Problem 1A. *Score:*

Suppose that x, y, p, q are real numbers with $x \geq 0$, $y \geq 0$, $p > 1$, $1/p + 1/q = 1$. Prove Young's inequality

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

Solution: Fix y and consider $xy - x^p/p - y^q/q$ as a function of x. It is at most zero for $x = 0$ and for x large, so it is enough to check it is at most zero at all critical points. Differentiation shows that the only critical point is $x = y^{p-1}$ when it is 0.

Problem 2A. *Score:*

Suppose $f : [0, 1] \to R$ is a continuous function with

$$\int_0^1 x^n f(x) dx = 0$$

for all integers n with $1 \leq n < \infty$. Prove that f is identically 0.

Solution: We can say that $f(x)$ is identically 0.

Let $F(x) = \int_x^1 f(s) ds$, so that

$$\int_0^1 x^m F(x) dx = 0$$

by integration by parts (since $F(1) = 0$) for all $m \geq 0$. By linearity of the integral, F is orthogonal to polynomials:

$$\int_0^1 P(x) F(x) dx = 0$$

whenever P is a polynomial. Let $\epsilon > 0$ and let P be a polynomial which approximates f within ϵ on the interval $[0, 1]$, by the Weierstrass approximation theorem. Then

$$\int_0^1 F(x)^2 dx = \int_0^1 F(x)P(x)dx + \int_0^1 F(x)(F(x) - P(x))dx \leq \sqrt{\int_0^1 F(x)^2 dx} \sqrt{\int_0^1 (F(x) - P(x))^2 dx}$$

by orthogonality and the Cauchy-Schwarz inequality. By choice of P,

$$\int_0^1 F(x)^2 dx \leq \epsilon \sqrt{\int_0^1 F(x)^2 dx}.$$

Cancelling a square root factor shows that

$$\int_0^1 F(x)^2 dx \leq \epsilon^2$$

and since $\epsilon > 0$ was arbitrary we have

$$\int_0^1 F(x)^2 dx = 0.$$

Since F is continuous it must vanish identically, and then $f(x) = -F'(x)$ must vanish identically as well.

Problem 3A. *Score:*

Suppose that X is an uncountable subset of the reals. Prove that there is a point of X that is a limit of a sequence of distinct points of X.

Solution: If not, then for every point of $x \in X$ we can find an integer n_x such that no point of X is within $1/n_X$ of x. Since X is uncountable, there is some integer n with an uncountable number of points such that $n_x = n$. But any 2 points of this set must be at distance at least $1/n$, so there are only countable number of them.

Problem 4A. *Score:*

(a). Show that there is a function $f(z)$, holomorphic (analytic) near $z = 0$, such that

$$f(z)^{10} = \frac{1}{\cos(z^5 + 2z^7)} - 1$$

for all z in a neighborhood of $z = 0$.

(b). Find the radius of convergence of its power series about $z = 0$. Your answer may involve a root of an explicitly given polynomial.

Solution:

(a). From the power series for the cosine function, we have

$$\cos(z^5 - z^7) = 1 - \frac{(z^5 - z^7)^2}{2!} + \cdots = 1 - \frac{z^{10}}{2} + z^{12} + \ldots$$

and therefore

$$\frac{1}{\cos(z^5 - z^7)} = 1 + \frac{z^{10}}{2} + \ldots .$$

Therefore $\frac{1}{\cos(z^5+2z^7)} - 1$ has a zero of order 10 at $z = 0$, and so

$$\frac{1}{\cos(z^5 + 2z^7)} - 1 = z^{10}g(z)$$

near $z = 0$, where g extends to a holomorphic function near $z = 0$ which does not vanish at $z = 0$. We may then let

$$f(z) = \exp\left(\frac{\log g(z))}{10}\right) ,$$

and this is holomorphic at $z = 0$.

(b). We have $\cos z = 0$ only at odd integer multiples of $\pi/2$, and $\cos z = 1$ only if $\sin z = 0$, which happens only at integer multiples of π. Having removed the singularity at $z = 0$, we have that f is holomorphic on the set where $|z^5 + 2z^7| < \pi/2$, so the largest radius of convergence is the positive root r of $x^5 + 2x^7 = \pi/2$, since $|z| < r$ implies $|z^5 + 2z^7| \leq |z|^5 + 2|z|^7 < r^5 + 2r^7 = \pi/2$, and $\cos(z^5 + 2z^7) = 0$ when $z = r$.

Problem 5A. *Score:*

Let $f(z)$ be a function holomorphic on the whole complex plane $\mathbb{C}$ such that $f(z) \in \mathbb{R}$ for all $z \in \mathbb{R}$. Show that $\overline{f(z)} = f(\bar{z})$ for all $z \in \mathbb{C}$.

Solution:

Let $g(z) = \overline{f(\bar{z})}$. By Cauchy-Riemann condition $g(z)$ is holomorphic and therefore $h(z) = f(z) - g(z)$ is also holomorphic on $\mathbb{C}$. One the other hand, $h(z) = 0$ for any $z \in \mathbb{R}$. Again Cauchy-Riemann equations imply that $h(z) \equiv 0$.

Problem 6A. *Score:*

Let T be a linear transformation of a vector space V into itself. Suppose that $T^{m+1} = 0$, $T^m \neq 0$ for some positive integer m. Show that there is a vector x such that $x, Tx, \ldots, T^m x$ are linearly independent.

Solution:

Pick x so that $T^m x \neq 0$. If the points are linearly dependent, choose a relation $a_k T^k x + ... + a_m T^m x = 0$ with $a_k \neq 0$ and k as large as possible. Applying T gives a similar relation with a larger k, contradiction.

Problem 7A. *Score:*

Suppose n is a positive integer and let f be the function $f(x) = (1, x, x^2, ...x^{n-1})$ from $\mathbb{R}$ to $\mathbb{R}^n$, Show that a hyperplane (of codimension 1) containing the points $f(1), f(2), ..., f(n)$ does not pass through the origin.

Solution: This is equivalent to showing that the points are linearly independent. So it is enough to show that the determinant formed by their coordinates is nonzero. But this is a Vandermonde determinant, which shows it is nonzero.

Problem 8A. *Score:*

Let A be an abelian group. Suppose that $a \in A$ and $b \in A$ have orders h and k, respectively, and that h and k are relatively prime.

Let r and s be integers. Show that if $ra = sb$ then $ra = sb = 0$.

Solution: Since h and k are relatively prime, they generate the unit ideal in $\mathbb{Z}$, so there exist integers x and y such that $xh + yk = 1$. Therefore,

$$ra = (xh + yk)ra = xh(ra) + yk(sb) = xr(ha) + ys(kb) = 0 \ ,$$

and therefore also $sb = 0$.

Problem 9A. *Score:*

Let $\mathbf{F}$ be a field and let X be a finite set. Let $R(X, \mathbf{F})$ be the ring of all functions from X to $\mathbf{F}$, endowed with the pointwise operations. What are the maximal ideals of $R(X, \mathbf{F})$?

Solution:

Let $R = R(X, \mathbf{F})$. For all $x \in X$ and $a \in \mathbf{F}$ let $\phi_{x,a} \colon X \to \mathbf{F}$ be the function given by $\phi_{x,a}(x) = a$ and $\phi_{x,a}(x') = 0$ for all $x' \neq x$.

Let I be an ideal of R, and let $S \subseteq X$ be the set

$$S = \{x \in X : f(x) \neq 0 \text{ for some } f \in I\} .$$

Then, for all $x \in S$, the ideal I contains the function $\phi_{x,1}$ since I contains some element f with $f(x) \neq 0$; then

$$\phi_{x,1} = \phi_{x,f(x)^{-1}} \cdot f \in I .$$

For any $f \colon X \to \mathbf{F}$ that vanishes at all $x \notin S$, we then have

$$f = \sum_{x \in S} f(x) f_{x,1} \in I ;$$

therefore $I = \{f \colon X \to \mathbf{F} : f(x) = 0 \text{ for all } x \notin S\}$.

Conversely, for any $S \subseteq X$ the set of all functions $X \to \mathbf{F}$ supported on S is an ideal of R. This therefore gives a bijection between the set of subsets of X and the set of ideals of R.

Therefore the set of maximal ideals of R is the set

$$\{\ker \psi_x : x \in X\} ,$$

where $\psi_x \colon R \to \mathbf{F}$ is the function that takes $f \in R$ to $f(x) \in \mathbf{F}$ (which is a ring homomorphism).

Problem 1B. *Score:*

Which of the following series converge? Give reasons.

1. $$\sum_{n=1}^{\infty} \frac{(2n)!(3n)!}{n!(4n)!} .$$

2. $$\sum_{n=2}^{\infty} \frac{1}{n^{1+1/(\log n)^2}} .$$

Solution: The first converges by the ration test, and the second diverges by comparison with the harmonic series.

Problem 2B. *Score:*

Suppose that f is a smooth function from the reals to the reals satisfying the differential equation

$$f'(x) = \sin(f(x))e^{-x^2}$$

Prove that f is bounded.

Solution:

$|f(a) - f(b)| \leq \int_a^b |f'(x)dx \leq \int_{-\infty}^{\infty} e^{-x^2} dx$ which is finite, so f is bounded.

Problem 3B. *Score:*

Let the function f be given by $f(x) = 0$ if x is irrational and $f(x) = 1/n^2$ if $x = m/n$ where m, n are coprime integers and $n > 0$. Show that there is a point where f is continuous but not differentiable.

Solution:

The function f is continuous at all irrational points, so in particular if the derivative exists at some point it must be 0. Suppose x is the limit of the numbers $x_m = 1/2^1 + 1/2^2 + \cdots + 1/2^{2^m}$ then $(f(x) - f(x_m))/(x - x_m)$ does not tend to 0 as m tends to infinity, so f is not differentiable at x.

Problem 4B. *Score:*

Evaluate

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)^2} \, dx \; .$$

Solution:

The integrand is the imaginary part of the function

$$f(z) = \frac{ze^{iz}}{(z^2-1)^2} \; ,$$

so we will use contour integration to evaluate this integral.

For all (real) $R > 1$ let C_R be the positively oriented contour consisting of the interval $[-R, R]$ on the real axis, together with the semicircle $|z| = R$, $\operatorname{Im} z \geq 0$. The function f is holomorphic except for double poles at $z = \pm i$, so we need to find its residue at $z = i$. Write

$$f(z) = \frac{g(z)}{(z-i)^2} \; , \qquad \text{where} \quad g(z) = \frac{ze^{iz}}{(z+i)^2} \; .$$

Then the residue of f at $z = i$ is the coefficient of $z - i$ in the Taylor series expansion of $g(z)$ about $z = i$, which is

$$\begin{aligned} g'(i) &= \left(\frac{(e^{iz} + ize^{iz})(z+i)^2 - 2(z+i)ze^{iz}}{(z+i)^4} \right) \Bigg|_{z=i} \\ &= \frac{(e^{-1} - e^{-1})(2i)^2 - 2(2i)ie^{-1}}{(2i)^4} \\ &= \frac{0 - 4i^2e^{-1}}{16i^4} \\ &= \frac{1}{4e} \,. \end{aligned}$$

Therefore

$$\oint_{C_R} f(z)\, dz = 2\pi i \cdot \frac{1}{4e} = \frac{\pi i}{2e} \,.$$

Since $|e^{iz}| = e^{-\operatorname{Im} z} \le 1$ for all z in the upper half plane, we have $|f(z)| \le R/(R^2-1)^2$ on the semicircle in C_R. The length of this semicircle is πR, so the contribution of the integral along the semicircle to the contour integral is bounded in absolute value by $\pi R^2/(R^2-1)^2$, which $\to 0$ as $R \to \infty$.

Therefore, in the limit as $R \to \infty$, the integral along the semicircle approaches 0, and we have

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)^2}\, dx = \operatorname{Im} \lim_{R\to\infty} \oint_{C_R} f(z)\, dz = \operatorname{Im} \frac{\pi i}{2e} = \frac{\pi}{2e} \,.$$

Problem 5B. *Score:*

Suppose that the complex function f is holomorphic and bounded for $\Re(z) > 0$. Prove that it is uniformly continuous for $\Re(z) > 1$.

Solution:

By the Cauchy integral formula the derivative is bounded in the region $\Re(z) > 1$, so the function is uniformly continuous in this region.

Problem 6B. *Score:*

Prove that a complex square matrix of finite order is diagonalizable. Give an example of a square matrix of finite order (over some other algebraically closed field) that is not diagonalizable.

Solution:

The minimal polynomial of a matrix of finite order n divides the polynomial $x^n - 1$. Over the complex numbers this has no repeated roots so the matrix is diagonalizable.

The matrix $\binom{11}{01}$ over a field of characteristic 2 has order 2 but is not diagonalizable.

Problem 7B. *Score:*

Find the number of conjugacy classes of complex 5 by 5 matrices such that all eigenvalues are 1.

Solution:

Putting the matrix in Jordan normal form show that the number of conjugacy classes is the number of partitions of 5, which is 7.

Problem 8B. *Score:*

Let G be a finite group and H be a subgroup.

(a) Show that the number of subgroups of G conjugate to H divides the index of H.

(b) Show that if

$$G = \bigcup_{g \in G} gHg^{-1}$$

then $G = H$.

Solution:

(a) Let X denote the set of all subgroups conjugate to H. Then G acts transitively on X and the stabilizer of $H \in X$ coincides the normalizer $N(H)$ of H. Then

$$|X| = \frac{|G|}{|N(G)|} = \frac{[G:H]}{[N(H):H]}.$$

(b) Since any subgroup contains the identity element we have

$$|\bigcup_{g \in G} gHg^{-1}| \leq |X||H| - |X| + 1.$$

Since $|X||H| \leq |G|$ we have $0 \leq 1 - |X|$. This implies $|X| = 1$, H is normal and therefore $H = G$.

Problem 9B. *Score:*

Let $p(z)$ be a polynomial with real coefficients such that $p(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$. Show that if the degree of $p(z)$ is d then $d!p(z) \in \mathbb{Z}[z]$.

Solution: Follows from the Lagrange interpolation formula in the points $0, 1, \ldots d$.

$$p(z) = \sum_{j=0}^{d} \frac{\prod_{i \neq j}(z - i)}{\prod_{i \neq q}(j - i)}.$$