho(\mathcal{A})$. These bounds are stated in terms of *regular triangulations*. This concept is reviewed in the beginning of Section 2. The proof of our result is given in Section 3.

2 Bounds in terms of regular triangulations

A *subdivision* of the pair $(Q, \mathcal{A})$ is a collection $\Delta = \{\mathcal{A}_1, \ldots, \mathcal{A}_m\}$ of subsets of $\mathcal{A}$ such that

- each polytope $Q_i = conv(\mathcal{A}_i)$ has the full dimension d for $i = 1, \ldots, m$,
- $Q = Q_1 \cup Q_2 \cup \ldots \cup Q_m$, and
- each intersection $Q_i \cap Q_j$ is a common proper face of Q_i and Q_j, for $1 \leq i < j \leq m$.

The subsets $\mathcal{A}_i$ are the *cells* of the subdivision Δ. We say that Δ is a *triangulation* of $(Q, \mathcal{A})$ if each cell $\mathcal{A}_i$ has cardinality $d+1$. An important subclass of triangulations is the class of *regular triangulations* (see e.g. Billera, Filliman and Sturmfels [1990], Gel'fand, Kapranov and Zelevinsky [1990], and Lee [1991]). In what follows we give an algebraic definition of regular triangulation, based on the results in Sturmfels [1991]. Let $y_1, \ldots, y_n, z, x_1, \ldots, x_d$ be variables, and consider the $\mathbf{Z}$-algebra homomorphism

$$\mathbf{Z}[y_1, y_2, \ldots, y_n] \rightarrow \mathbf{Z}[x_1, \ldots, x_d, z], \quad y_i \mapsto z \cdot x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_d^{a_{id}}, \tag{2.1}$$

with $a_i = (a_{i1}, \ldots, a_{id})$ as before. We define the *toric ideal* $I_\mathcal{A}$ to be the kernel of map (2.1). The toric ideal $I_\mathcal{A}$ is generated by all homogeneous polynomials of the form

$$y_1^{\mu_1} y_2^{\mu_2} \cdots y_n^{\mu_n} - y_1^{\nu_1} y_2^{\nu_2} \cdots y_n^{\nu_n}, \tag{2.2}$$

where $\mu_1 a_1 + \ldots + \mu_n a_n = \nu_1 a_1 + \ldots + \nu_n a_n$ and $\mu_1 + \ldots + \mu_n = \nu_1 + \ldots + \nu_n$. The relations (2.2) correspond to affine dependencies of $\mathcal{A}$.

We fix $\omega \in \mathbf{N}^n$. For each polynomial $f \in \mathbf{Z}[y_1, \ldots, y_n]$ we consider $f(t^{\omega_1} y_1, \ldots, t^{\omega_n} y_n)$ as a univariate polynomial in the parameter t. Its leading coefficient $init_\omega(f)$ is a polynomial in $\mathbf{Z}[x_1, \ldots, x_n]$, called the *initial form* of f with respect to ω. We define the *initial ideal* of $I_\mathcal{A}$ as

$$init_\omega(I_\mathcal{A}) \quad := \quad \langle\, init_\omega(f) \;:\; f \in I_\mathcal{A} \,\rangle. \tag{2.3}$$

For sufficiently generic ω the initial ideal $init_\omega(I_\mathcal{A})$ is generated by monomials. We now suppose that this is the case. In the language of Gröbner basis theory: the vector ω represents a *term order* for $I_\mathcal{A}$. A monomial $y_1^{\mu_1} \cdots y_n^{\mu_n}$ is said to be *standard* if $y_1^{\mu_1} \cdots y_n^{\mu_n}$ does not lie in $I_\mathcal{A}$. Let Δ_ω denote the collection of all $(d+1)$-subsets $\{a_{i_0}, a_{i_1}, \ldots, a_{i_d}\}$ of $\mathcal{A}$ for which all powers of the square-free monomial $y_{i_0} y_{i_1} \cdots y_{i_d}$ are standard.

Theorem 2.1 *(Sturmfels [1991]) The set Δ_ω is a triangulation of $(Q, \mathcal{A})$.*

A triangulation Δ of $(Q_\mathcal{A})$ is called *regular* if $\Delta = \Delta_\omega$, for some $\omega \in \mathbf{N}^n$. For examples of non-regular triangulations see Billera, Filliman and Sturmfels [1990], Figure 1 and Lee [1991], Figures 2 and 6.

Fix a term order $\omega \in \mathbf{N}^n$, and let Δ_ω be the corresponding regular triangulation. Each cell $\mathcal{A}_i \in \Delta_\omega$ generates an affine lattice $\mathbf{Z}\{\mathcal{A}_i\}$ of rank d. The quotient of lattices $\mathbf{Z}^d / \mathbf{Z}\{\mathcal{A}_i\}$ is a finite abelian group of order $v(\mathcal{A}_i)$. By standard results on finite abelian groups (Hungerford [1974], Cor. 2.7), there exist unique positive integers $m_1, m_2, \ldots, m_d$, called *invariant factors*, such that m_{i-1} divides m_i for $i = 2, \ldots, n$, and

$$\mathbf{Z}^d/\mathbf{Z}\{\mathcal{A}_i\} \quad \simeq \quad \mathbf{Z}/m_1\mathbf{Z} \oplus \mathbf{Z}/m_2\mathbf{Z} \oplus \ldots \oplus \mathbf{Z}/m_d\mathbf{Z}. \tag{2.4}$$

We have $m_1 m_2 \cdots m_d = v(\mathcal{A}_i)$. Let $even(\mathcal{A}_i)$ denote the number of invariant factors m_i which are even. If the integer $v(\mathcal{A}_i)$ is odd, or equivalently, if $even(\mathcal{A}_i) = 0$, then we call $\mathcal{A}_i$ an *odd cell.* We now state the new result of this note.

Theorem 2.2 *Let $(c_{ij}) \in \mathcal{U}_\mathcal{A}$, and let ρ be the number of real roots $\mathbf{x} \in (\mathbf{R}^*)^d$ of the system (1.5) for sufficiently small $t > 0$. This number satisfies the inequalities*

$$\#\, odd\ cells\ in\ \Delta_\omega \quad \leq \quad \rho \quad \leq \quad \sum_{\mathcal{A}_i \in \Delta_\omega} 2^{even(\mathcal{A}_i)}. \tag{2.5}$$

This theorem has the following two corollaries.

Corollary 2.3 *For any configuration $\mathcal{A} \subset \mathbf{N}^d$, the number $\rho(\mathcal{A})$ is bounded below by the number of odd cells in any regular triangulation of $(Q, \mathcal{A})$.*

Corollary 2.4 *Fix $(c_{ij}) \in \mathcal{U}_\mathcal{A}$ and let Δ_ω and ρ as in Theorem 2.2.*

(a) If each cell in Δ_ω is odd, then the lower and upper bounds in (2.5) agree and ρ coincides with the total number of cells in Δ_ω.

(b) If each cell $\mathcal{A}_i \in \Delta_\omega$ has unit volume $v(\mathcal{A}_i) = 1$, then all complex roots of (1.5) are real for $t \to 0$, and we have $\rho = \rho(\mathcal{A}) = v(\mathcal{A})$.

We illustrate these results for three classes of examples.

Example 2.5 (Dense univariate polynomials) Let $c_0, c_1, \ldots, c_n$ be fixed real numbers, let $\omega = (\omega_0, \ldots, \omega_n) \in \mathbf{N}^{n+1}$ such that $\omega_{i-1} + \omega_{i+1} \neq 2\omega_i$ for $i = 1, \ldots, n-1$, and let $t > 0$ be sufficiently small. We are interested in the number ρ of real roots of the polynomial

$$f_t(x) \quad := \quad c_0 t^{\omega_0} + c_1 t^{\omega_1} x + c_2 t^{\omega_2} x^2 + \ldots + c_n t^{\omega_n} x^n. \tag{2.6}$$

Let $(i_1, \omega_{i_1}), (i_2, \omega_{i_2}), \ldots (i_k, \omega_{i_k})$ be the vertices of the polygon $conv\{\, (i, \omega_i), (i, 0) : i = 0, 1, \ldots, n\}$. We may suppose $0 = i_1 < i_2 < \ldots < i_k = n$. Let f_{odd} denote the number of differences $i_{j+1} - i_j$ which are odd. Then $f_{odd} \leq \rho \leq 2(k-1) - f_{odd}$.

If all differences $i_{j+1} - i_j$ are odd, then $f_{odd} = \rho$ for all choices of coefficients $c_i \in \mathbf{R}^*$. For instance, choose arbitrary non-zero real numbers $c_0, c_1, \ldots, c_7$, and consider

$$f_t(x) \quad = \quad c_0 t^3 + c_1 t^4 x + c_2 t^3 x^2 + c_3 t^1 x^3 + c_4 t^1 x^4 + c_5 t^3 x^5 + c_6 t^2 x^6 + c_7 t^2 x^7. \tag{2.7}$$

For small parameter values $t > 0$, the polynomial $f_t(x)$ has precisely $f_{odd} = 3$ real roots.

Example 2.6 (The cross-polytope) Consider the system of polynomial equations

$$\sum_{j=0}^{n} c_{ij} \prod_{k=j+1}^{n} x_k + \sum_{j=0}^{n} d_{ij}\, x_0 x_1 \cdots x_n \prod_{k=1}^{j} x_k \quad = \quad 0 \qquad (i = 1, 2, \ldots, n+1) \tag{2.8}$$

where the c_{ij}, d_{ij} are real coefficients, and the product over the empty set is defined to be 1. The corresponding set $\mathcal{A} \subset \mathbf{N}^{n+1}$ is unimodularly equivalent to the vertices of the regular $(n+1)$-dimensional cross-polytope. For $n = 2$ the cross-polytope $Q = conv(\mathcal{A})$ is the octahedron. There is a canonical regular triangulation of $\mathcal{A}$

into 2^n simplices of unit volume. The number of complex roots of (2.8) equals 2^n, and, by Corollary 2.8, this number coincides with the maximum number $\rho(\mathcal{A})$ of real roots.

Example 2.7 (Bivariate systems) Fix $d = 2$. We write $f(\Delta_\omega)$ for the number of *triangles* (= cells) in the regular triangulation Δ_ω of $\mathcal{A} = \{a_1, \ldots, a_n\} \subset \mathbf{N}^2$. An easy count of triangles in planar graphs shows that $f(\Delta_\omega) \leq 2n - 5$. Theorem 2.2 implies

$$\rho \quad \leq \quad 2^2 \cdot f(\Delta_\omega) \quad \leq \quad 8n - 20.$$

Thus for $d = 2$ fixed, the number of real roots of (1.5) for $t \to 0$ is bounded above by a linear polynomial in n. Note that the Khovanskii upper bound (1.3) is exponential in n.

It is easy to see that the number $\rho(\mathcal{A})$ is bounded below by a quadratic polynomial in n. Generalizing Example 1.4, we let $f(x)$ and $g(y)$ be univariate polynomials of degree n, each having n distinct positive roots. Then the bivariate system $f(x) + g(y) = f(x) - g(y) = 0$ has n^2 real roots, but its support set $\mathcal{A}$ has only cardinality $2n + 1$. This shows that in general the number ρ in Theorem 2.2 grows much slower than $\rho(\mathcal{A})$. But how much?

We formulate this as a problem of asymptotic complexity. Let $\rho(d, n)$ denote the maximum of the numbers $\rho(\mathcal{A})$ where $\mathcal{A}$ runs over all n-element subsets of $\mathbf{N}^d$.

Problem 2.8 *For fixed $d \geq 2$, does $\rho(d, n)$ grow polynomially or exponentially in n ?*

3 The proof

Our proof of Theorem 2.2 is algorithmic. We describe an explicit procedure for computing the ρ real roots of (1.5) for $t \to 0$. We start with a "subroutine" for the base case $n = d + 1$.

Proof of Theorem 2.2 (Part I: $n = d + 1$). We show that (2.5) holds for the number of roots of (1.1), for all $(c_{ij}) \in \mathcal{U}_\mathcal{A}$. Using elementary row operations, we transform the $d \times (d + 1)$-coefficient matrix (c_{ij}) into a unit matrix plus one extra column. We thus rewrite the system (1.1) in the form

$$\gamma_1 \mathbf{x}^{a_1} - \mathbf{x}^{a_{d+1}} \;=\; \gamma_2 \mathbf{x}^{a_2} - \mathbf{x}^{a_{d+1}} \;=\; \ldots \;=\; \gamma_d \mathbf{x}^{a_d} - \mathbf{x}^{a_{d+1}} \;=\; 0, \qquad (3.1)$$

where the γ_i are $\mathbf{Q}$-linear combinations of the old coefficients c_{ij}. Equivalently, we have

$$\gamma_1\, \mathbf{x}^{a_1 - a_{d+1}} \;=\; \gamma_2\, \mathbf{x}^{a_2 - a_{d+1}} \;=\; \ldots \;=\; \gamma_d\, \mathbf{x}^{a_d - a_{d+1}} \;=\; 1. \qquad (3.2)$$

We now compute the *Smith normal form* of the $d \times d$-exponent matrix $(a_1 - a_{d+1}, \ldots, a_d - a_{d+1})$. This means we construct invertible integer $d \times d$-matrices U and V such that

$$V \cdot \big(a_1 - a_{d+1}, a_2 - a_{d+1}, \ldots, a_d - a_{d+1}\big) \cdot U \;=\; diag(m_1, m_2, \ldots, m_d), \quad (3.3)$$

where m_{i-1} divides m_i for all i. As in (2.4), we have $\mathbf{Z}^d / \mathbf{Z}\{\mathcal{A}\} \simeq \mathbf{Z}/m_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/m_d\mathbf{Z}$, and $m_1 m_2 \cdots m_d = v(\mathcal{A})$, the normalized volume of the simplex $Q = conv(\mathcal{A})$.

The invertible matrix $U = (u_1, \ldots, u_n)$ defines a monoidal transformation of coordinates $x_i \mapsto \mathbf{z}^{u_i}$. In the new coordinates $\mathbf{z} = (z_1, \ldots, z_d)$ our system (3.3) equals

$$\tilde{\gamma}_1\, z_1^{m_1} \;=\; \tilde{\gamma}_2\, z_2^{m_2} \;=\; \ldots \;=\; \tilde{\gamma}_d\, z_d^{m_d} \;=\; 1, \qquad (3.4)$$

where the $\tilde{\gamma}_i$ are obtained from the γ_j via the monoidal transformation defined by V.

By construction, the number of real roots of (1.1) equals the number of real roots of (3.4). This number is bounded above by $2^{\#\{m_i : m_i \text{ even}\}}$. If $\nu = m_1 m_2 \cdots m_d$ is odd, then at least one of the roots is real. This completes our proof for the case $n = d+1$.

For the general case we need the following description of the regular triangulation Δ_ω.

Lemma 3.1 *A subset $\mathcal{B} \subset \mathcal{A}$ is a cell of Δ_ω if and only if there exist $\lambda_0, \lambda_1, \ldots, \lambda_d \in \mathbf{Q}$*

$$\text{such that} \qquad \sum_{j=1}^{d} \lambda_j a_{ij} + \lambda_0 \quad \begin{cases} = \omega_i & \text{if } a_i \in \mathcal{B}, \\ > \omega_i & \text{if } a_i \notin \mathcal{B}. \end{cases} \tag{3.5}$$

Proof This follows from the equivalent definitions in Billera, Filliman and Sturmfels [1990], (4.4) and Lee [1991], Section 4. □

Proof of Theorem 2.2 (Part II: $n > d+1$). We view the complex roots of (1.5) as the branches of a vector-valued algebraic function of t as $t \to 0$. The number of branches equals $v(\mathcal{A}) = \sum_{\mathcal{B}\in\Delta_\omega} v(\mathcal{B})$. For each cell $\mathcal{B}$ of Δ_ω we get $v(\mathcal{B})$ branches. These are computed using the following transformation.

We choose $\lambda_0, \lambda_1, \ldots, \lambda_d \in \mathbf{Q}$ as in Lemma 3.1. We substitute $x_i \cdot t^{-\lambda_i}$ for x_i and multiply each equation in (1.5) by $t^{-\lambda_0}$ to get the equivalent system

$$\frac{1}{t^{\lambda_0}} \cdot f_i\Big(t; \frac{x_1}{t^{\lambda_1}}, \ldots, \frac{x_d}{t^{\lambda_d}}\Big) \quad = \quad \sum_{a_j \in \mathcal{B}} c_{ij} \mathbf{x}^{a_j} + \sum_{a_\ell \notin \mathcal{B}} c_{i\ell}\, t^{\gamma_\ell}\, \mathbf{x}^{a_\ell}, \qquad (i = 1, 2, \ldots, d). \tag{3.6}$$

The exponents γ_ℓ are positive rational numbers.

For $t = 0$ the system (3.6) has $v(\mathcal{B})$ complex roots. We may assume that there are no multiple roots, if necessary, by shrinking the Zariski open set $\mathcal{U}_\mathcal{A}$. Let $\rho_\mathcal{B}$ denote the number of real roots at $t = 0$. By the Implicit Function Theorem, the system (3.6) has $\rho_\mathcal{B}$ real roots for all parameters t in a small neighborhood of the origin 0.

We have shown that for $t \to 0$ the number of real roots of (1.5) equals

$$\rho \quad = \quad \sum_{\mathcal{B}\in\Delta_\omega} \rho_\mathcal{B}.$$

By part I each $\rho_\mathcal{B}$ satisfies the desired upper and lower bound. Since these bounds are additive with respect to the cells of Δ_ω, the proof of Theorem 2.2 is complete. □

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