Math 123, Section 1: Fall 1994, J. Strain.

Final Exam, 8 December 1994.

Name:

$\alpha > 0$

Problem 1. *(6 points)*
(a) Suppose $\alpha 0$ and let f be a continuous function on R with $|f(y)| \leq |y|^2$. Find $\epsilon 0$ such that every solution y of $y' = -\alpha y + f(y)$ with $|y(0)| < \epsilon$ exists for all $x \geq 0$.
(b) Show that solutions of $y' = -y + y^2$ may not exist for all $x \geq 0$ if $y(0)$ is too large.

Solution:

Problem 2. *(4 points)*
Suppose A is an $n \times n$ matrix whose eigenvalues λ_j all have nonzero real parts. Show that every solution y of $y' = Ay$ satisfies either $||y(x)|| \to \infty$ or $||y(x)|| \to 0$ as $x \to +\infty$.

Solution:

Problem 3. *(4 points)*

(a) Suppose $f \in C^0(R^n; R^n)$ satisfies $f(0) = 0$ and $||f(x) - f(y)|| \leq ||x - y||$. Show that the solution of $y' = f(y)$ satisfies $||y(x)|| \leq e^x ||y_0||$ and exists for all $x \in R$.

(b) Let

$$y_1(x) = \cos(\sqrt{100 - x})$$

$$y_2(x) = \sin(\sqrt{100 - x}).$$

Can $y = (y_1, y_2)$ solve a 2×2 autonomous system $y' = f(y)$ with $f \in C^1(R^2; R^2)$? If so, find such a system; otherwise, explain why not.

Solution:

Problem 4. *(4 points)*
(a) Suppose $f \in C^1(R)$ and

$$y_1' = f(y_2)$$

$$y_2' = f(y_3)$$

$$y_3' = f(y_1)$$

with $y_1(0) = y_2(0) = y_3(0)$. Show that $y_1 = y_2 = y_3$ for all x.
(b) Show that

$$y' = -2y$$

with $y(0) = 0$ has only the trivial solution.

Solution:

Problem 5. *(6 points)*
Let

$$P = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Find
(a) $Y(x) = e^{Px}$
(b) the solution of $y' = Py$ with $y(0) = (1, 0)^T$
(c) the solution of $y' = Py + (1, 0)^T$ with

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} y(0) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} y(1) = 0.$$

Solution:

Problem 6. *(6 points)*
Suppose f is a C^1 function on $I \times R$ where $I = [0,1]$ and $|f| + |f_x| + |f_y| \le M$. We say that a function y_ϵ which is continuous and piecewise-C^1 on I is an ϵ-solution of

$$y' = f(x,y), \qquad y(0) = 0$$

if $y_\epsilon(0) = 0$ and

$$|y'_\epsilon(x) - f(x, y_\epsilon(x))| \le \epsilon$$

at any $x \in I$ where y'_ϵ exists.
(a) Suppose for each $\epsilon 0$ there is an ϵ-solution y_ϵ which is C^1 on I. Show that there is a sequence $\epsilon_n \to 0$ and a continuous function y such that $y_{\epsilon_n} \to y$ uniformly on I.
(b) Show that if y is the limit from (a) then $y(0) = 0$ and y is a solution of $y' = f(x,y)$.
(c) Show that

$$y^h(x) = 0 \qquad 0 \le x \le h$$

$$y^h(x) = \int_0^{x-h} f(s, y^h(s))ds \qquad h \le x \le 1$$

defines an ϵ-solution for any $\epsilon 0$ (depending on h).

$\varepsilon > 0$

Solution: